REU 2018 Day 6 Ben Brubaker

(Linear) Algebraic Monoids

Rough idea: Last week, the group $\underbrace{GL_n(F)}_{\text{general linear group of } n\times n \text{ invertible matrices}/F}$ for F a field

general linear group
of $n\times n$ invertible matrices $/F$

$$GL_n(F) \subseteq \underbrace{Mat_n(F)}$$

$n\times n$ matrices $/F$,
a monoid under multiplication.
$GL_n(F)$ are the group of units (invertible elements) inside.

Try to understand rep'n theory of monoids like $Mat_n(F)$ in order to infer properties of their unit group (like $GL_n(F)$).

Let's go back to...

S_n = symmetric group on n letters.

We know its representation theory, for example. Its irreducible representations are in bijection with partitions $\lambda \vdash n$

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \quad \lambda_1 + \lambda_2 + \dots = n.$$

Moreover, given such an irrep λ (irreducible rep'n)

$\dim(V_\lambda)$ = # of standard tableaux of shape λ

χ_λ = character of V_λ is a function on conjugacy classes in S_n ($\leftrightarrow$ cycle types for S_n)

e.g. $\mu = (4,2,1) \vdash 7 \leftrightarrow$ cycletype $\underbrace{(abcd)}_{4}\underbrace{(ef)}_{2}\underbrace{(g)}_{1}$

$$\chi_\lambda(\mu) = \sum_{\substack{\text{standard} \\ \text{tableaux } T \\ \text{of shape } \lambda}} wt_T(\mu)$$

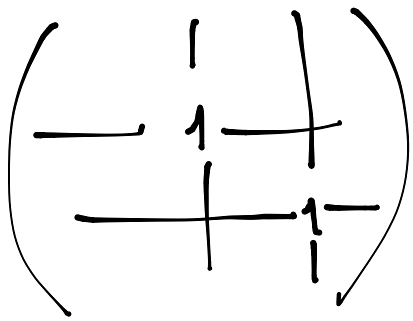
some known combinatorial rule!

S_n is the group of units in a monoid —

the rook monoid R_n

= {arrangements of non-attacking rooks on an $n \times n$ chessboard}

= {$n \times n$ matrices of 1's and 0's, at most one 1 in each row and column}



In particular, $S_n \cong$ $n\times n$ permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}$$

but R_n also contains noninvertible matrices like $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Consider representations of the rook monoid

$$\phi : M \longrightarrow \mathrm{End}(V)$$
$$m \longmapsto \phi(m), \text{ a } k\times k \text{ matrix if } \dim V = k$$

It's still true for R_n that rep'ns always decompose into direct sums of irreducible rep'ns.

The irreducible rep'ns are in bijection with partitions of size $\leq n$, and there are character formulas, etc. (Solomon)

REU Exercise 14

(a) Find all irreducible rep'ns of S_3

(b) —— " ———————— R_3

Exercise in thinking about ways to construct rep'ns. So if you know one way, find another, or replace $n=3$ by $n=4$.

(c) How many elements are there in R_n?

REU Problem 6 (roughly)

Study and classify irreducible rep'ns of monoids M, or equivalently, (modules over their) monoid algebras $\mathbb{C}[M]$, and their generalizations.

Elements of $\mathbb{C}[M]$ are just formal linear combinations of the form

$$\sum_{i=1}^{k} c_i [m_i], \quad m_i \in M, \; c_i \in \mathbb{C}$$

with $[m_i]\cdot[m_j] = [\underbrace{m_i * m_j}_{\text{multiplication in } M}]$

Which monoids should we study?

ANS: Ones that contain finite Coxeter groups.

For example, S_3 is a Coxeter group since S_3 is generated by involutions, say $r_1=(1,2)$, $r_2=(2,3)$

e.g. $r_1 \circ r_2 = (123)$

$$r_1 \circ r_2 \circ r_1 = (13) = r_2 \circ r_1 \circ r_2$$

S_3 has a presentation as

omitting the circles $r_1 \circ r_2$

$$\langle r_1, r_2 \mid r_1^2 = r_2^2 = 1,\ r_1 r_2 r_1 = r_2 r_1 r_2 \rangle$$

$$(\text{or } (r_1 r_2)^3 = 1)$$

Encode the presentation in a graph:

vertices = generators

If generators r_i, r_j

have $(r_i r_j)^2 = 1$, don't connect them

$(r_i r_j)^3 = 1$, connect them

$(r_i r_j)^4 = 1$, connect them with a double edge

and weirder things happen; we'll worry about how to decorate them later.

EXAMPLE: S_3 has this graph

$$\underset{1}{\bullet} — \underset{2}{\bullet}$$

What are the finite Coxeter groups?

Check that S_n has presentation with graph

$$\underset{\substack{1\\(12)}}{\bullet} — \underset{\substack{2\\(23)}}{\bullet} — \underset{\substack{3\\(34)}}{\bullet} — \cdots — \underset{\substack{n-1\\(n-1,n)}}{\bullet}$$

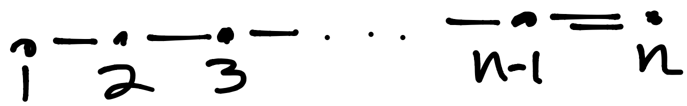
A finite Coxeter group is a (finite) group with a presentation

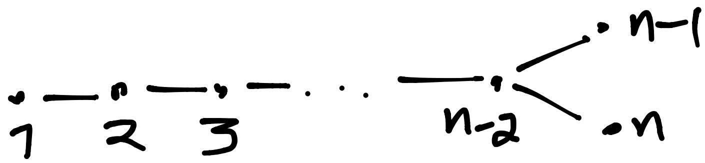
$$\langle r_1, \dots, r_k \mid r_i^2 = 1,\ (r_i r_j)^{m_{ij}} = 1 \rangle$$

r_i are involutions

$m_{ij} \in \{2, 3, \dots\}$

Others:

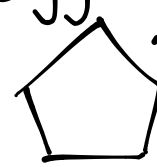
B_n 

D_n 

+ 9 more called

$E_6, E_7, E_8, F_4, G_2, H_3, H_4, I_2(k)$

dihedral groups = symmetries of regular k-sided polygon



$k=5$

(and any product

$W_1 \times W_2 \times \dots \times W_k$

of such groups)

e.g.

$$B_3 = \langle r_1, r_2, r_3 \mid r_i^2 = 1,\ r_1 r_2 r_1 = r_2 r_1 r_2,\ r_2 r_3 r_2 r_3 = r_3 r_2 r_3 r_2,\ r_1 r_3 = r_3 r_1 \rangle$$

$$= S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$$

semidirect product

B_3 has a representation acting on $\mathbb{R}^3$ by permuting/negating coordinates

= the hyperoctahedral group of all signed permutation matrices

like
$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

Let's explore the monoids associated to B_n, the symplectic rook monoids

$$\text{“}RSp_{2n}\text{”} := \left\{ A \in R_{2n} \;\middle|\; \begin{array}{l} \text{either} \\ AJA^T = A^TJA = 0 \\ \text{or } AJA^T = A^TJA = J \end{array} \right\}$$

where $J =$ blockmatrix $\left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & -1 \\ & \iddots & \\ -1 & & 0 \end{matrix} & 0 \end{array}\right)$

<u>REU Exercise 15</u>

(a) Determine the elements of RSp_4, and show it contains an isomorphic copy of B_2.

(b) What is the order (cardinality) of RSp_{2n}?

RSp_{2n} was studied in 2007 (Li-Li-Cao)
but interesting questions remain.
For example, R_n-representations or RSp_{2n}-representations
can be restricted to a rep'n of the
Coxeter group (S_n or B_n) inside.

S_n ← Coxeter group A_{n-1}

$$\mu \vdash k \leq 3 \qquad V_\mu\Big|_{S_3} = \bigoplus_{\nu \vdash 3} \underbrace{W_\nu}_{S_3\text{-irrep.}}{}^{c_\nu} \quad c_\nu \leftarrow \text{multiplicity}$$

What is this map?

REU Exercise #16

(a) Describe the image of irreducible rep'ns of R_3 under restriction to S_3. What about R_n to S_n?

(b) Same question for RSp_{2n}?

Why do we care?

$GL_n(\mathbb{F}_q)$, $\mathbb{F}_q$ a finite field with $q = p^k$ elements, $k \geq 1$

contains a subgroup

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = \text{upper triangular invertible } n \times n \text{ matrices}$$

= the Borel subgroup

THM (Bruhat decomposition)

$$GL_n(\mathbb{F}_q) = \bigsqcup_{w \in S_n} BwB$$

where $BwB := \{ b_1 w b_2 : b_1, b_2 \in B \}$
a double coset

e.g. $BeB = B\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}B = B$ (boring!)

Similarly, the symplectic group

$$Sp_{2n}(\mathbb{F}_q) = \bigsqcup_{w \in B_n} BwB$$

$\overset{\text{DEF'N}}{\Vert}$

$$\{A \in GL_{2n}(\mathbb{F}_q) : A^T J A = J\}$$

where $B :=$ upper triangular matrices in $Sp_{2n}(\mathbb{F}_q)$.

Consider the algebra over $\mathbb{C}$ gen'd by the characteristic functions of the B-double cosets BwB, with multiplication by convolution.

This algebra is called the
(Iwahori-)Hecke algebra $\mathcal{H}(G,B)$.

Why would anyone consider this?
(in 3 reasons)

Reason #1: $G(\mathbb{F}_q)$ $(G = GL_n, Sp_{2n})$
rep'n theory is hard
(main result of Deligne-Lusztig
to construct them via
fancy cohomology theory)

Better to try to break up the
set of all irreducible rep'ns into
digestible chunks.

Consider repns with atleast one nonzero B-fixed vector.

$$\phi : G \longrightarrow \mathrm{Aut}(V)$$

$$g \longmapsto \phi(g)(v) : \text{map on vectors in } V$$

Want $v \in V$ s.t. $\phi(b)v = v \ \forall b \in B$

Reason #2:

THM (Borel-Matsumoto)

The map $V \longmapsto V^B$ = B-fixed vectors gives a bijection

$$\left\{\begin{matrix}\text{irreps of } G(\mathbb{F}_q) \\ \text{with a B-fixed vector}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{irreps of} \\ \mathcal{H}_n(G,B)\end{matrix}\right\}$$

Reason #3:

$$\mathcal{H}(GL_n(\mathbb{F}_q), B) \cong \mathbb{C}[S_n]$$

$$\mathcal{H}(Sp_{2n}(\mathbb{F}_q), B) \cong \mathbb{C}[B_n]$$

Conclusion: Reps of $GL_n(\mathbb{F}_q)$ with B-fixed vectors are indexed by $\lambda \vdash n$

What about rep'ns of monoids?
Is there a similar story to tell?

REU Problem #6 (more precisely)

(a) Is there a Borel-Matsumoto Theorem for monoids?

e.g. $M_n = \mathrm{Mat}_n(\mathbb{F}_q)$,

$\cup$

$B := \{$upper triangular invertible matrices$\}$

have $M_n = \bigsqcup_{r \in R_n} BrB$

So $\mathcal{H} = \mathcal{H}(M_n, B)$ = convolution algebra as before

(b) Describe irreducible rep'ns of $\mathcal{H}(MSp_{2n}, B)$ where MSp_{2n} is monoid with units $Sp_{2n}(\mathbb{F}_q) \cdot \mathbb{F}_q^\times$ when $n=2$. General n?

scalar matrices

Defined more precisely in Li-Li-Cao "Alg. monoids & Renner monoids" Example 12