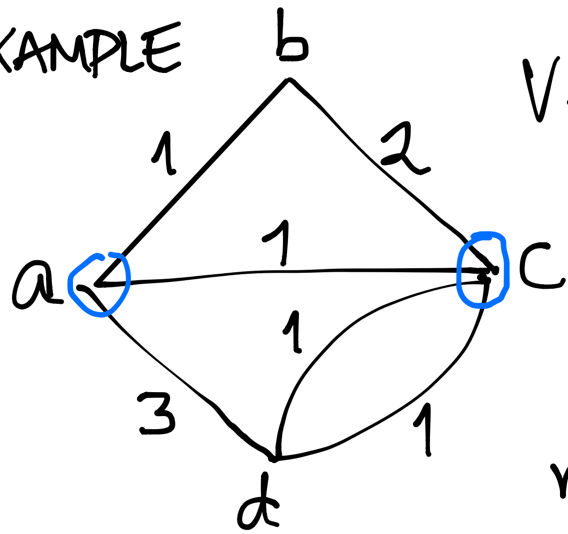
REU 2018 Day 3 S. Chepuri

Electrical Networks

DEF'N: A resistor network is a finite graph (V, E) with a specified set $\emptyset \neq B \subset V$ called boundary vertices and a real nonnegative weight c_e, called conductance, associated to each edge $e \in E$. The other vertices are called internal vertices $I = V \setminus B$

The resistance $r_e = \frac{1}{c_e}$.

EXAMPLE



$V = \{a, b, c, d\}$

$B = \{a, c\}$

$I = \{b, d\}$

$r_{ab} = \frac{1}{1} = 1$

$r_{bc} = \frac{1}{2}$

DEF'N: A potential function is a function $V \xrightarrow{f} \mathbb{R}_{\geq 0}$

The voltage across an edge xy is $V_{xy} := f(x) - f(y)$

Ohm's Law: $V = IR$

voltage (V), current (I), resistance (R)

In our notation, $v_{xy} = i_{xy} r_{xy}$

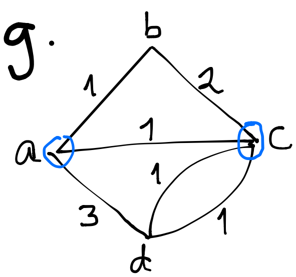
or $i_{xy} = v_{xy} c_{xy}$

where i_{xy} is the current flowing across xy.

NOTE: As we've defined them, v_{xy} and i_{xy} have direction, i.e. $v_{yx} = -v_{xy}$, $i_{yx} = -i_{xy}$. Not true for $r_{xy} = r_{yx} = \frac{1}{c_{xy}} = \frac{1}{c_{yx}}$.

Given a network and potential function, we can use Ohm's law to define all the currents.

e.g.



$f(a)=1$

$f(b)=1/3$

$f(c)=0$

$f(d)=3/5$

$\Rightarrow v_{ab}=f(a)-f(b)=\frac{2}{3}$

$v_{bc}=f(b)-f(c)=1/3$

$i_{ab}=V_{ab}c_{ab}=2/3$

$i_{bc}=V_{bc}c_{bc}=2/3$

Kirchhoff's current law (KCL)

For any node in an electrical circuit, the sum currents flowing in is zero.

We are interested in potential functions where KCL holds for all internal vertices.

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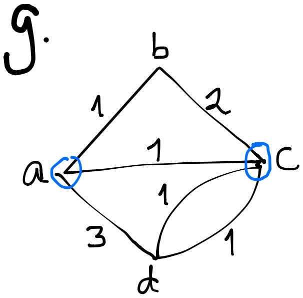
e.g.



$i_{ab} + i_{cb} =$

$i_{ab} - i_{bc} =$

$\frac{2}{3} - \frac{2}{3} = 0$ ✓

THEOREM Given a resistor network and a potential function defined on the boundary vertices, there is a unique extension of the potential function to all vertices, insisting on KCL at all internal vertices.

Dirichlet Problem:

Find this unique extension.

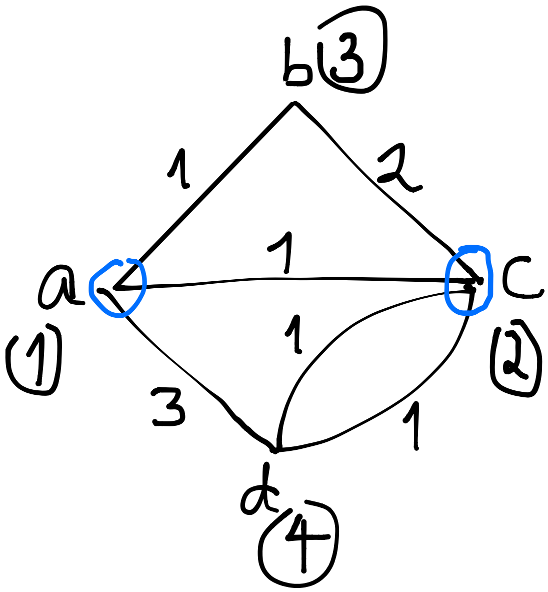
How to find it?

We construct the Kirchhoff matrix of the network as follows.

Index $v \in B$ as $\{1, 2, \dots, m\}$ and $v \in I$ as $\{m+1, \dots, n\}$

$$K_{ij} = \begin{cases} \displaystyle\sum_{\substack{\text{edges } e \\ \text{between } i \text{ and } j}} c_e & i \neq j \\ \displaystyle -\sum_{\substack{\text{edges } e \\ \text{incident to } i}} c_e & i = j \end{cases}$$

(= negative of $L(G)$ from Day 2 if all $c_{ij} = 1$)

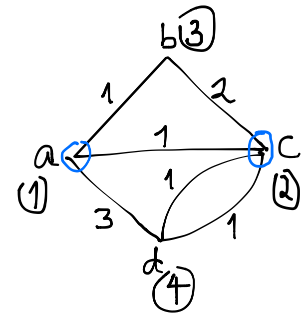


$$K = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 2 & 2 \\ 1 & 2 & -3 & 0 \\ 3 & 2 & 0 & -5 \end{bmatrix} \end{array}$$

Given a potential function as a vector $v \in \mathbb{R}^n_{\geq 0}$, then Kv gives the current flowing into each vertex.

e.g. $$\begin{bmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 2 & 2 \\ 1 & 2 & -3 & 0 \\ 3 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1/3 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -43/15 \\ 43/15 \\ 0 \\ 0 \end{bmatrix}$$

e.g.



$$-5(1)+1(0)+1\left(\tfrac{1}{3}\right)+3\left(\tfrac{3}{5}\right)$$

$$=-(1+1+3)(1)+1(0)+1\left(\tfrac{1}{3}\right)+3\left(\tfrac{3}{5}\right)$$

$$=\overset{c_{ac}\,(f(c)-f(a))}{1(0-1)}+1\left(\tfrac{1}{3}-1\right)+3\left(\tfrac{3}{5}-1\right)$$

= current flowing into 1.

We can divide K into 4 parts

$$K=\begin{array}{c c} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \end{array}$$

and same for potential $\begin{array}{c} m \\ n-m \end{array}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ B^t x + Cy \end{bmatrix}$$

Solution to Dirichlet Problem

Given x, want to find y such that $B^t x + Cy = 0$.

Hence $y = -C^{-1} B^t x$.

NOTE: C is invertible by the theorem.

We also found the current flowing into the boundary vertices.

In fact, we can think about the map directly from potential on the boundary vertices to the boundary currents

$$Ax + By = Ax - BC^{-1}B^t x$$
$$= \underbrace{(A - BC^{-1}B^t)}_{\text{Schur complement}} x$$

DEF'N: The response matrix L of a network is the Schur complement of the Kirchhoff matrix.

Inverse Problem

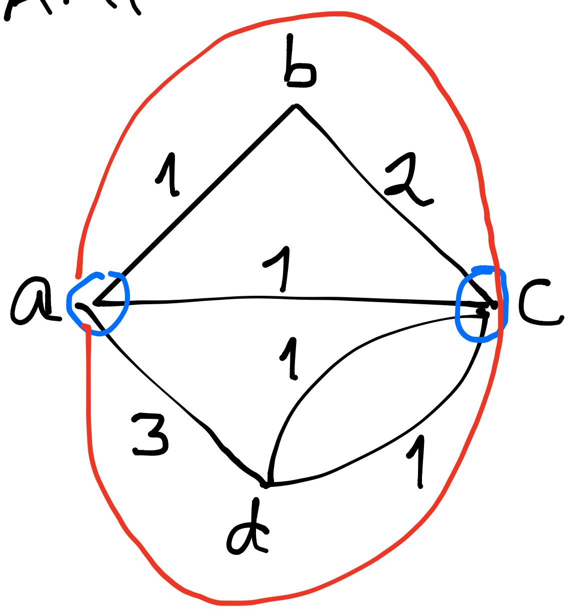
a) Given a response matrix and a network with unknown conductances, when can we uniquely recover the conductances?

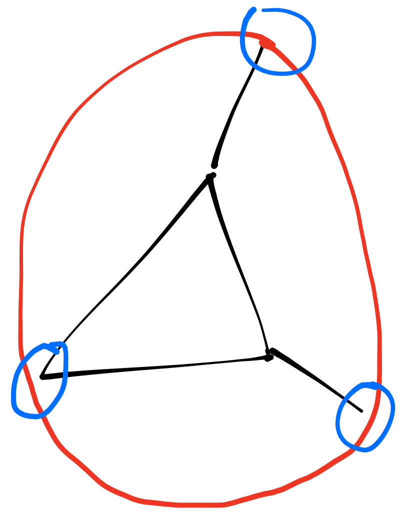
b) Which matrices are response matrices?

Curtis-Ingerman-Morrow (1998) solved this for circular planar resistor networks (cprn).

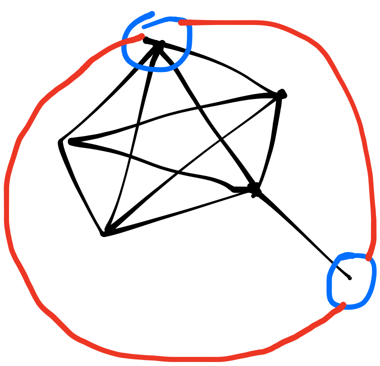
<u>DEF'N</u>: A cprn is a network that can be embedded in a disk, with all boundary vertices on the disk boundary.

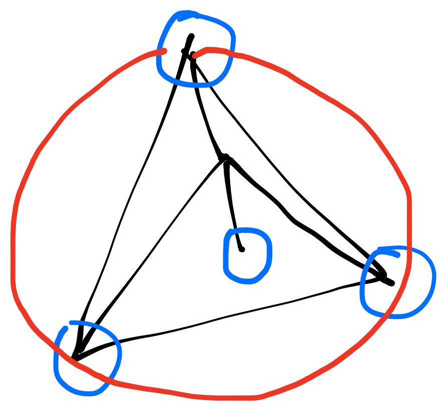
EXAMPLES



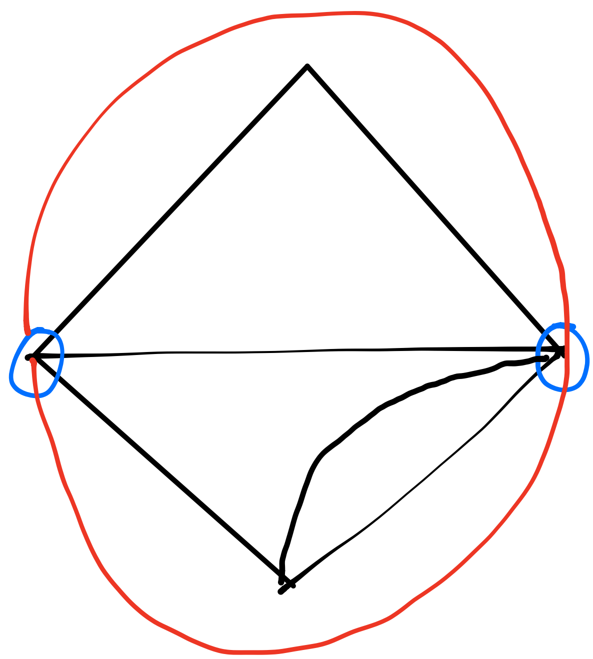


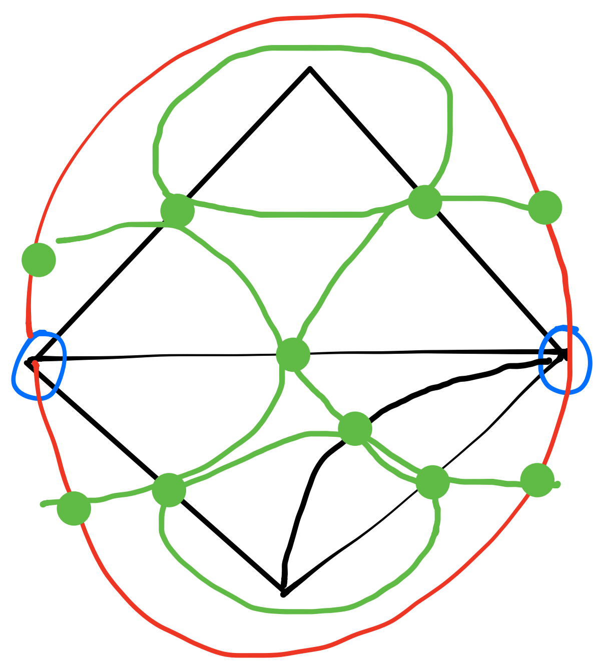
NON-EXAMPLES



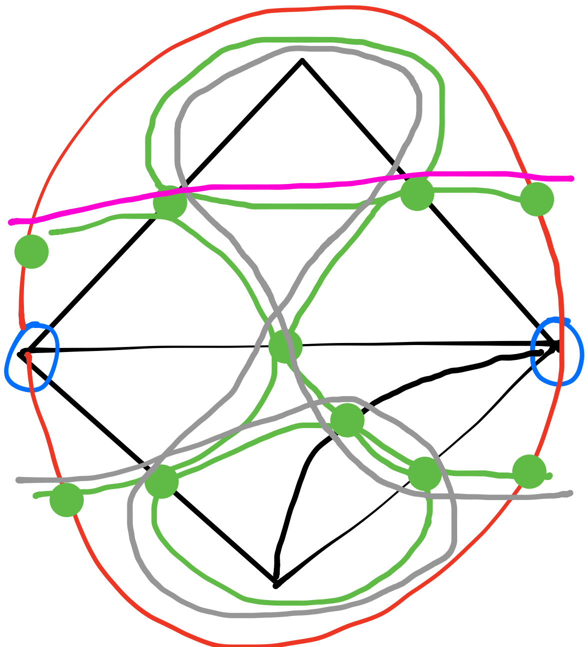


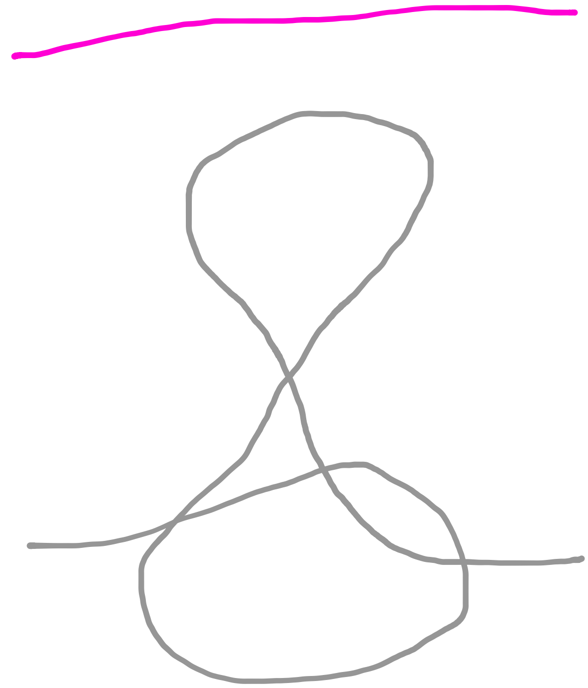
DEF'N/EXAMPLE of medial graph

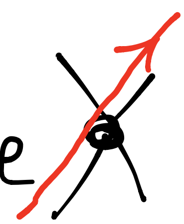




Two strands:





Draw strands from boundary vertices to each other passing through internals like 

DEFN: A cpm is critical if the medial graph has these properties:

1) no closed loops
2) no self intersecting strands
3) no 2 strands intersect more than once

THEOREM: The following local moves do not change the response matrix:

parallel transformation

series transformation

loop removal

pendant removal

Y-Δ transformation

DEF'N: Two cprn's are electrically equivalent if they have the same response matrix.

THEOREM (Curtis-Ingerman-Morrow 1998)

(1) Any cprn is electrically equivalent to a critical cprn

(2) Any 2 electrically equivalent cprn's can be connected by the local moves. If both are critical, then only Y-Δ moves are needed.

(3) The conductances of a cprn can be recovered uniquely if and only if it is critical.

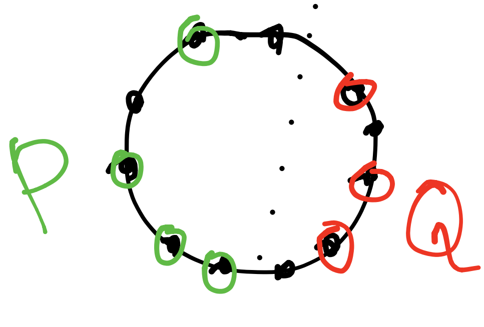
(4) L is the response matrix of a cprn, with B labeled clockwise, if and only if

(a) L is symmetric

(b) rows sum to 0

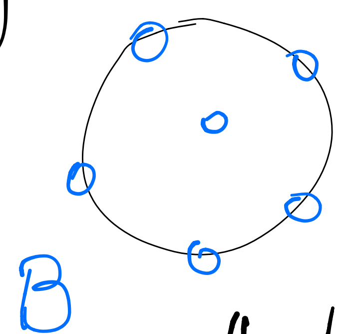
(c) for any $P, Q \subset B$ disjoint, with $|P|=|Q|$, having no $a<b<c<d$ with $a,c \in P$, $b,d \in Q$, then $\det L_P^Q \geq 0$

submatrix of L with rows P columns Q

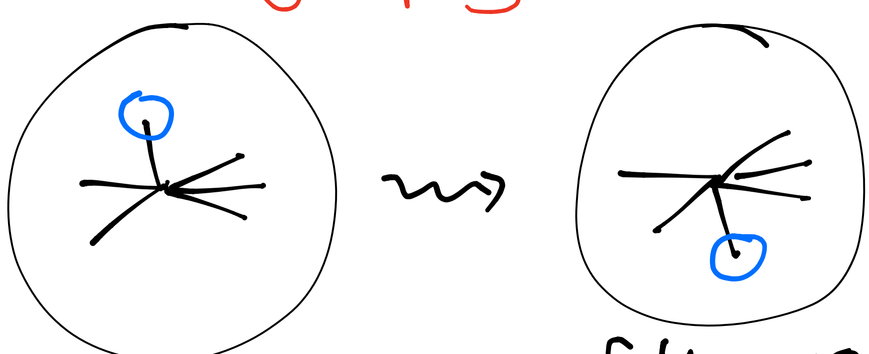


We seek an analogue for networks that are not quite cprn's.

We'll consider those that have all but one boundary vertex on the disk boundary.



Introduce a new local move called antenna-jumping



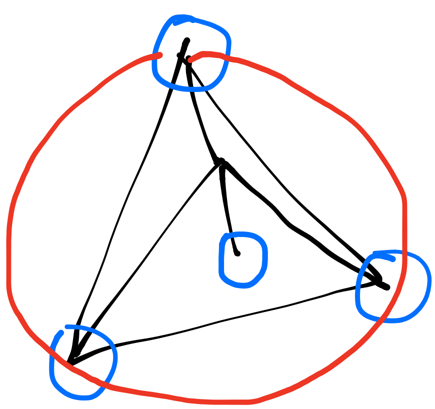
It's not a sequence of the previous local moves, but preserves response matrix.

If we want an analogue of (2) in theorem, we need to add antenna-jumping.

PROPOSED DEF'N: A network is critical if # of edges can't be reduced using a local move.

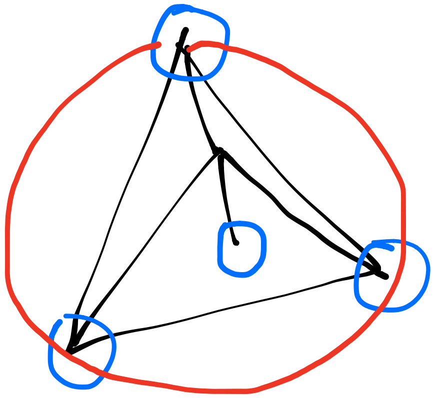
REU Exercise 7

Show the # of edges in this



can't be reduced by applying local moves.

EXAMPLE



has response matrix

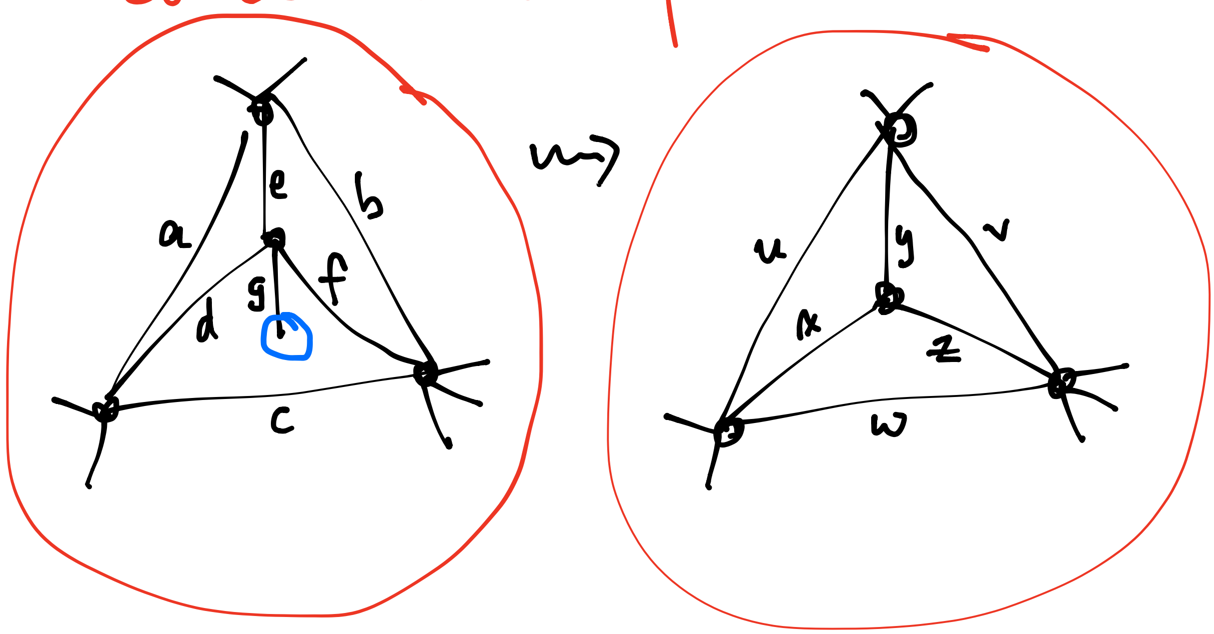
$$L = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

determined by the 6 circled entries (using $L^t = L$, rows sum to zero).
But there are 7 edges/conductances, so L cannot recover them uniquely.

So with the proposed definition, (3) in theorem would fail.

Let's introduce a new local move:

antenna absorption



REU Exercise 8

Find u, v, w, x, y, z so the response matrix is unchanged.

REU Problem 3(a)

Does an analogue of (3) now hold? If not, how to modify def'n of critical to fix it?

Problem 3(b)

Is the analogue of (2) true?

Problem 3(c)

Find a description of the response matrices for these new kinds of networks in a cone.