ON RESOLUTIONS, MINIMAL AND VIRTUAL

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Abstract. The purpose of this paper is serve as a gentler introduction to virtual resolutions which were first introduced in [BZES17]. However, before discussing virtual resolutions, we show why they are important. This is done by comparing them to minimal resolutions of modules over the standard graded polynomial ring over a field.

1. Introduction

In n-dimensional projective space, there is a correspondence between certain sheaves and finitely generated modules over the standard graded polynomial ring in $n + 1$ variables over a field. Because of this, the geometry of varieties can be investigated through modules over a graded polynomial ring. Homological tools can be used to study the modules and therefore understand properties of the geometry of the varieties.

Section 2 concentrates on introducing tools such as minimal free resolutions that are useful in studying modules. We discuss invariants such as the Betti numbers of a module and mention how Betti numbers are able to explain properties of the modules. The Hilbert function and polynomial are defined in this section and the connection with Castelnuovo–Mumford regularity is described. This section lays the groundwork of attempting to generalize many of these notions to the more general toric variety case.

Toric varieties are defined and discussed in Section 3. In this section, the Cox ring is defined and the correspondence between subvarieties of a toric variety and certain ideals of the Cox ring is discussed. Just as subvarieties of projective space can be understood by studying modules of a polynomial ring, subvarieties of toric varieties can be better understood by studying certain modules over the Cox ring. An emphasis is placed on understanding the multigrading of the Cox ring through the group action of a torus.

In a product of projective spaces or, more generally, in a toric variety, the tools introduced in Section 2 do not as closely reflect the geometry of subvarieties. A proposed replacement of minimal free resolutions called virtual resolutions is given in Section 4. We define virtual resolutions and give two different ways of constructing them. A couple of examples are also provided in this section.

Finally, in Section 5 a couple of possible research directions are discussed. As virtual resolutions are rather new objects of study, there are many directions to pursue. However, we restrict our attention to questions about points in $\mathbb{P}^1 \times \mathbb{P}^1$.

2. Syzygies

In this section we will discuss some results and techniques well known for investigating syzygies of modules in projective space $\mathbb{P}^n$. Throughout this section, let $S = \mathbb{k}[x_0, x_1, \ldots, x_n]$ be the standard graded polynomial ring in $n+1$ variables over an algebraically closed field $\mathbb{k}$ and let $\mathfrak{m} = \langle x_0, ..., x_n \rangle$ be the maximal ideal.

We are able to use modules over S to investigate subvarieties of $\mathbb{P}^n$ because of the following theorem.

Theorem 2.1. *There is a bijective correspondence*

$$\{\textit{nonempty closed subvarieties of } \mathbb{P}^n\} \longleftrightarrow \{\mathfrak{m}\textit{-saturated radical homogeneous ideals}\}.$$

2.1. Minimal Free Resolutions.

Free resolutions were orginially studied by David Hilbert in the 19th century. We start by defining a minimal resolution and providing a few examples. This discussion is largely following [Eis95] and [Eis05].

Definition 2.2. Let R be a ring. A **projective resolution** of an R-module M is a complex

$$F\colon \quad F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_n} F_n \xleftarrow{\varphi_{n+1}} \cdots$$

of projective R-modules F_i such that $H_0(F) = M$ and otherwise F has no homology. We say F is a **free resolution** if all the F_i are free and is a **graded free resolution** if R is a graded ring, all the F_i are graded free modules, and the maps are all homogeneous maps of degree 0. If, for some $n < \infty$, we have $F_{n+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq n$, then we shall say that F is a **finite** resolution of **length** n.

It is straightforward to see that every finitely generated module M over S has a free resolution. To construct one, simply let F_0 be the free module generated by a set of generators on M. The kernel of the map $F_0 \to M$ is called the **first syzygy module** of M. Next let F_i be the free module on a set of generators of the ith syzygy of M and repeat this process. The ith syzygy module is the image of the map $F_i \to F_{i-1}$ and the elements of these syzygy modules are called syzygies.

If M is a graded S-module, then define $M(a)$ to be the module M shifted by a. That is, $M(a)_n$, the nth graded component of $M(a)$ is M_{a+n}, the $(a+n)$th graded component of M.

Example 2.3. Let $S = \mathbb{k}[x, y]$ and $M = S/\langle x, y\rangle = \mathbb{k}$. Following the procedure above, $F_0 = S$. Now let $K_0 = \ker(F_0 \to M)$. Then $K_0 = \langle x, y\rangle$ and thus F_1 is free of rank 2.

Let $\varphi_1\colon S^2(-1) \to S$ be defined by $\varphi_1(a, b) = ax + by$. Notice that this is a degree zero map. $K_1 = \ker(\varphi_1) = \langle (y, -x)\rangle$ is generated by a single element so F_2 has rank 1.

In order for φ_2 to be a degree zero map, $F_2 = S(-2)$ and $\varphi_2\colon 1 \mapsto (y, -x)$. This map is injective so $K_3 = 0$ which means that $F_3 = 0$ so our free resolution has length 2. The free resolution is

$$F\colon \quad S \xleftarrow{\begin{bmatrix} x & y \end{bmatrix}} S^2(-1) \xleftarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} S(-2) \longleftarrow 0,$$

where the maps over the arrows are denoted by the corresponding matrices. This particular example is the Koszul complex of length 2. ⋄

It is not true in general that free resolutions are unique. Of course, we could always take the direct sum of a complex F with the shifted trivial complex $0 \leftarrow S \xleftarrow{\text{id}} S \leftarrow 0$ to yield a new free resolution. If S is a graded ring (such as a polynomial ring), there is however, one "smallest" free resolution, called the minimal resolution, in the following sense. If F and G are two minimal resolutions of M, then there is a graded isomorphism of complexes $F \to G$ that induces the identity on M.

Definition 2.4. A free resolution F of an S-module M is **minimal** if the image of each $\varphi_n \colon F_n \to F_{n-1}$ is contained in $\mathfrak{m}F_{n-1}$. In other words, a free resolution F is minimal if all the maps of the complex $\Bbbk \otimes_S F$ are zero.

In fact, it turns out that every free resolution contains the minimal resolution as a direct summand. Looking at Example 2.3 above, it is easy to see that $F \otimes_S \Bbbk$ has all zero maps (because all entries in the matrices are positively graded) so F is the minimal resolution of $\Bbbk$. Next is a slightly more interesting example of a minimal free resolution.

Example 2.5. Let $S = \Bbbk[x, y]$. Let $\mathfrak{m}$ be the maximal ideal $\langle x, y\rangle$ in S. Following the exact same procedure as in Example 2.3, a resolution of $\mathfrak{m}^2 = \langle x^2, xy, y^2\rangle$ is

$$S \xleftarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x \end{bmatrix}} S(-3)^2 \longleftarrow 0.$$

Notice that all entries of the matrices are contained in $\mathfrak{m}$ so this resolution is indeed minimal. ◇

Up until this point in this paper, all examples have had finite minimal free resolutions. It turns out that for finitely generated modules over S, this will always be the case.

Theorem 2.6 (Hilbert Syzygy Theorem)**.** *If $S = \Bbbk[x_0, ..., x_n]$, then every finitely generated graded S-module has a finite graded resolution of length $\leq n + 1$, by finitely generated free modules.*

2.2. **The Hilbert Function.**

Hilbert also defined the Hilbert function which is a way to measure invariants of a module.

Definition 2.7. Let M be a finitely generated module over $S = \Bbbk[x_0, x_1, ..., x_n]$. Recall the degree d component of M be denoted by M_d. The **Hilbert function** of M is the numerical function

$$H_M(d) := \dim_{\Bbbk}(M_d).$$

Let F be a finite free resolution of M of length r. Since each φ_i is a degree zero map, extracting the dth degree component of M and the F_i, there is an exact sequence of vector spaces

$$F_d\colon \quad (F_0)_d \longleftarrow (F_1)_d \longleftarrow \cdots \longleftarrow (F_r)_d \longleftarrow 0.$$

So the Hilbert function can be computed as

$$H_M(d) = \sum_{i=0}^{r} (-1)^i \dim_{\Bbbk}(F_i)_d. \tag{2.7.1}$$

Example 2.8. Revisiting Example 2.5 where $S = \Bbbk[x, y]$ and $M = S/\mathfrak{m}^2$, notice that the zero degree component of M is $M_0 = \Bbbk$, so $H_M(0) = 1$. Moving to the degree 1 component of M, since $M_1 = \operatorname{span}_{\Bbbk}\langle x, y\rangle$, then $H_M(1) = 2$. Finally, $M_n = 0$ for $n \geq 2$ so $H_M(n) = 0$ for $n \geq 2$. This is summarized below:

$$H_M(d) = \begin{cases} 1 & \text{for } d = 0, \\ 2 & \text{for } d = 1, \\ 0 & \text{for } d \geq 2. \end{cases}$$

◇

In particular, in Example 2.8, the Hilbert function stabilized to zero. Does this always happen? It turns out the answer is no. There is good news, though. Hilbert discovered that the infinite values of the Hilbert function can be encoded in a polynomial.

Theorem 2.9. *If M is a finitely generated graded module over S, then for $d \gg 0$, $H_M(d)$ agrees with a polynomial of degree $\leq n + 1$.*

Definition 2.10. The polynomial in Theorem 2.9, denoted $P_M(d)$, is called the **Hilbert polynomial** of M.

2.3. **Betti Numbers.** Because minimal free resolutions are unique up to graded isomorphism, the twists of the free modules at each step in the resolution are independent of the choice of minimal free resolution. These data are called the **Betti numbers**. If F is a free complex of R-modules, where

$$F\colon \quad F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow \cdots$$

with $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$, then the $\beta_{i,j}$ are called the ith graded Betti numbers of F. The ith total Betti number is $\beta_i = \sum_j \beta_{i,j}$. If F is a free resolution of a module M, then this means that M is generated in degrees $\beta_{0,j}$. In Example 2.5, the only nonzero Betti numbers are

$$\beta_{0,0} = 1, \quad \beta_{1,2} = 3, \quad \beta_{2,3} = 2.$$

However, simply listing Betti numbers can be cumbersome. We can better encode the Betti numbers in a Betti diagram or Betti table where the entry in column i and row j is $\beta_{i,i+j}$. The Betti table of Example 2.5 is:

	0	1	2
0	1	-	-
1	-	3	2

where for clarity the zeros in the table have been replaced with dashes. Also notice that we could have extended this table to the right and down as far as we would like to with dashes, but it makes more sense to only include the nonzero part of the Betti table. The projective dimension of a module can immediately be seen by looking at the width of the Betti table of the module. In the case of this example, the projective dimension is 2. We will see in Section 2.4 that the height of the Betti table has to do with the regularity of the module.

An easy combinatorial argument gives $\dim_{\Bbbk} S_d = \binom{n+d}{d}$. Putting this together with the above discussion of Betti numbers and (2.7.1), we get the following.

Proposition 2.11. *Suppose $S = \Bbbk[x_0, ..., x_n]$ and M is a graded S-module. If M has a finite free resolution*

$$F: \quad F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_r$$

where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$ are all finitely generated, then

$$H_M(d) = \sum_{i=0}^{r} (-1)^i \sum_j \binom{n+d-\beta_{i,j}}{d}.$$

Notice that we will always be able to find such a finite free resolution by Theorem 2.6. This proposition says exactly what the Hilbert polynomial of M is. An interesting question to ask is if there is an invariant of M that tells us when $H_M(d) = P_M(d)$. We will answer this question in Section 2.4.

Notice that the Hilbert Syzygy Theorem (Theorem 2.6) only gives a bound on the length of a free resolution of a module. Another natural question to ask is whether there are invariants of a module that determine the length of a resolution. In the graded case, where projective modules are free modules, the depth of the module and ring can be chosen to play the role of these invariants.

Theorem 2.12 (Auslander–Buchsbaum formula)**.** *Let R be a graded ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R-module. If M is of finite projective dimension, then*

$$\operatorname{pdim}(M) = \operatorname{depth}(\mathfrak{m}, R) - \operatorname{depth}(\mathfrak{m}, M).$$

Consider the ideal I of a finite number of points in $\mathbb{P}^2$. If $S = \Bbbk[x, y, z]$, then the Auslander–Buchsbaum formula tells us that the projective dimension of S/I is 2 so the projective dimension of I is 1. We also have the following useful tool for ideals of projective dimension 1.

Theorem 2.13 (Hilbert–Burch)**.** *Suppose that an ideal I in a Noetherian ring R admits a free resolution of length 1:*

$$0 \longleftarrow I \longleftarrow G \xleftarrow{M} F \longleftarrow 0.$$

If the rank of the free module F is t, then the rank of G is $t+1$, and there exists a nonzerodivisor a such that $I = aI_t(M)$, where $I_t(M)$ is the ideal generated by the $t+1$ maximal minors of the matrix representation of M with respect to given bases of F and G. In fact, the generator of I that is the image of ith basis vector of G is a times the ith minor of M. Furthermore, the depth of $I_t(M)$ is 2.

Conversely, given a nonzerodivisor $a \in R$ and given a $(t+1) \times t$ matrix M with entries in R such that $\operatorname{depth}(I_t(M)) \geq 2$, the ideal $I = aI_t(M)$ admits a free resolution of length one as above. The ideal I has depth 2 precisely when the element a is a unit.

One way to prove the converse of this theorem is to construct the complex $0 \leftarrow I \leftarrow F \leftarrow G \leftarrow 0$ and then check that it has no homology. In general, though, it is not always easy to check that a given free complex is a resolution. However, by the following theorem, it suffices to count the ranks of free modules and depth of ideals generatered by maximal minors. If φ is a map of free modules, denote by $I(\varphi)$ the ideal generated by the maximal minors of the matrix representation of φ with respect to some given basis.

Theorem 2.14 (Buchsbaum–Eisenbud). *Let R be a Noetherian ring. A complex of free R-modules*

$$F: \quad F_0 \xleftarrow{\varphi_1} F_1 \longleftarrow \cdots \longleftarrow F_{r-1} \xleftarrow{\varphi_r} F_r \longleftarrow 0.$$

is exact if and only if $\operatorname{rank}\varphi_{i+1} + \operatorname{rank}\varphi_i = \operatorname{rank} F_i$ *and* $\operatorname{depth} I(\varphi_i) \geq i$ *for every i.*

2.4. **Castelnuovo–Mumford Regularity.** We now examine the question: When does the Hilbert function become polynomial? This leads us to the notion of Castelnuovo–Mumford regularity. From now on, when we say regularity, we mean Castelnuovo–Mumford regularity. There are a couple of equivalent ways of defining regularity for S-modules. As seen below, regularity is not really a property of a module, but rather a property of a complex.

Definition 2.15. Let

$$F: \quad F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_i \longleftarrow \cdots$$

be a graded complex of free S-modules with $F_i = \bigoplus_j S(-a_{i,j})$. The regularity of F is

$$\operatorname{reg} F := \sup_{i,j}\{a_{i,j} - i\}.$$

If M is an S-module, and F is the minimal resolution of M we define $\operatorname{reg} M := \operatorname{reg} F$.

If the Betti diagram of a module is known, then we can instantly find the regularity of the module. The regularity is simply the height of the Betti diagram. This is because the entry in column i and row j in the Betti diagram is defined to be $\beta_{i,j-i}$.

To answer the question of when the Hilbert function becomes polynomial, we have the following.

Theorem 2.16. *Let M be a finitely generated graded module over the polynomial ring $S = \Bbbk[x_0, ..., x_n]$.*

(1) The Hilbert function $H_M(d)$ agrees with the Hilbert polynomial $P_M(d)$ for $d \geq \operatorname{reg} M + 1$.
(2) More precisely, if M is a module of projective dimension δ, then $H_M(d) = P_M(d)$ for $d \geq \operatorname{reg} M + \delta - n$.

It is also possible to define the regularity in terms of local cohomology. Following [BS98]'s introduction of local cohomology, if M is an R-module, let

$$\Gamma_I(M) = \bigcup_{n\in\mathbb{N}} (0 :_M I^n).$$

That is, $\Gamma_I(M)$ is the set of elements of M that are annihilated by some power of I. This is a submodule of M and if f is a homomorphism of R-modules $M \to N$, then $f(\Gamma_i(M)) \subset$

$\Gamma_i(N)$ so f induces a homomorphism $\Gamma_I(M) \to \Gamma_I(N)$ and it is easy to check that $\Gamma_I(-)$ is functorial. In fact, Γ_I is a left-exact covariant functor. So it makes sense to give the ith right derived functor of Γ_I a name.

Definition 2.17. For $i \in \mathbb{N}_0$, the ith right derived functor of Γ_I is denoted by H_I^i and is called the i**th local cohomology functor** with respect to I.

Then the regularity of a module can be defined in the following equivalent way.

Theorem 2.18. *Let M be a finitely generated graded S-module and let d be an integer. The following conditions are equivalent:*

(i) $d \geq \operatorname{reg} M$

(ii) $d \geq \max\{e | \operatorname{H}_{\mathfrak{m}}^i(M)_e \neq 0\} + i$ *for all* $i \geq 0$.

(iii) $d \geq \max\{e | \operatorname{H}_{\mathfrak{m}}^0(M)_e \neq 0\}$ *and* $\operatorname{H}_{\mathfrak{m}}^i(M)_{d-i+1} = 0$ *for all* $i \geq 0$.

3. Toric Varieties

Toric varieties are among the most concrete varieties since they have much more combinatorial structure than other abstract varieties. In this section we will concentrate on the correspondence between the geometry of a toric variety and algebra. Specifically we will look at the Cox ring. Much of the content of this section can be found in [CLS11].

Definition 3.1. A **toric variety** is an irreducible variety V that contains a torus $T \cong (\mathbb{k}^*)^m$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.

Affine toric varieties can be constructed from strongly convex rational cones. General toric varieties can be glued together from affine toric varieties and their combinatorial data is encoded in a fan Σ of cones. Let $\Sigma(1)$ denote the set of one-dimensional cones (rays) in Σ. Following [Cox95], define the **total coordinate ring** or **Cox ring** as

$$S = \mathbb{k}[x_\rho | \rho \in \Sigma(1)].$$

In fact, given a fan, the toric variety of the fan can be constructed using the quotient construction as outlined in [CLS11]. If the cardinality of $\Sigma(1)$ is n, then the toric variety will be $(\mathbb{k}^n \backslash Z)/\sim$ where Z is some exceptional set in $\mathbb{k}^n$ and points in $\mathbb{k}^n \backslash Z$ are identified if they are in the same orbit of a specific group action $G \times \mathbb{k}^n \to \mathbb{k}^n$ where $G \subset (\mathbb{k}^*)^n$. The exceptional set can be defined as the vanishing set of a specific ideal in the Cox ring. Namely, define

$$x^{\hat{\sigma}} := \prod_{\rho \in \Sigma(1) \backslash \sigma} x_\rho.$$

Then $Z = V(B)$, where

$$B := \langle x^{\hat{\sigma}} | \sigma \in \Sigma \rangle \subset S.$$

This is called the **irrelevant ideal** of the Cox ring. If τ is a face of σ, then $x^{\hat{\sigma}}$ divides $x^{\hat{\tau}}$ so in computing B, it suffices to index over all maximal faces of Σ. Additionally, the group action will induce a multigrading on the Cox ring. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_k)$ and $\boldsymbol{\lambda}^{\boldsymbol{a}} = \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_k^{a_k}$. Then if $G \cong (\Bbbk^*)^k$ and $\boldsymbol{\lambda} \in G$ acts on $(t_1, ..., t_n) \in (\Bbbk^*)^n$ by

$$\boldsymbol{\lambda} \cdot (t_1, ..., t_n) = (\boldsymbol{\lambda}^{\boldsymbol{a_1}} t_1, \boldsymbol{\lambda}^{\boldsymbol{a_2}} t_2, ..., \boldsymbol{\lambda}^{\boldsymbol{a_n}} t_n),$$

then we say

$$\deg(x_i) = \boldsymbol{a_i}. \tag{3.1.2}$$

In this way the geometry can be changed into algebra and tools from commutative algebra (such as those described in Section 2) can be used to explore properties of the toric variety.

Theorem 3.2. *Let X_Σ be a simplicial toric variety. Then there is a bijective correspondence*

$$\{\textit{nonempty closed subvarieties of } X_\Sigma\} \longleftrightarrow \{B\textit{-saturated radical homogeneous ideals}\}.$$

Another way to interpret the multigrading of S is through the class group of S. The class group $\mathrm{Cl}(X)$ is the free abelian group generated by the codimension one subvarieties of X. Under the Orbit-Cone Correspondence (Theorem 3.2.6 in [CLS11]), one dimensional cones in the fan of a toric variety X_Σ correspond to codimension one orbits. The closure of this codimension one orbit is a prime divisor so each ray $\rho \in \Sigma$ corresponds to a prime divisor D_ρ. Defining $\deg(x_\rho) = D_\rho$, the Cox ring is graded by Weil divisors.

The grading by the class group matches with the grading given by the weights of the G action. If M is the group of characters for the toric variety X_Σ, then there is a short exact sequence (for toric varieties without torus factors)

$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where if $\chi^m \in M$ and u_ρ is the minimal generator of the ray ρ, then

$$\mathrm{div}(m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

The grading of x_ρ is given by the image of e_ρ (the basis vector that ρ corresponds to in $\mathbb{Z}^{\Sigma(1)}$) in $\mathrm{Cl}(X_\Sigma)$. This grading matches the grading from the weights of the $(\Bbbk^*)^k \cong G$ action as described in (3.1.2) above. In fact, the group G that acts on $\Bbbk^n \backslash Z$ is the group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \Bbbk^*)$. In the examples that follow we will simply state G for concreteness. However, it is not hard to find G from the following lemma from [CLS11].

Lemma 3.3. *Let X_Σ be the toric variety of a fan Σ. Let $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \Bbbk^*) \subset (\Bbbk^*)^{\Sigma(1)}$. Then given a basis $e_1, e_2, ..., e_n$ of M, and minimal generators μ_ρ of $\rho \in \Sigma(1)$*

$$G = \left\{ (t_\rho) \in (\Bbbk^*)^{\Sigma(1)} | \prod_{\rho \in \Sigma(1)} t_\rho^{\langle e_i, \mu_\rho \rangle} = 1 \textit{ for } 1 \le i \le n \right\}.$$

As a first example, the toric variety $\mathbb{P}^n$ can be constructed using the quotient construction.

Example 3.4. Let $e_1, e_2, ..., e_n$ be the standard basis vectors of the lattice $\mathbb{Z}^n$. Define $e_0 = -e_1 - e_1 - \cdots - e_n$. The fan corresponding to $\mathbb{P}^n$ consists of cones generated by all proper subsets of $\{e_0, e_1, ..., e_n\}$. So in this case, the Cox ring is $\Bbbk[x_0, x_1, ..., x_n]$. To recover $\mathbb{P}^n$, first start with $\mathbb{k}^{n+1}$. Now remove the exceptional set and then quotient out by the group action. Notice that every maximal face will not contain exactly one ray. So the irrelevant ideal $B = \langle x_0, x_1, ..., x_n \rangle$ and $Z = V(B) = 0$. The group $G = \{(\lambda, \lambda, ..., \lambda) \in \mathbb{k}^{n+1} | \lambda \in \mathbb{k}^*\} \cong \mathbb{k}^*$ acts on $\mathbb{k}^{n+1}$ by scalar multiplication. So the toric variety is

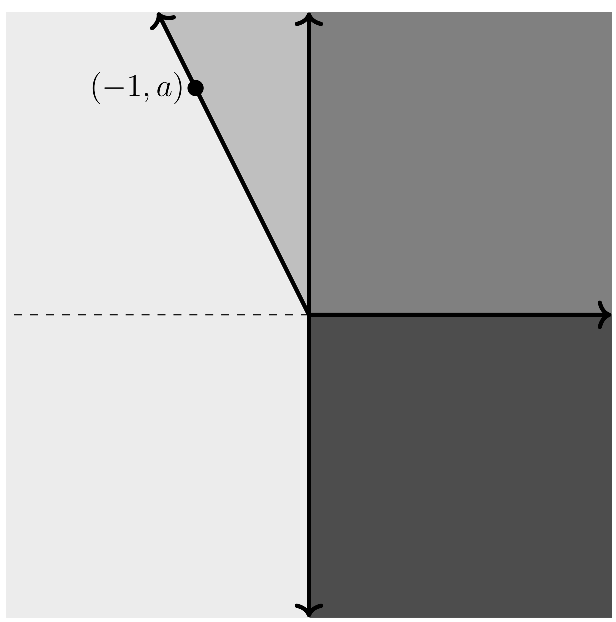
$$(\mathbb{k}^{n+1} \backslash 0)/\mathbb{k}^* = \mathbb{P}^n_{\mathbb{k}}.$$

Since $\mathbb{k}^*$ acts on $\mathbb{k}^{n+1}$ by $\lambda \cdot (t_0, t_1, ..., t_n) = (\lambda t_0, \lambda t_1, ..., \lambda t_n)$, then $\deg(x_i) = 1$ for every $0 \leq i \leq n$. This is the standard grading on $\mathbb{k}[x_0, ..., x_n]$ and the irrelevant ideal is the maximal ideal. ◇

Example 3.4 makes it clear that Theorem 2.1 is a special case of Theorem 3.2 where $X_\Sigma = \mathbb{P}^n$ and $B = \mathfrak{m}$.

In Example 3.4, the grading on the Cox ring is the standard grading, but in general this does not always happen. Next we examine the Hirzebruch surface $\mathbb{F}_a$.

Example 3.5. The fan $\Sigma \subset \mathbb{R}^2$ of the Hirzebruch surface $\mathbb{F}_a$ consists of the cones Cone(e_1, e_2), Cone$(e_2, -e_1 + ae_2)$, Cone$(-e_1 + ae_2, -e_2)$ and Cone$(-e_2, e_1)$ where e_1, e_2 are the standard basis vectors of $\mathbb{R}^2$.



Let $u_1 = -e_1 + ae_2$, $u_2 = e_1$, $u_3 = -e_2$, and $u_4 = e_2$. These are the rays of Σ. The Cox ring of $\mathbb{F}_a$ is $S = \mathbb{k}[x_1, x_2, x_3, x_4]$. The irrelevant ideal is

$$B = \langle x_1x_4, x_1x_3, x_2x_3, x_2x_4 \rangle = \langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle$$

so $Z = V(B) = \Bbbk^2 \times \{0,0\} \cup \{0,0\} \times \Bbbk^2$. The group $G = \{(\lambda, \lambda, \mu, \lambda^{-a}\mu) | \lambda, \mu \in \Bbbk^*\} \cong (\Bbbk^*)^2$ acts on $\Bbbk^4$ via coordinate-wise multiplication. So in this case the grading of S is

$$\begin{aligned} \deg(x_1) &= (1,0) \\ \deg(x_2) &= (1,0) \\ \deg(x_3) &= (0,1) \\ \deg(x_4) &= (-a,1). \end{aligned}$$

As a final remark to this example, we note that if $a = 0$, then $\mathbb{F}_a \cong \mathbb{P}^1 \times \mathbb{P}^1$. So it is immediate that the irrelevant ideal of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\langle x_1x_4, x_1x_3, x_2x_3, x_2x_4 \rangle$, which has primary decomposition $\langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle$. ⋄

4. Virtual Resolutions

As shown in Section 2, minimal resolutions are a good way to study modules over projective space. However, in products of projective space, it turns out that minimal resolutions contain too much algebraic structure. In particular, they contain information that does not matter geometrically. This results in minimal resolutions being longer and the ranks of the free modules being larger than needed. The reason for this difference is that in $\mathbb{P}^n$, the vanishing set of the irrelevant ideal is the origin. Recall from Example 3.4 that in this case, the irrelevant ideal is the maximal ideal.

On the other hand, consider the simplest case of a product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ where the first coordinate is parametrized by x_1, x_2 and the second is parametrized by y_1, y_2. The irrelevant ideal is no longer the maximal ideal. Instead, as noted at the end of Example 3.5, the irrelevant ideal is $(x_1, x_2) \cap (y_1, y_2) = (x_1y_1, x_1y_2, x_2y_1, x_2y_2)$. The vanishing set of this ideal is more complicated than just the origin. Recall that the definition of the minimal free resolution required the image of φ_i to be contained in $\mathfrak{m}F_{i-1}$. The irrelevant ideal will play a similar role in the definition of a virtual resolution.

Before continuing any further we shall establish some notational conventions. For the remainder of this paper, $\boldsymbol{n} = (n_1, n_2, ..., n_k)$ and $\mathbb{P}^{\boldsymbol{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. Also denote the Cox ring by S and the irrelevant ideal by B. As mentioned in Section 3, the Cox ring is graded by the class group. In a product of projective spaces, all Weil divisors are Cartier so the class group coincides with the Picard group. Therefore S is graded by the Picard group $\operatorname{Pic}(X)$.

Definition 4.1. Let $X = \mathbb{P}^{\boldsymbol{n}}$. A **virtual resolution** of a $\operatorname{Pic}(X)$-graded S-module M is a complex of $\operatorname{Pic}(X)$-graded free S-modules

$$F : F_0 \longleftarrow F_1 \longleftarrow \cdots$$

such that the $H_0(F)$ is isomorphic to the B-saturation of M, $(M : B^\infty) = \bigcup_{i \geq 1}(M : B^i)$ and the higher homology groups are supported on B.

An equivalent definition of a virtual resolution of M is a complex of free S-modules that when sheafified becomes a locally free vector bundle resolution of the sheaf $\tilde{M}$.

4.1. **Multigraded Regularity.** In $\mathbb{P}^{\boldsymbol{n}}$, the Hilbert function and polynomial of modules are no longer functions of a single variable, but rather a function of k variables. So it is harder to talk about regularity in the previous sense. Instead, [MS04] defines a multigraded regularity.

Definition 4.2. Let X be a smooth projective toric variety and let S denote the Cox ring of X. Let $\mathbb{N}\mathcal{C}$ be the semigroup generated by a finite subset $\mathcal{C} = \{\mathbf{c_1}, ..., \mathbf{c_l}\}$ of $\mathrm{Pic}(X)$. For $\boldsymbol{m} \in \mathrm{Pic}(X)$, we say that a $\mathrm{Pic}(X)$-graded S-module M is $\boldsymbol{m}$-regular if the following conditions are satisfied:

(a) $H^i_B(M)_{\boldsymbol{p}} = 0$ for all $i \geq 1$ and all $\boldsymbol{p} \in \bigcup(\boldsymbol{m} - \lambda_1\boldsymbol{c}_1 - \cdots - \lambda_l\boldsymbol{c}_l + \mathbb{N}\mathcal{C})$ where the union is over all $\lambda_1, ..., \lambda_l \in \mathbb{N}$ such that $\lambda_1 + \cdots \lambda_l = i - 1$.
(b) $H^0_B(M)_{\boldsymbol{p}} = 0$ for all $\boldsymbol{p} \in \bigcup_{1 \leq j \leq l}(\boldsymbol{m} + \boldsymbol{c}_j + \mathbb{N}\mathcal{C})$.

We set $\mathrm{reg}(M) := \{\boldsymbol{p} \in \mathrm{Pic}(X) | M \text{ is } \boldsymbol{p}\text{-regular}\}$.

Notice that if $X \subset \mathbb{P}^n$ so that $\mathrm{Pic}(X) = \mathbb{Z}$, then by Theorem 2.18 this definition is the same as the single-graded regularity defined in Section 2.4.

A difference between the single-graded and multigraded regularity is that the multigraded regularity is not a single element, but rather a set of elements. This difference isn't so large as we can think of the single-graded regularity as a set as well. Indeed, if M is p-regular, then M is q-regular for all $q \geq p$. In general, the multigraded regularity cannot be defined by a single minimal element, but we can still talk about the minimal elements of the multigraded regularity. This is because it follows from Definition 4.2 that if M is $\boldsymbol{p}$-regular, then M is $\boldsymbol{q}$-regular for all $\boldsymbol{q} \succeq \boldsymbol{p}$ where $\succeq$ denotes the component-wise partial ordering on $\mathbb{Z}^k$.

The multigraded regularity acts like the single graded regularity in another useful way. If $\boldsymbol{p} \in \mathrm{reg}(M)$, then the Hilbert function matches the Hilbert polynomial for all $\boldsymbol{q} \succeq \boldsymbol{p}$.

4.2. **Virtual Resolutions of a Pair.** [BZES17] showed that there are a couple of different ways of constructing virtual resolutions. One way is to use the multigraded Castelnuovo–Mumford regularity.

Let M be an S-module and let $\boldsymbol{d}$ be an element in the regularity of M. Then the **virtual resolution of the pair** $(M, \boldsymbol{d})$ is obtained from the minimal resolution of M by finding the minimal free resolution of M and removing all twists not less than $\boldsymbol{n} + \boldsymbol{d}$. This can only decrease the length of the minimal resolution so the virtual resolution of a pair is always at least as short as the minimal resolution. Also notice that we are justified in saying "the" virtual resolution of a pair since as the minimal resolution of the module M is unique up to isomorphism, so too is the virtual resolution of the pair $(M, \boldsymbol{d})$.

Example 4.3. Consider an ideal I of three points $([1:1],[1:4])$, $([1:2],[1:5])$, and $([1:3],[1:6])$ (where $[a:b]$ is a set of homogeneous coordinates in $\mathbb{P}^1$) in general position in $\mathbb{P}^1 \times \mathbb{P}^1$. *Macaulay2* [M2] shows that the B-saturation of I is $J = (I : B^\infty)$ is

$$\begin{gathered}\langle 3x_0x_2 + x_1x_2 - x_0x_3, 40x_1^2x_2 + 6x_0^2x_3 - 13x_0x_1x_3 - 3x_1^2x_3, 6x_0^3 - 11x_0^2x_1 + 6x_0x_1^2 - x_1^3, \\ 120x_1x_2^2 - 34x_1x_2x_3 - 2x_0x_3^2 + 3x_1x_3^2, 120x_2^3 - 74x_2^2x_3 + 15x_2x_3^2 - x_3^3\rangle\end{gathered}.$$

The minimal resolution of J is

$$S \longleftarrow \begin{matrix} S(0,-3) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \\ \oplus \\ S(-1,-1) \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^2 \\ \oplus \\ S(-3,-1)^2 \end{matrix} \longleftarrow \begin{matrix} S(-2,-3) \\ \oplus \\ S(-3,-2) \end{matrix} \longleftarrow 0.$$

By examining the Hilbert function and Hilbert polynomial, it can be seen that for all $(a, b) \succeq (2, 0)$, $H_J(a, b) = P_J(a, b)$. Indeed, it turns out that (2,0) is in the regularity of J. The virtual resolution of the pair $(J, (2, 0))$ is

$$S \longleftarrow \begin{matrix} S(-1,-1) \\ \oplus \\ S(-2,-1) \\ \oplus \\ S(-3,0) \end{matrix} \longleftarrow S(-3,-1)^2 \longleftarrow 0.$$

The above virtual resolution of the pair has shorter length and is thinner than the minimal resolution. $\diamond$

Notice by the definition of the virtual resolution of a pair, that if the minimal resolution of a module M is known, then given an element $\boldsymbol{d}$ in the regularity of M, it is very easy to find the virtual resolution of the pair $(M, \boldsymbol{d})$. However, the full minimal resolution need not be computed in order to find the virtual resolution of the pair $(M, \boldsymbol{d})$. Rather the same strategy as outlined in Section 2 used to find the minimal resolution of M can be used, but we can simply omit any generators of degree $\boldsymbol{n} + \boldsymbol{d}$ or higher. This is a computationally easier way to compute the virtual resolution of the pair and is the content of Algorithm 4.4 in [BZES17].

In fact, it appears that the multigraded regularity of a module and the module's set of virtual resolutions are intimately related. Not only can we construct a virtual resolution if we know an element in the regularity of a module, but [BZES17] showed that the set of virtual resolutions of a module determine the regularity of the module as well.

4.3. **Another Way of Constructing a Virtual Resolution.** In the case where the variety is zero-dimensional (i.e. a set of points in $\mathbb{P}^{\boldsymbol{n}}$), there is another way of finding a virtual resolution. First, for $\mathbb{P}^{\boldsymbol{n}}$, write the Cox ring as $\Bbbk[x_{1,0}, ..., x_{1,n_1}, x_{2,0}, ..., x_{2,n_2}, ..., x_{k,n_k}]$. Then for a vector $\boldsymbol{a} \in \mathbb{N}^k$ let $B^{\boldsymbol{a}} = \bigcap_{i=1}^{k} \langle x_{i,0}, ..., x_{i,n_i} \rangle^{a_i}$. We will call $B^{\boldsymbol{a}}$ a power of the irrelevant ideal.

Theorem 4.4. *If $X \subset \mathbb{P}^{\boldsymbol{n}}$ is a set of points and I is the corresponding B-saturated S-ideal, then there exists $\boldsymbol{a} \in \mathbb{N}^k$ such that the minimal free resolution of $S/(I \cap B^{\boldsymbol{a}})$ is a virtual resolution of S/I.*

Example 4.5. Consider the B-saturated ideal J from Example 4.3. Recall that the variety of this ideal is a set of three points in general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Finding the minimal free resolution of $J \cap B^{(0,1)}$ recovers the same virtual resolution as in Example 4.3. ⋄

A natural question to ask is if there are other ways of creating virtual resolutions. This is a possible research direction. It appears that there may be a way to create a virtual resolution of an ideal I by finding a resolution of an ideal created by taking a subset of the generators of I.

Example 4.6. Consider the ideal generated by the first and second generators of the B-saturated ideal mentioned in Example 4.3,

$$\langle 3x_0x_2 + x_1x_2 - x_0x_3, 40x_1^2x_2 + 6x_0^2x_3 - 13x_0x_1x_3 - 3x_1^2x_3\rangle.$$

The minimal resolution of this ideal gives the virtual resolution of the ideal of the three points from Example 4.3

$$S \xleftarrow{\varphi_1} \begin{matrix} S(-1,-1) \\ \oplus \\ S(-1,2) \end{matrix} \xleftarrow{\varphi_2} S(-2,-3) \longleftarrow 0.$$

Macaulay2 [M2] shows that the maps between these resolutions are given by the matrices

$$\varphi_1 = \begin{bmatrix} x_0x_2 + (1/3)x_1x_2 - (1/3)x_0x_3 & x_1^2x_2 + (3/20)x_0^2x_3 - (13/40)x_0x_1x_3 - (3/40)x_1^2x_3 \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} -(20/3)x_1^2x_2 - x_0^2x_3 + (13/6)x_0x_1x_3 + (1/2)x_1^2x_3 \\ (20/3)x_0x_2 + (20/9)x_1x_2 - (20/9)x_0x_3 \end{bmatrix}$$

so this complex is a Koszul complex. ⋄

5. Possible Research Directions

Now that virtual resolutions have been introduced and their utility has been discussed, some possible avenues of research may be examined.

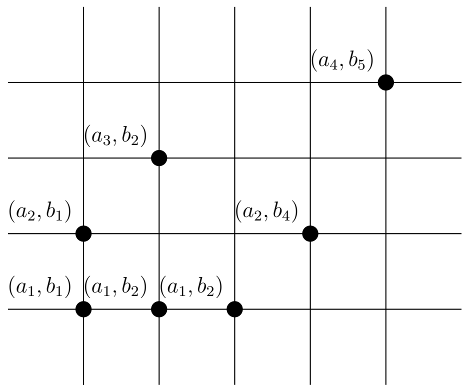
First, call a variety **arithmetically Cohen–Macaulay** (ACM) if its coordinate ring is Cohen–Macaulay (the depth of every ideal in the ring is equal to the ideal's codimension). In [GVT15], Elena Guardo and Adam Van Tuyl are able to classify which sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ are ACM. By the Auslander–Buchsbaum formula, ACM sets of points have projective dimension equal to their codimension. That is, in the case of $\mathbb{P}^1 \times \mathbb{P}^1$, ACM sets of points have projective dimension $2 = \dim(\mathbb{P}^1 \times \mathbb{P}^1)$. We call a set of points in $\mathbb{P}^{\boldsymbol{n}}$ virtually ACM if there is a virtual resolution of $\boldsymbol{n}$ of length $|\boldsymbol{n}| = n_1 + \cdots + n_k$.

Points in $\mathbb{P}^1 \times \mathbb{P}^1$ may be placed on a ruling according to their coordinates in each copy of $\mathbb{P}^1$ in the following way. There are two projections $\pi_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

$$\pi_1(a,b) = a, \qquad \pi_2(a,b) = b.$$

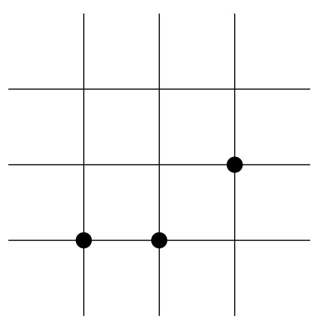
First we make a grid of horizontal and vertical lines. The horizontal lines correspond to the first copy of $\mathbb{P}^1$ and the vertical lines correspond to the second copy of $\mathbb{P}^1$. Two points $p, q \in \mathbb{P}^1 \times \mathbb{P}^1$ lie on the same horizontal line if $\pi_1(p) = \pi_1(q)$, that is if the first coordinates of the points match. They lie on the same vertical line if $\pi_2(p) = \pi_2(q)$ (if the second coordinates of the points match). By permuting the horizontal lines, we can always make the grid so that the number of points on each horizontal line decreases from bottom to top.

For example, the set of points $(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_4), (a_3, b_2), (a_4, b_5)$ can be represented on the following grid (here the points are labeled so it is easier to see where each came from; the points need not always be labeled):



Of course, the sets of points that are ACM will also be virtually ACM because the minimal free resolution of a set of points is also a virtual resolution. Therefore the notion of virtually ACM is weaker than the notion of ACM. Computations in *Macaulay2* [M2] show that there are sets of points that are virtually ACM, but not ACM. Below is the smallest example of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are virtually ACM, but not ACM. In fact, according to Proposition 1.2 in [BZES17], points in $\mathbb{P}^{\boldsymbol{n}}$ will always be virtually ACM.

Example 5.1. Consider the set of three points $X = \{([1:1],[1:4]), ([1:1],[1:5]), ([1:2],[1:6])\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. These sets of points lie in the following configuration on the ruling.



The B-saturation of the ideal corresponding to these points is

$$J = \langle 120x_2^3 - 74x_2^2x_3 + 15x_2x_3^2 - x_3^3, 180x_1x_2^2 - 81x_1x_2x_3 + x_0x_3^2 + 8x_1x_3^2,$$
$$6x_0x_2 - 6x_1x_2 - x_0x_3 + x_1x_3, 2x_0^2 - 3x_0x_1 + x_1^2\rangle,$$

and J has minimal free resolution

$$S \longleftarrow \begin{matrix} S(-1,-1) \\ \oplus \\ S(-2,0) \\ \oplus \\ S(0,-3) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow \begin{matrix} S(-2,-1) \\ \oplus \\ S(-1,-3)^2 \\ \oplus \\ S(-2,-2) \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0.$$

The Auslander–Buchsbaum formula then tells us that X cannot be ACM.
However, X has the virtual resolution

$$S \longleftarrow \begin{matrix} S(-1,-1) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0,$$

which is a Koszul complex. Since the length of the above virtual resolution is $2 = \dim(\mathbb{P}^1 \times \mathbb{P}^1)$, the set of points X is virtually ACM. As this set of points has a virtual resolution that is a Koszul complex, we say this set of points is a virtual complete intersection. ◇

As mentioned in Examples 4.6 and 5.1, there are virtual resolutions of sets of three points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are Koszul. In fact, computations in *Macaulay2* [M2], seem to imply that most sets of 3 and 4 points in $\mathbb{P}^1 \times \mathbb{P}^1$ are virtual complete intersections. It would be interesting to classify when sets of points are virtual complete intersections.

Another thing to explore is when the minimal resolution of an ideal generatered by a subset of minimal generators of an ideal form a virtual resolution. For example in Example 5.1, the virtual resolution is actually the minimal resolution of the ideal generatered by a $(1,1)$-form and a $(1,2)$-form, both of which are in a minimal generating set of J. Is there always a virtual resolution that consists of a free resolution of a subset of a minimal set of generators of an ideal?

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