REU 2018 Day 8 Gregg Musiker
F-polynomials to Infinity

1. (Finite) F-polynomials
2. Stable Cluster Variables (REU 2016)
3. Asymptotic triangulations & Coxeter transformations
4. Continued Fraction Interpretations
5. REU Problem #8:
   Relate Infinite F-polynomials to 3 & 4

① F-polynomials

$$\begin{tikzcd} & & 1 \\ 0 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 3 \end{tikzcd} \xrightarrow[\mu_2]{} \begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[ur] \arrow[dr] & 2 \arrow[l] \\ & 3 \arrow[u] \end{tikzcd}$$

D_4 quiver

$$\Big\downarrow \mu_3$$

$$\boxed{\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ 3 \arrow[u] & 2 \arrow[l] \end{tikzcd}} = \begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[ur] & 2 \arrow[d] \\ & 3 \arrow[ul] \end{tikzcd}$$

therefore mutation-equivalent to a D_4-quiver

FACT: The oriented cycle

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[dr] & \\ n \arrow[u] & & 2 \arrow[d] \\ \cdots \arrow[u] & \cdots & 3 \end{tikzcd}$$

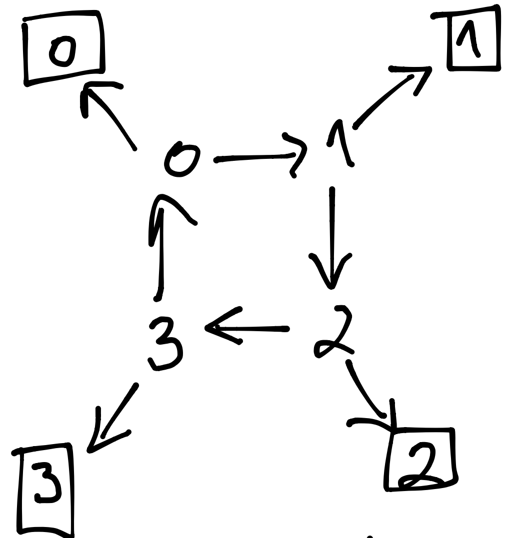
is always mutation-equivalent to a quiver of type D_4

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ 3 \arrow[u] & 2 \arrow[l] \end{tikzcd}$$

F-polynomials:

Step 1: For every vertex $i \in Q$, adjoin a new vertex $\boxed{i}$ and an arrow $i \to \boxed{i}$

(**WARNING**: Sage uses convention $i \leftarrow \boxed{i}$!!)



<u>Step 2</u>: Mutate Q as usual, never at the new frozen vertices $\boxed{i}$, but taking into account the new vertices & arrows.

Use initial alphabet

$$\{x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n\}$$

<u>EXAMPLE</u>: Applying μ_0 gives

$$x_0 x_0' = y_0 x_1 + x_3 \Rightarrow x_0' = \frac{y_0 x_1 + x_3}{x_0}$$

STEP 3: Set all $x_i = 1$, and replace x_i, x_i' with F_i, F_i's

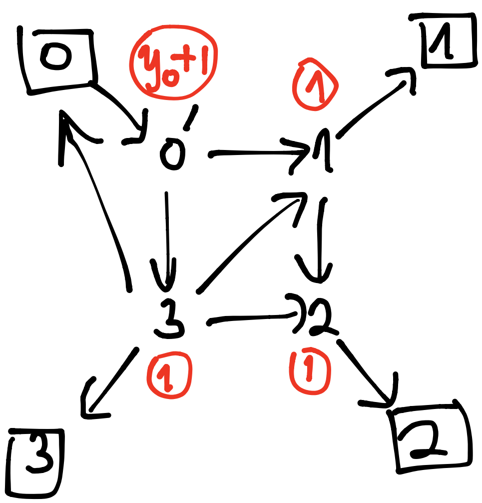
i.e. $x_0 x_0' = y_0 x_1 + x_3 \Rightarrow x_0' = \dfrac{y_0 x_1 + x_3}{x_0}$

becomes

$$F_0 F_0' = y_0 F_1 + F_3$$

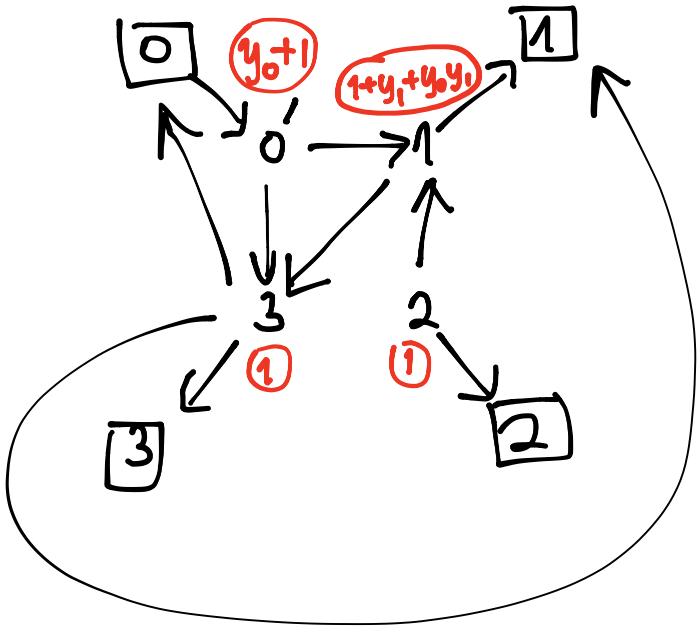
with $F_0 = F_1 = F_2 = F_3 = 1$ $\Rightarrow F_0' = y_0 + 1$

After mutation at 0,

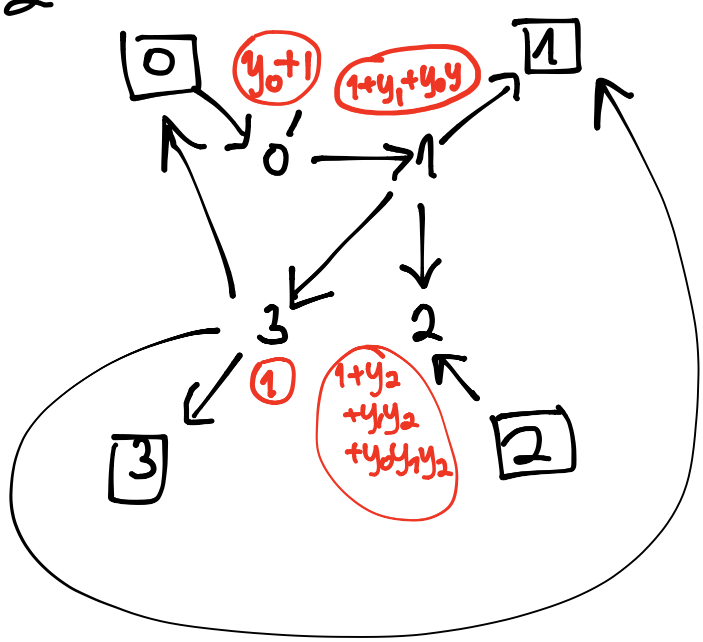


F-polynomials in red

Next do μ_1:



Then do μ_2:



Do μ_3, then μ_0:

$$F_3' = 1+y_1+y_0y_1+y_3y_0y_1$$

$$F_0'' = \frac{1+y_1+y_0y_1+y_0}{1+y_0} = 1+y_1$$

$$F_2' = y_2(1+y_1+y_0y_1)+1$$

Computation:

If you mutate

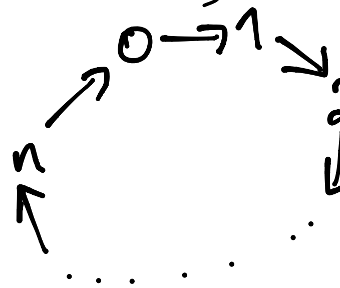
$$\underbrace{0\,1\,2\,3\,0\,1\,2\,3\cdots 0\,1\,2\,3\,0}_{21 \text{ mutations total}}$$

you will get back to F-polynomials 1, 1, 1, 1.

Only 12 nontrivial F-polynomials will appear during the process:

$$
\begin{array}{lll}
1+y_0 & 1+y_1+y_0y_1 & 1+y_2+y_1y_2+y_0y_1y_2 \\
1+y_1 & 1+y_2+y_1y_2 & \vdots \\
1+y_2 & 1+y_3+y_2y_3 & \vdots \ \text{(cyclic} \\
1+y_3 & 1+y_0+y_3y_0 & \quad \text{shifts)}
\end{array}
$$

More generally,



D_{n+1} has $(n+1)n$ nontrivial F-polynomials in general.

Define $\tilde{A}_{1,2}$ as

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

(not ayclic)

and more generally

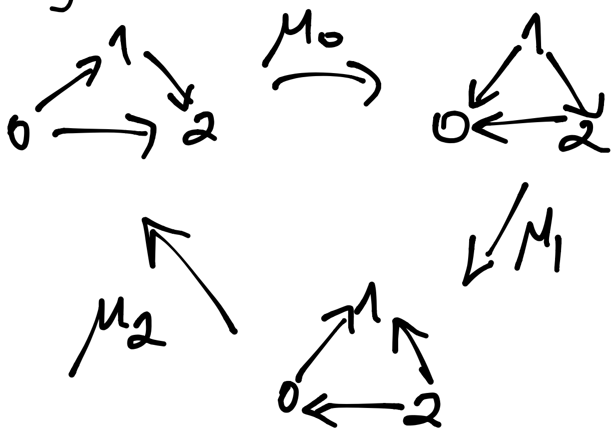
$$\tilde{A}_{1,n} = \begin{tikzcd} & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[dr] & \\ 0 \arrow[ur] \arrow[rrrr] & & & & n \end{tikzcd}$$

---

e.g. $\tilde{A}_{1,1} = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r] & 1 \end{tikzcd}$ = Kronecker quiver

Starting with

$\tilde{A}_{1,2}$



Similarly mutating $0, 1, 2, \ldots, n$ on $\tilde{A}_{1,n}$ it is periodic.

With initial cluster x_0, x_1, x_2,

let $x_0' = x_3$

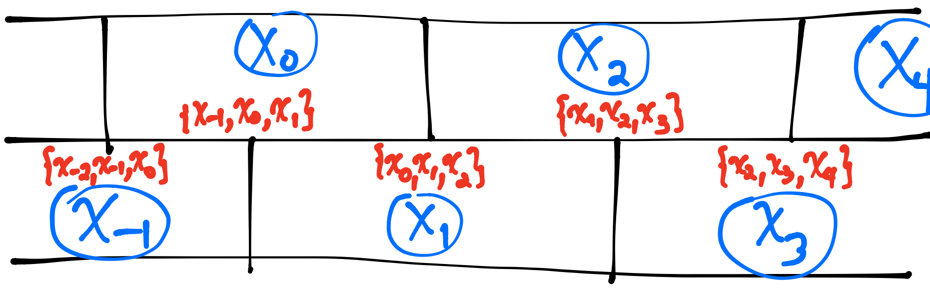
$x_1' = x_4$ and we'll get

$x_2' = x_5$ $\quad x_m x_{m-3} = x_{m-1} x_{m-2} + 1$

$\vdots$

Also define F-polynomials F_m for $m \in \mathbb{Z}$ accordingly, with $F_0 = F_1 = F_2 = 1$.

The double brick wall:

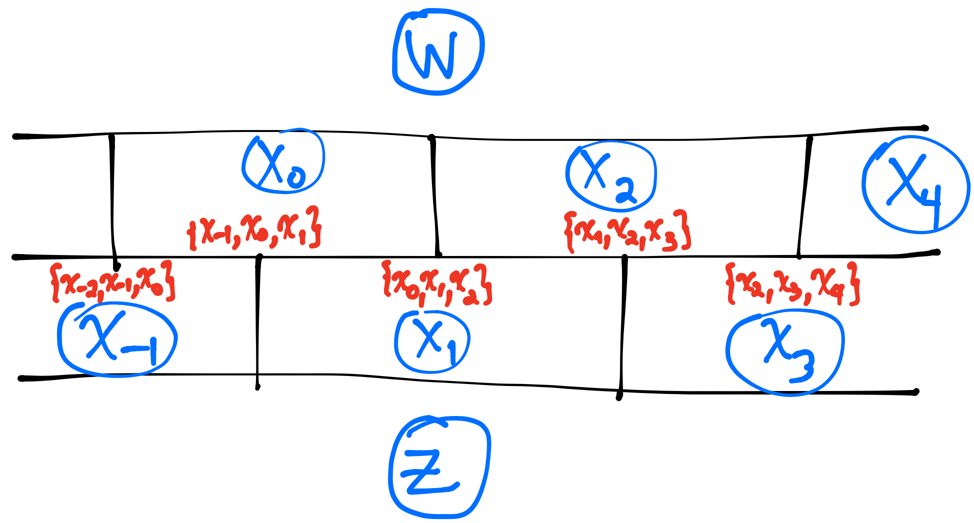


What if we mutate $\{x_0, x_1, x_2\}$ not at 0 or 2, but at 1?

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \xrightarrow{\mu_1} \begin{tikzcd} & 1' = W \arrow[dl] & \\ 0 \arrow[rr, shift left] \arrow[rr, shift right] & & 2 \arrow[ul] \end{tikzcd}$$

$$x_1 W = x_0 + x_2$$

$$W = \frac{x_0 + x_2}{x_1}$$



If we mutate 0 then 2, we get

$$z = \frac{x_3+x_1}{x_2} = \frac{x_1x_2+1+x_0x_1}{x_0x_2}$$

Naive attempt to compute $\lim_{m\to\infty} F_m$:

- $1+y_0$
- $1+y_1+y_0y_1$
- $1+y_2+y_1y_2+y_0y_2+2y_0y_1y_2+y_0^2y_1y_2$

$\vdots$

- $1+2y_2+\cdots$
- $1+3y_2+\cdots$
- $1+4\,y_2+\cdots$

The coefficients on y_2 are getting bigger and bigger – not good!

Second attempt

Essentially see Canakci-Schiffler §7.
(also see N. Reading)

$$\lim_{m\to\infty} \frac{F_m}{F_{m-1}} = \frac{a+\sqrt{b}}{c}$$

Third attempt (stable cluster variables)

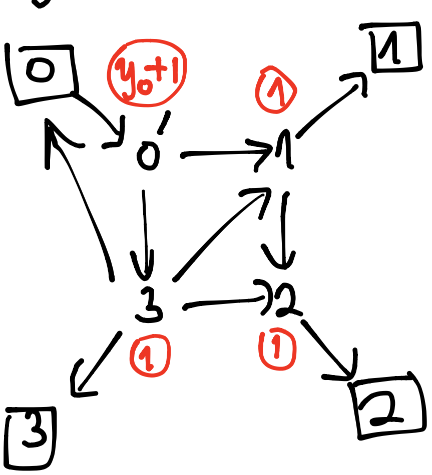
Each term in F_m is replaced with another monomial term, in such a way that the resulting $\tilde{F}_m$ converge as power series.

STEP 1: Define C-matrices as we mutate

$$C_m = \begin{array}{c} \\ \boxed{1} \\ \boxed{2} \\ \vdots \\ \boxed{n} \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & n \end{matrix} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

with $(\boxed{i}, j)$-entry $= \left(\#\text{ arrows } j \to \boxed{i}\right)$, negative for arrows $j \leftarrow \boxed{i}$

e.g.



$$C_1 = \begin{array}{c} \\ \boxed{0} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Step 2: For every monomial

$$C_{\bar{a}} \, y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$$

in the F-polynomial F_m,

replace it with

$$C_{\bar{a}} \, y_1^{b_1} y_2^{b_2} \cdots y_n^{b_n}$$

where $\bar{b} = -C_m^{-1} \bar{a}$. (Note: C_m is always invertible)

e.g. $\tilde{F}_1 = 1 + y_0$

$\tilde{F}_2 = 1 + y_1 + y_0 y_1$

$\tilde{F}_3 = 1 + y_2 + y_1 y_2 + y_0 y_2 + 2 y_0 y_1 y_2 + y_0^2 y_1 y_2$

$\vdots$

$\vdots$

$$\widetilde{F}_7 = 1 + y_0 y_1 + y_0 y_1^2 y_2 + 2 y_0 y_1 y_2 + 2 y_0^2 y_1^2 y_2 + \cdots$$

$$\widetilde{F}_8 = 1 + y_1 y_2 + y_1 y_2^2 y_0 + 2 y_1 y_2 y_0 + 2 y_1^2 y_2^2 y_0 + \cdots$$

$\vdots$

REU Exercise #19^{+}

(a) Prove that in this $\widetilde{A}_{1,2}$ example, mutating $0, 1, 2, 0, 1, 2, \ldots$, one has

$$\lim_{m \to \infty} \widetilde{F}_{3m} = 1 + \frac{y_1 y_2 + y_2}{(1 - y_0 y_1 y_2)^2}$$

(and $\widetilde{F}_{3m+1}, \widetilde{F}_{3m+2}$ look like cyclic shifts)

(b) Show for $\tilde{A}_{1,n}$

$$\begin{tikzcd} & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \\ 0 \arrow[ur] \arrow[urrrr] & & & & \end{tikzcd}$$

mutating at $0 1 2 \cdots n \, 0 1 2 \cdots n \cdots$,

calling the results F_m for $m \in \mathbb{Z}$

$$F_m F_{m-n-1} = F_{m-1} F_{m-n} + 1$$

REU Problem 8(a)

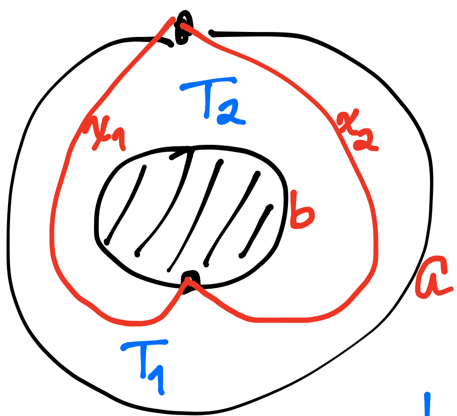
(c) Prove the analogous conjecture for $\tilde{A}_{1,n}$, i.e.

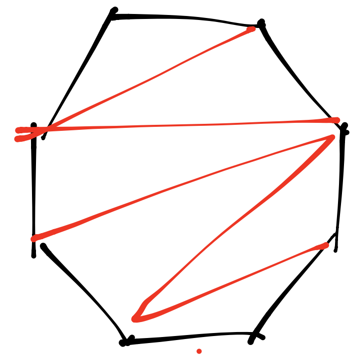
$$\lim_{m \to \infty} \tilde{F}_{(n+1)m} = 1 + \frac{y_1 y_2 y_3 \cdots y_n + y_2 y_3 \cdots y_n + \cdots + y_n}{(1 - y_0 y_1 \cdots y_n)^2}$$

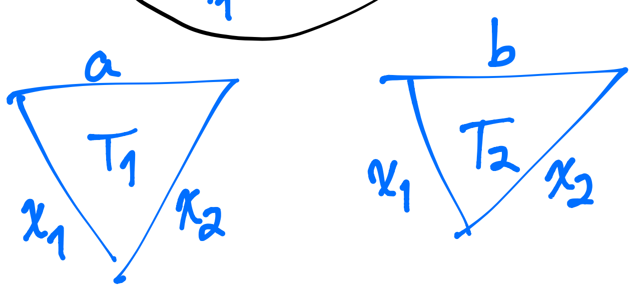
REMARK: Grace Zhang's REU 2016 report proved for $\tilde{A}_{1,1}$ $0 \rightrightarrows 1$ that

$$\lim_{m \to \infty} \tilde{F}_{2m} = 1 + \frac{y_1}{(1 - y_0 y_1)^2}$$

(Asymptotic) triangulations of an annulus





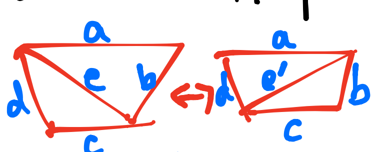


<u>FACT</u>: You can relate certain cluster algebras to triangulations of surfaces

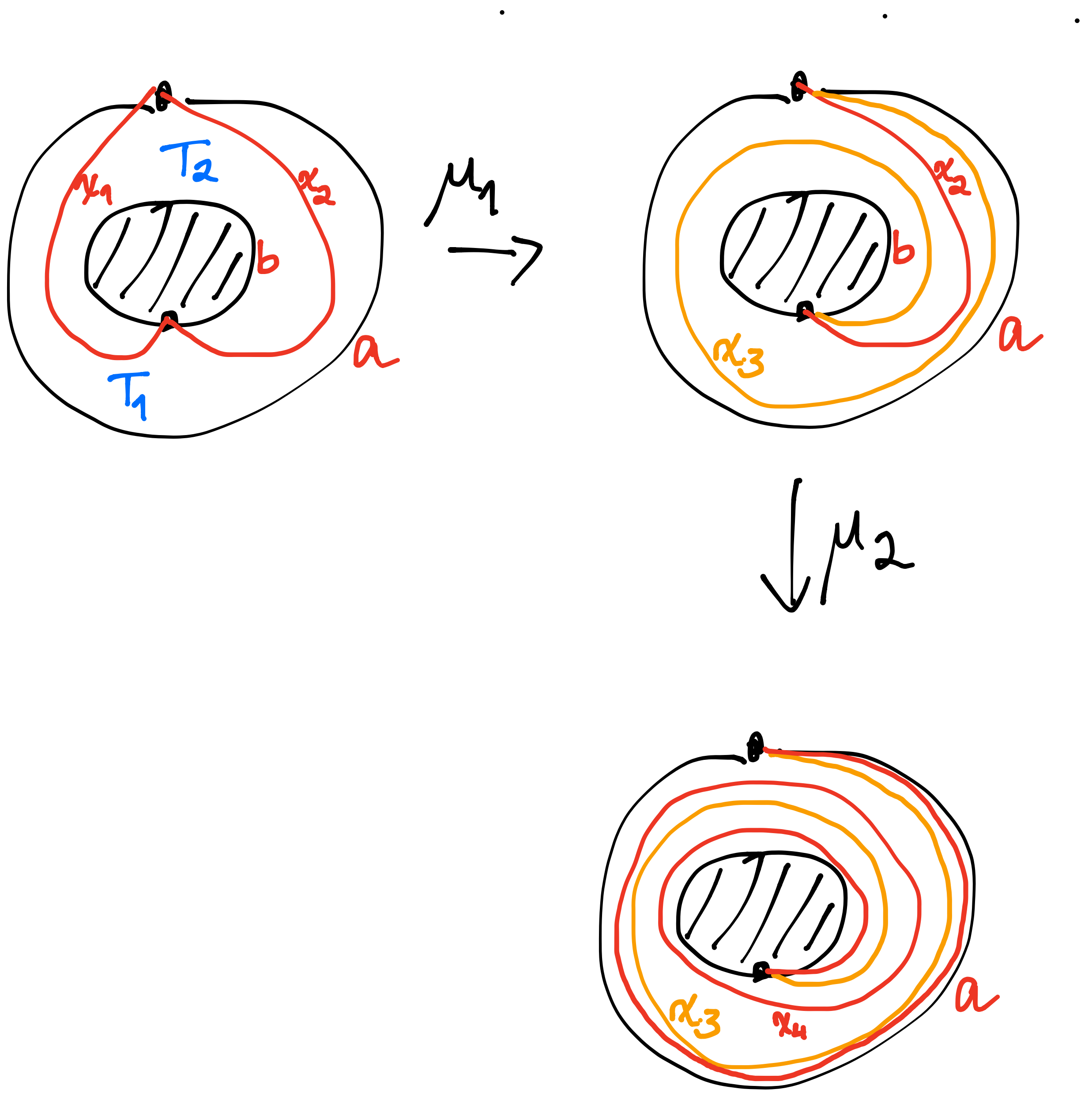
triangulations $\leftrightarrow$ clusters

arcs $\leftrightarrow$ cluster variables

quadrilateral flips $\leftrightarrow$ mutations



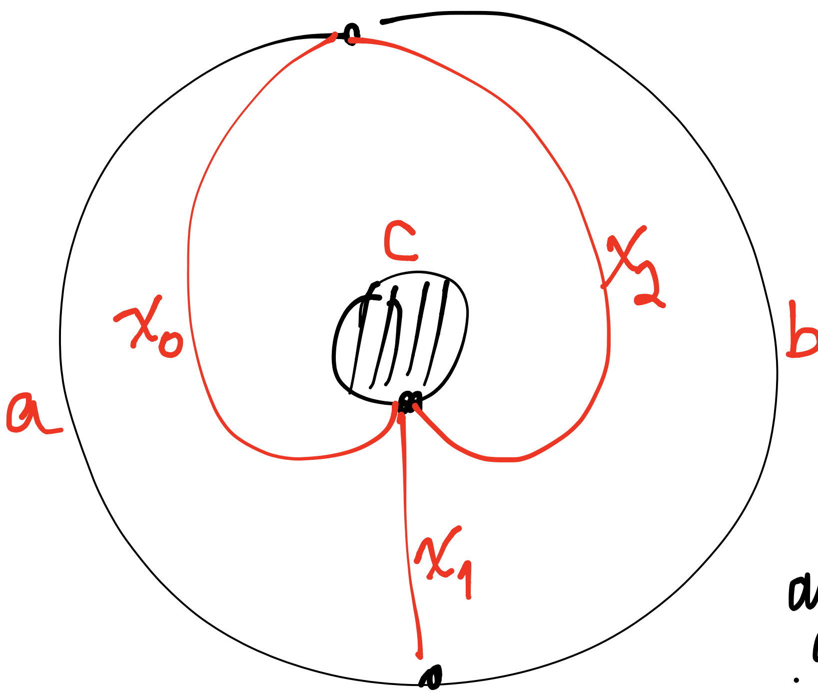
$ee' = ac + bd$



T_2
α_1
α_2
b
a
T_1
μ_1
α_2
b
α_3
a
μ_2
α_3
α_4
a

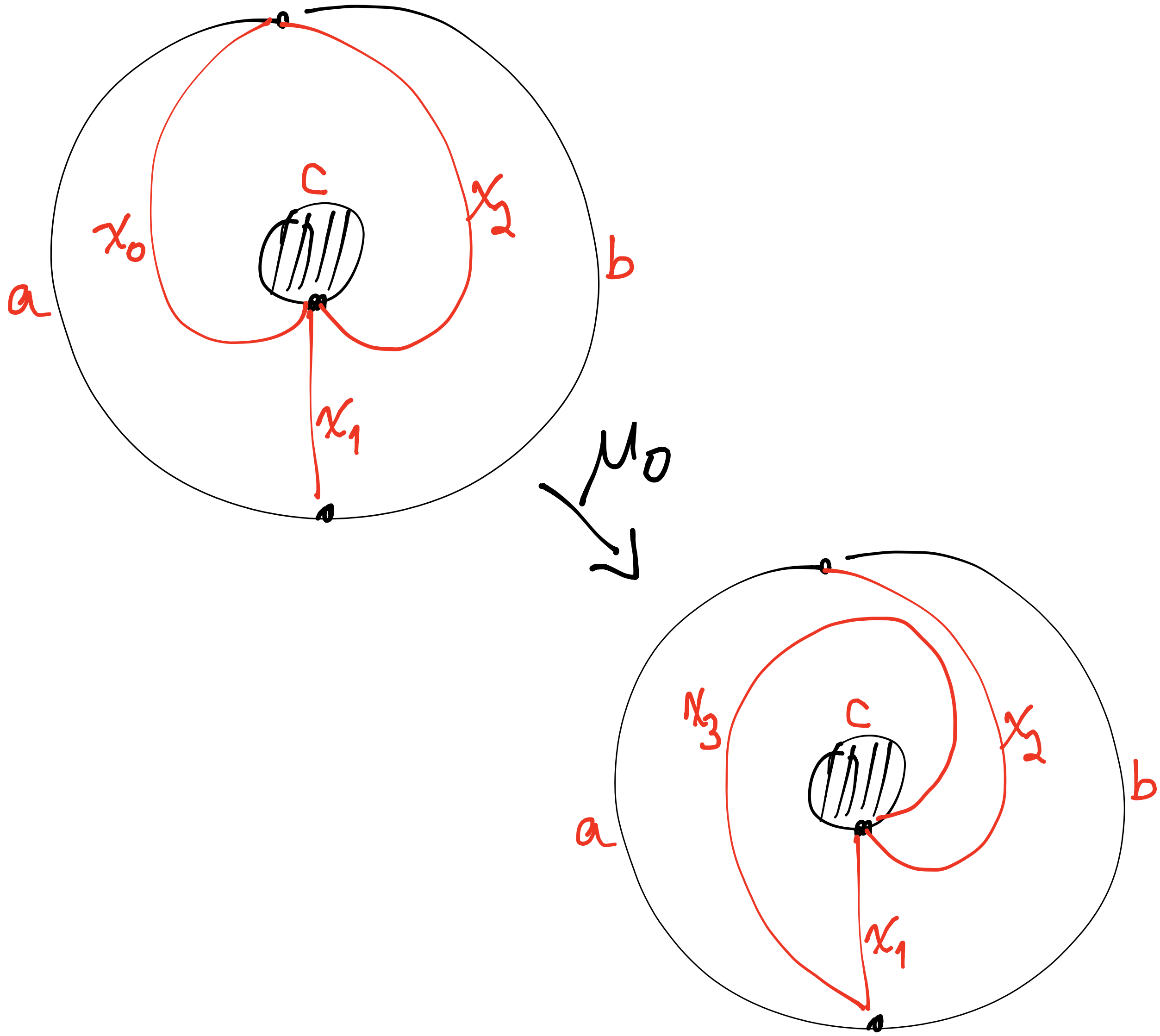
$\tilde{A}_{1,2}$
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$\rightsquigarrow$ annulus with 1 marked point inner
2 marked points outer

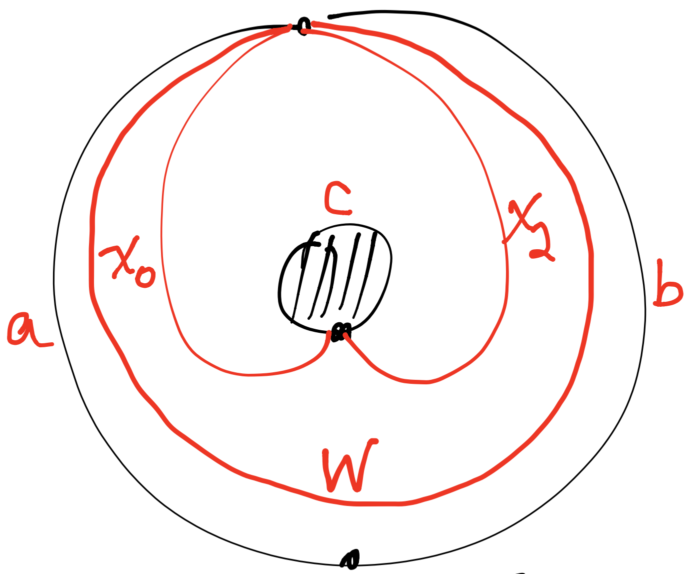


boundary arcs a, b, c ignored

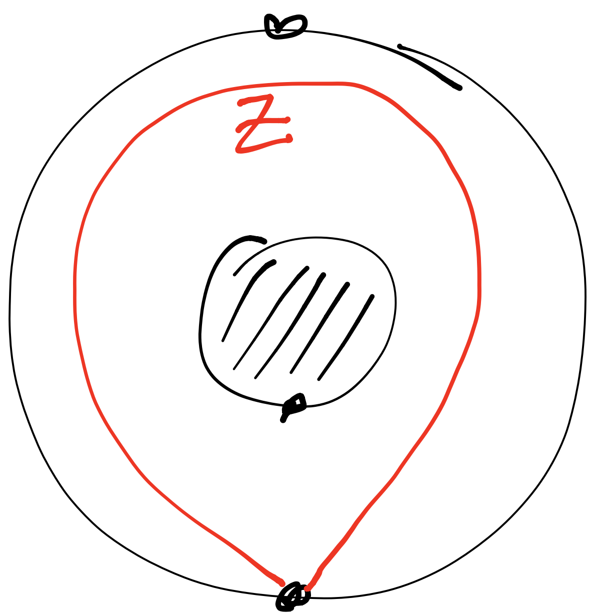
Mutate at 0 = flip at x_0

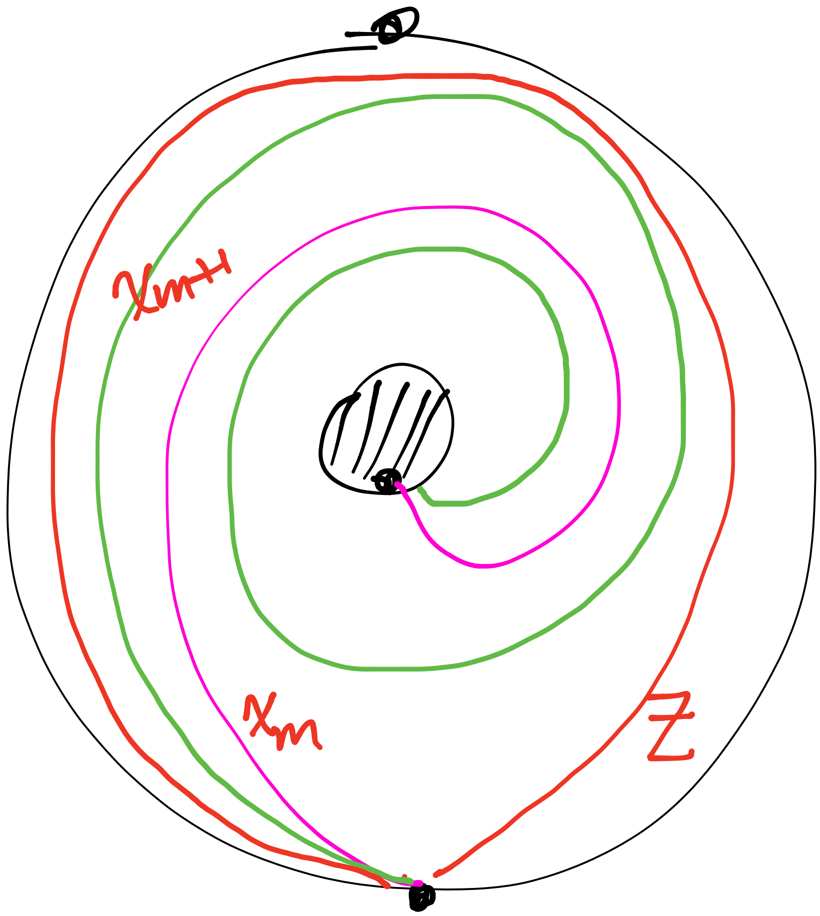


What should W look like?



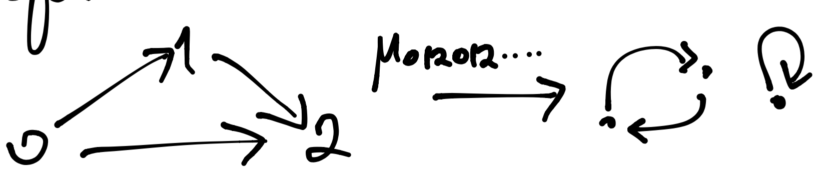
What about Z?





"DEF'N": In an asymptotic triangulation, the winding numbers can go to ∞!

The quiver "breaks"



REU Problem 8(b)

Connect stable cluster variables to asymptotic arcs.

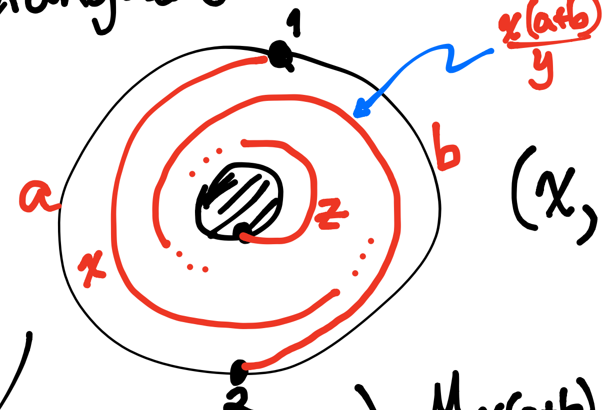
REU Exercise #20

(a) Prove that

$$W = \frac{x_0 + x_2}{x_1} \text{ also equals } \frac{x_{2k} + x_{2k+2}}{x_{2k+1}} \quad \forall k \in \mathbb{Z}$$

Prove a similar identity for Z.

(b) In Appendix B.2 of Vogel, it is argued that for $\tilde{A}_{1,2}$ there is an asymptotic triangulation



$$\Big(x, \underbrace{\frac{x(a+b)}{y}}_{=\,Y}, z\Big)$$

$$\Big(x, \frac{x(a+b)}{y}, z\Big) \xrightarrow{\mu_x} \Big(\frac{(a+b)^2}{y}, \frac{x(a+b)}{y}, z\Big)$$

$$\Big(x, \frac{x(a+b)}{y}, z\Big) \xrightarrow{\mu_{\frac{x(a+b)}{y}}} (x, y, z)$$

Rewrite the algebraic transformations of Vogel's Fig. 24 in terms of (x, Y, z).

REU Problem #8(b) (refined)

If you plug in stable cluster variables, do they obey these exchanges?

Continued fractions

$$[a_0, a_1, a_2, \dots, a_n]$$

$$:= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

e.g.

$$\underbrace{[1, 1, \dots, 1]}_{k \text{ times}} = \frac{Fib_k}{Fib_{k-1}}$$

$$1 + \frac{1}{1} = 1$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2}$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3}$$

golden ratio

$$\alpha := [1,1,1,\dots] = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} = \frac{1+\sqrt{5}}{2}$$

since

$$\alpha = 1 + \frac{1}{\alpha}$$

$$\Longrightarrow \alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\beta := [2,2,\dots]$$

has $\beta = 2 + \frac{1}{\beta}$

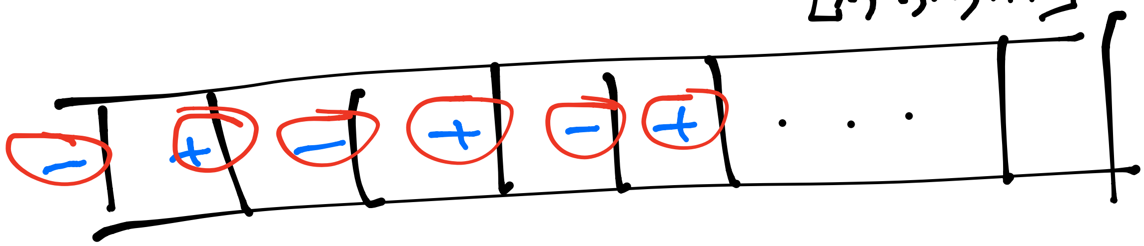
$$\beta^2 - 2\beta - 1 = 0$$

$$\beta = 1 + \sqrt{2}$$ silver ratio

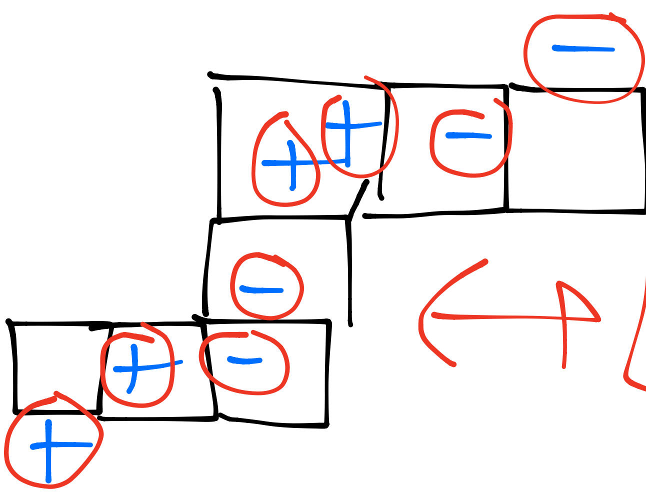
$$[n,n,\dots] = \frac{n + \sqrt{n^2+4}}{2}$$ metallic mean

Canakci-Schiffler directly relate cluster variables and F-polynomials to snake graphs and continued fractions

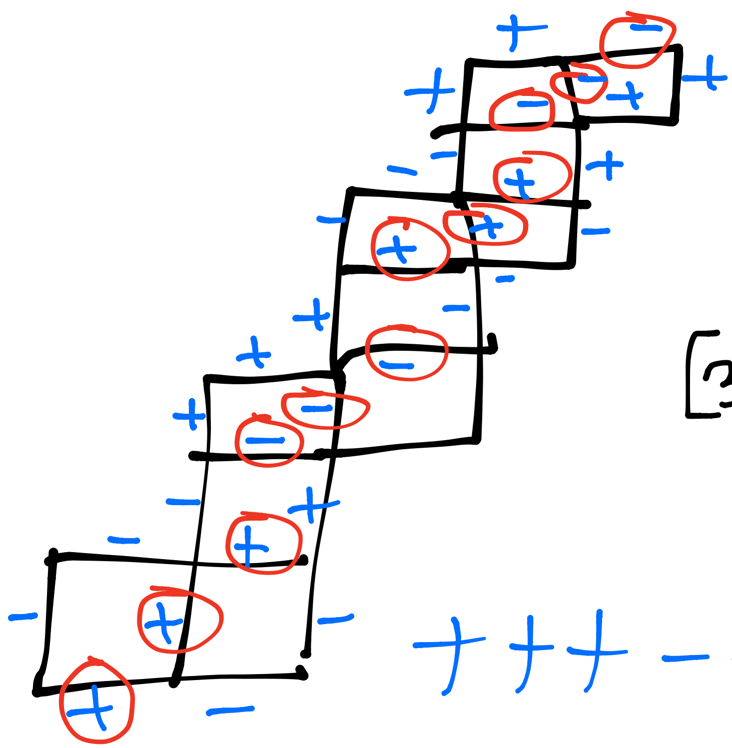
$[1,1,1,\dots]$



$$[a_0, a_1, \dots, a_n] \longleftrightarrow \underbrace{+++}_{a_0}\ \underbrace{--}_{a_1}\ \underbrace{++++}_{a_2} \dots$$



$\longleftrightarrow [2,2,2,2]$



$[3, 3, 3, \ldots]$

$+ + + - - - + + + - - -$

REU Problem 8(c)

Relate F-polynomials (of Canakci-Schiffler) from ∞ continued fractions $[n, n, n, \ldots]$ to stable cluster variables for $\tilde{A}_{1,n}$

Or other infinite periodic continued fractions?