REU 2018 Day 1

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Cluster algebras and dimer interpretations

A cluster algebra $\mathcal{A}$ is a subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ defined by generators and relations, starting with the initial seed cluster

$$\underbrace{\{x_1, \ldots, x_n\}}_{\text{cluster}} \longrightarrow \underbrace{\boxed{\mu_j}}_{\substack{\text{mutation in} \\ j^{th} \text{ direction}}} \longrightarrow \{x_1, x_2, \ldots, x_j', \ldots, x_n\}$$

generating new generators, a priori, infinitely many.

EXAMPLE 1

$\mathcal{A} \subset \mathbb{Q}(x_1, x_2)$ with initial cluster $\{x_1, x_2\}$ and binomial exchange relation $x_n x_{n-2} = x_{n-1} + 1$ for $n \geq 3$

So $x_3 = \dfrac{x_2 + 1}{x_1}$

$$x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_2 + 1 + x_1}{x_1 x_2}$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_2 + 1 + x_1}{x_1 x_2} + 1}{\frac{x_2+1}{x_1}}$$

a miracle occurs!

$$= \frac{x_2 + 1 + x_1 + x_1 x_2}{x_1 x_2} \cdot \frac{x_1}{x_2 + 1} = \frac{x_1 + 1}{x_2}.$$

Can check $x_6 = \frac{x_5+1}{x_4} = \dots = x_1$

$$x_7 = \dots = x_2$$

5-periodic !

Thus $\mathcal{A}$ has algebra generators

$$x_1, x_2, \underbrace{x_3, x_4, x_5}_{\text{Laurent polynomials in } x_1, x_2}$$

THEOREM [Fomin-Zelevinsky 2001]

For any cluster algebra, the distinguished generators, called cluster variables are Laurent polynomials in the initial cluster.

THEOREM [Lee-Schiffler 2013
Gross-Hacking Keel-Kontsevich]

The Laurent polynomials have **positive** integer coefficients.

EXAMPLE 2

$\mathcal{A} \subset \mathbb{Q}(x_1, x_2)$ with initial cluster $\{x_1, x_2\}$

and $x_n x_{n-2} = x_{n-1}^2 + 1$

$$x_3 = \frac{x_2^2+1}{x_1}$$

$$x_4 = \frac{x_3^2+1}{x_2} = \frac{\left(\frac{x_2^2+1}{x_1}\right)^2+1}{x_2} = \frac{(x_2^2+1)^2 + x_1^2}{x_1^2 x_2}$$

$$x_5 = \frac{x_4^2+1}{x_3} = \frac{\left[\frac{(x_2^2+1)^2+x_1^2}{x_1^2 x_2}\right]^2+1}{\left(\frac{x_2^2+1}{x_1}\right)}$$

$$= \frac{(x_2^2+1)^3+x_1^4+2x_1^2+2x_1^2x_2^2}{x_1^3x_2^2}$$

x_6 is even more complicated, but still a Laurent polynomial.

Let $x_1 = x_2 = 1$. Then

$$x_3 = \frac{1^2+1}{1} = 2$$

$$x_4 = \frac{2^2+1}{1} = 5$$

$$x_5 = \frac{5^2+1}{2} = 13$$

$$x_6 = \frac{13^2+1}{5} = \frac{170}{5} = 34$$

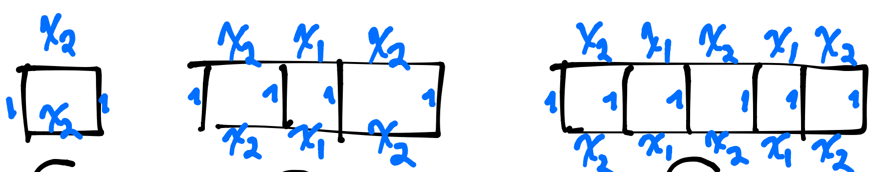
$$x_7 = \frac{34^2+1}{13} = 89$$

These are every other Fibonacci number

$\underline{1}, 1, \underline{2}, 3, \underline{5}, 8, \underline{13}, 21, \underline{34}, 55, \underline{89}, \ldots$

REU Exercise #1

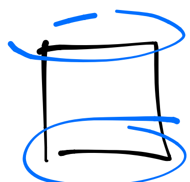
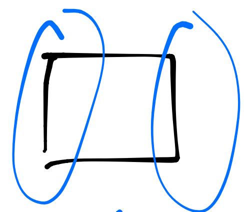
Define $G_m = 2 \times m$ grid graph



G_2 G_4 G_6

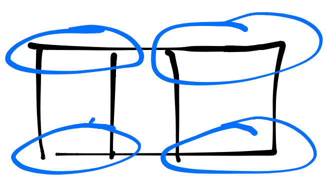
with weights on horizontal edges by x_2, x_1 alternating

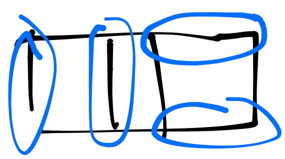
DEFN: A **dimer** or **perfect matching** on a graph G is a set of edges touching each vertex exactly once

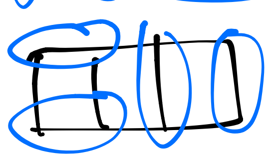
e.g. G_2  or 

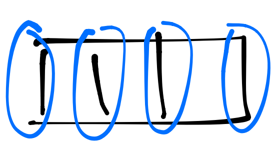
x_2^2 $\qquad$ 1

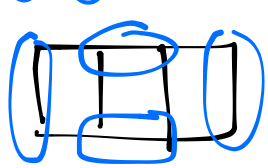
G_4

 x_2^4

 x_2^2

 x_2^2

 1

 x_1^2

(a) Prove that if $x_n x_{n-2} = x_{n-1}^2 + 1$ for $n \geq 3$, then x_n is this Laurent polynomial in $\{x_1, x_2\}$:

$$x_n = \frac{1}{x_1^{n-2} x_2^{n-3}} \sum_{\substack{\text{dimer } M \\ \text{of } G_{2n-4}}} x(M)$$

$$\text{where } x(M) = \prod_{e \in M} x(e)$$

(b) Prove the easy COROLLARY:

$$x_1 = x_2 = 1 \Rightarrow x_n = F_{2n-4}$$

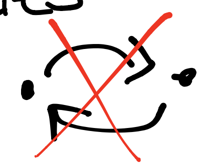
where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

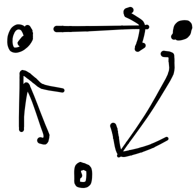
(END REU Exercise #1)

More general cluster algebras

A *quiver* Q is a directed graph.

We'll assume Q has no 1-cycles 

no 2-cycles 

(but  is fine).

If Q has n vertices, then it defines a cluster algebra $\mathcal{A}=\mathcal{A}(Q)$ inside $\mathbb{Q}(x_1,\dots,x_n)$ with initial cluster $\{x_1,x_2,\dots,x_n\}$.

For a cluster

$$\underset{Q}{\{u_1, \dots, u_n\}} \xrightarrow{\mu_j} \underset{Q' = \mu_j(Q)}{\{u_1, \dots, u_j', \dots, u_n\}},$$

$Q' = \mu_j(Q)$ is defined by these rules

1) For every 2-path $i \to j \to k$ in Q,
add a new arrow $i \longrightarrow k$ in Q'

e.g. $i \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}\; j \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix}\; k$ in Q

$\rightsquigarrow$ $i \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}\; k$ in Q'

2) Reverse direction of arrows incident to j

3) Delete 2-cycles

EXAMPLE 3

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} \arrow[r] & \overset{3}{\bullet} \end{tikzcd} \quad \overset{\mu_1}{\rightsquigarrow} \quad \begin{tikzcd} \overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l] \arrow[r] & \overset{3}{\bullet} \end{tikzcd}$$

$$\overset{\mu_3}{\rightsquigarrow} \quad \begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} & \overset{3}{\bullet} \arrow[l] \end{tikzcd}$$

$$\overset{\mu_2}{\rightsquigarrow} \quad \begin{tikzcd} \overset{1}{\bullet} \arrow[rr, bend right] & \overset{2}{\bullet} \arrow[l] & \overset{3}{\bullet} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[rr, bend left] & \underset{2}{\bullet} \arrow[l] & \underset{3}{\bullet} \arrow[l] \end{tikzcd} \quad \overset{\mu_2}{\rightsquigarrow} \quad \begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left, "\times"] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd} = 1 \to 2 \to 3$$

cancel 2-cycles

Can check $\mu_j^2 = id$ always

The variables mutate as follows:

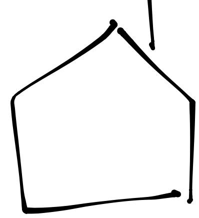
$$x_j x_j' = \prod_{\substack{i \to j \\ \text{(ingoing)}}} x_i^{\#\text{arrows } i \to j} + \prod_{\substack{j \to k \\ \text{(outgoing)}}} x_k^{\#\text{arrows } j \to k}$$

EXAMPLE 1 revisited

$$Q = 1 \longrightarrow 2 \qquad \{x_1, x_2\}$$

$$\begin{array}{ccc}
 & Q & \\
x_1' x_1 = 1 + x_2 \quad \swarrow \mu_1 & & \mu_2 \searrow \quad x_2' x_2 = x_1 + 1 \\
x_1' = \frac{x_2+1}{x_1} = x_3 \quad 1 \longleftarrow 2 & & 1 \longleftarrow 2 \quad x_2' = \frac{x_1+1}{x_2} = x_5 \\
x_2 x_2'' = x_3 + 1 \quad \downarrow \mu_2 & & \downarrow \\
1 \longrightarrow 2 & & 1 \longrightarrow 2 \\
x_2'' = \frac{x_3+1}{x_2} = x_4 & &
\end{array}$$

(closes up to a pentagon)



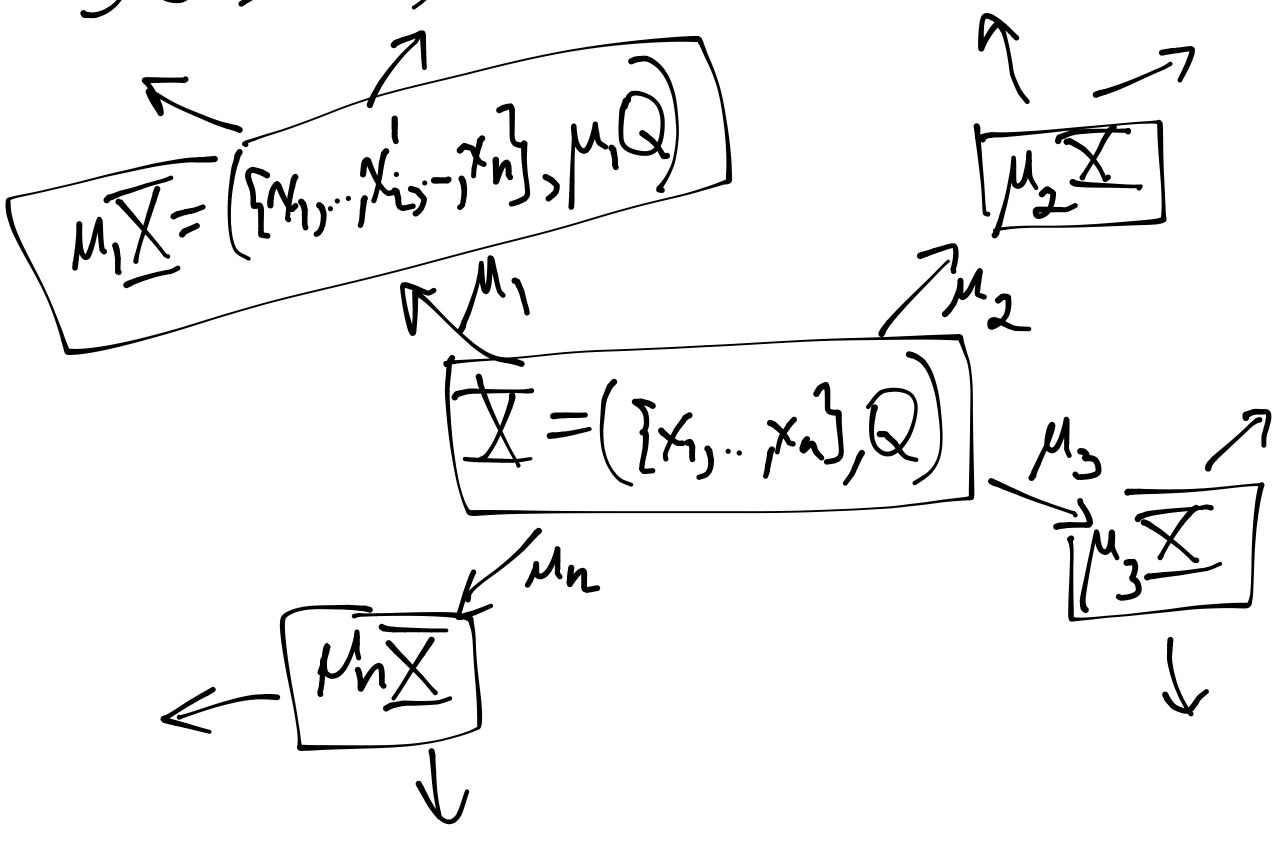
EXAMPLE 2 revisited

$Q = 1 \rightrightarrows 2$ Kronecker quiver

$$x_n x_{n-2} = x_{n-1}^2 + 1$$

For a general cluster algebra

$\mathcal{A} = \mathcal{A}(Q)$

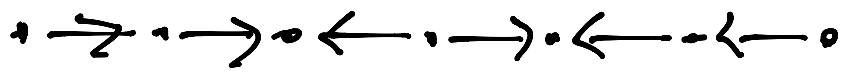


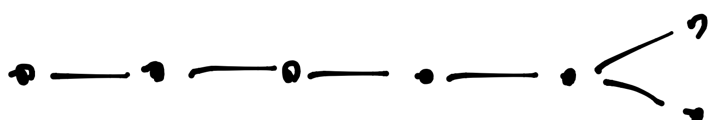
THEOREM [Fomin-Zelevinsky 2002]

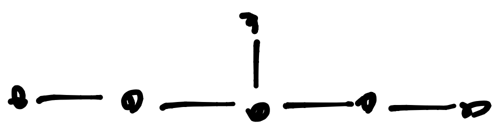
Finite type classification:

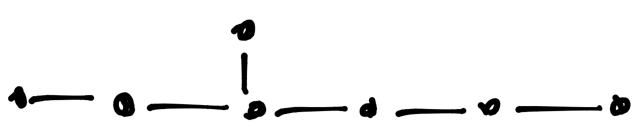
Which quivers Q give $\mathcal{A}(Q)$ with only finitely many cluster variables?

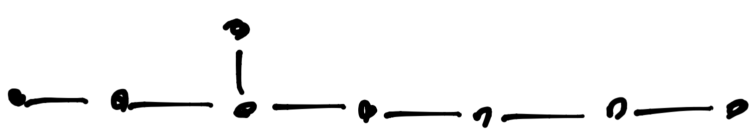
Those which are (mutation equivalent to) an orientation of a Dynkin diagram

A_n 

D_n 

E_6 

E_7 

E_8 

(Also B_n, C_n, F_4, G_2 which are not $\mathcal{A}(Q)$'s)

REU PROBLEM 1 (roughly)

Give a new combinatorial interpretation of the cluster variables in cluster algebras of finite type using double dimers and triple dimers

Recommended:

Get an account at COCALC (cocalc.com) so you can use SAGE to experiment with cluster algebras!

REU Exercise #2

a) Consider

$$Q = 1 \to 2 \to 3 \quad (\text{type } A_3)$$

and all its possible mutation sequences. Show that only a finite # of quivers are reached (including

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}$$

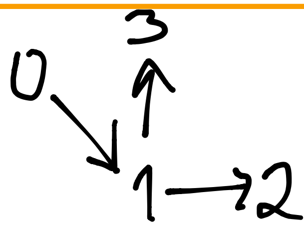
, for example)

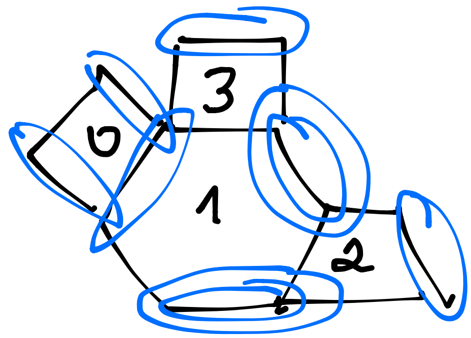
b) Same question for the cluster variables, and show how the clusters are connected to one another

$$\{x_1, x_2, x_3\} \xrightarrow{\mu_2} \left\{x_1, \frac{x_1 + x_3}{x_2}, x_3\right\}$$

~~c), d) Do the same for D_4 quiver~~ $\begin{tikzcd}[row sep=small] & & 4 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 3 \end{tikzcd}$

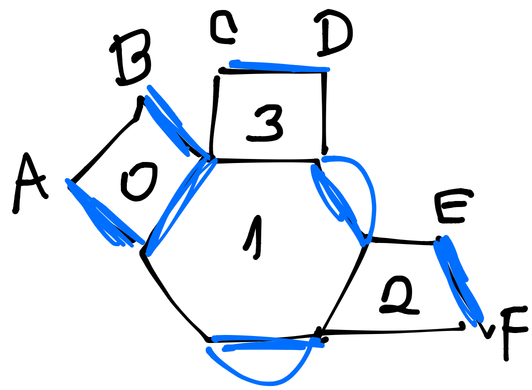
removed!





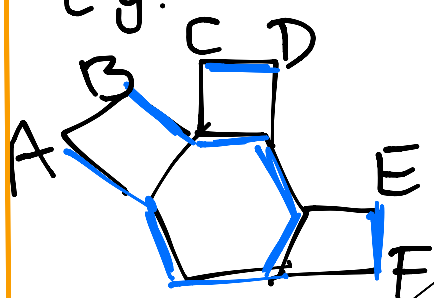
REU Exercise #3

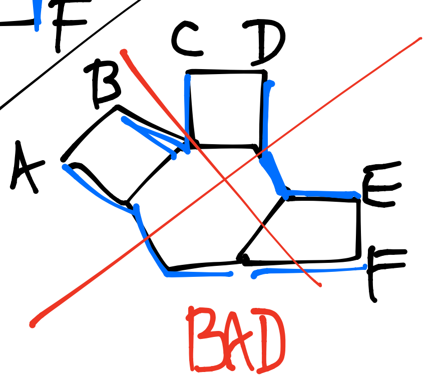
a) Verify there are 10 mixed double dimer configurations with paths as shown

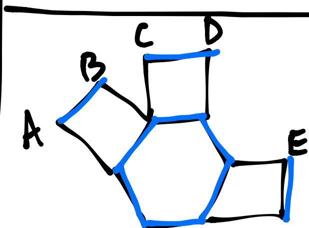


A mixed/double dimer configuration with paths $A \to B$, $C \to D$, $E \to F$

e.g.

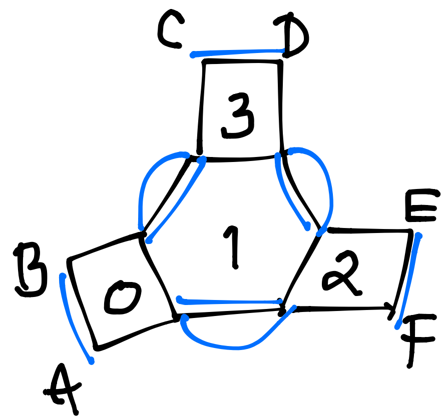






has coefficient 2^1 because of 1 cycle b)

$$\begin{tikzcd} & 3 & \\ 0 & 1 \arrow[l] \arrow[r] \arrow[u] & 2 \end{tikzcd}$$

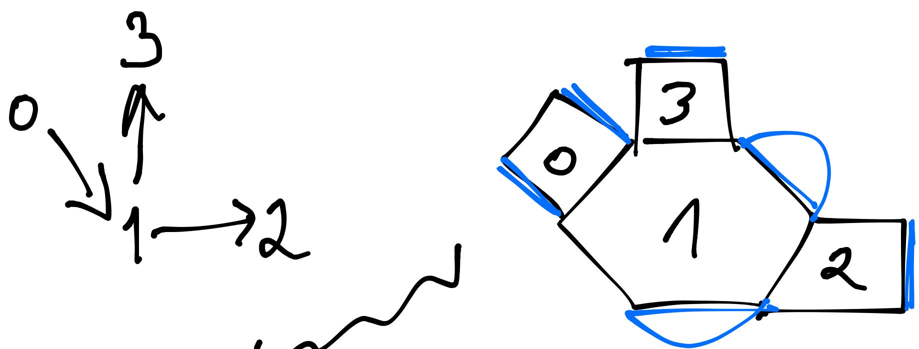


Show that there are 13 mixed configurations here
(and one will get coefficient 2)

DEF'N: The principal extension of Q adds $\boxed{i'} \to i$ for every vertex i of Q.

The new vertex i' is tracked by variable y_i.

The F-polynomials set all $x_i = 1$, leaving only y_i's. (not allowed to mutate at i' vertices)



$$\text{F-poly} = 1 + y_1 + y_0 + 2y_0y_1 + y_0y_1y_3 + \dots$$

CONJ:

For any acyclic D_n-quiver, there exists a graph made up of squares + one hexagon such that mixed dimer/double dimer configurations satisfying certain connectivity of 1-valent vertices have

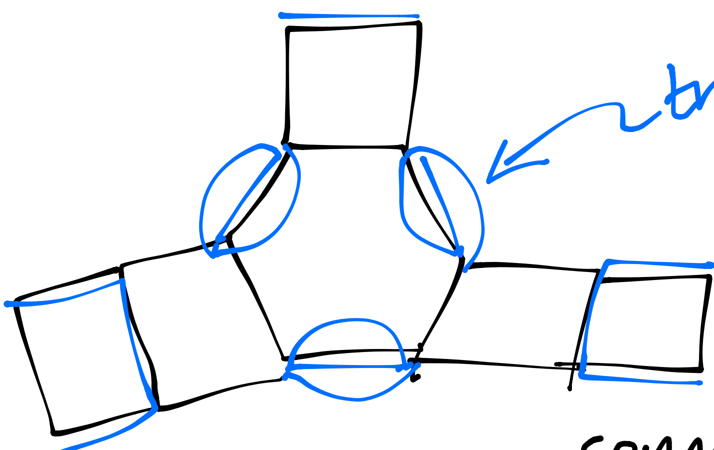
$$\text{F-polynomial} = \sum_D y(D)\, 2^{\#\text{cycle components}}$$

(cf. Theorem 4.1 in Kenyon-Pemantle, Thao Tran)

Type E_6

$$\begin{array}{ccccccccc} & & & & 6 & & & & \\ & & & & \uparrow & & & & \\ 1 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow & 4 & \leftarrow & 5 \end{array} \qquad \frac{(\text{some numerator})}{x_1 x_2^2 x_3^3 x_4^2 x_5 x_6}$$

F-poly has 181 terms



triple dimers!

connectivity rule?

Readings, e.g. by Thao Tran, that help to clarify will be distributed by email.

REU Problem #1

a) Using Thao Tran's work, or otherwise, flesh out and prove the above conjecture for acyclic D_n cluster algebras.

b) Extend interpretation and prove to the case of non-acyclic type D_n cluster algebras.

c) Same for type E.