Algebraic Monoids and Renner Monoids

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Abstract We collect some necessary concepts and principles in the theory of linear algebraic monoids which apply to further investigation on other topics such as the classification of reductive monoids, representations of algebraic monoids, monoids of Lie type, cell decompositions, monoid Hecke algebra, and monoid schemes. We use classical monoids as examples to demonstrate notions.

Keywords Algebraic monoid • Renner monoid • Classical monoid • Rook monoid

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1 Introduction

The Putcha-Renner theory of linear algebraic monoids is a big subject, which is built on linear algebraic groups, torus embeddings, and semigroups [61, 82]. Over the last three decades the theory has made significant progress in different fields: reductive monoids, Renner monoids, finite monoids of Lie type, monoids on groups with BN-pairs, group embeddings, monoid schemes, semisimple monoids,

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$\mathscr{J}$-irreducible monoids, combinatorics, and classical algebraic monoids [4, 12, 36, 40, 51, 61, 62, 64, 71, 75, 77, 80]. Unfortunately, the theory has a marketing problem as Solomon mentioned in [85], which is a very engaging introduction to the theory.

The aim of this survey paper is two-fold. We first give an introduction to the theory of linear algebraic monoids, and then focus on the recent developments in Renner monoids, with the intent to attract readers with interests in algebraic groups, combinatorics, Lie theory, and semigroup theory. We state the main theorems and provide sources instead of giving proofs. Occasionally, for some statements of conclusions we give short arguments.

Classical algebraic monoids are a special class of linear algebraic monoids. Throughout the paper, classical algebraic monoids are used as examples extensively to demonstrate important concepts.

The following section is devoted to algebraic monoids in general, including definitions, methods to construct algebraic monoids, classical monoids, $\mathscr{J}$-class structures, irreducible algebraic monoids, Putcha lattices, and classical rook monoids. In the next section we describe reductive monoids, with emphasis on Jordan decomposition, parabolic subgroups, type maps, and $\mathscr{J}$-irreducible monoids. The finial section records various recent results on Renner monoids such as definitions and properties, classical Renner monoids, standard form of elements in a Renner monoid, reduced row echelon form, length function, generators and defining relations, orders, conjugacy classes, generating functions, and generalized Renner monoids.

2 Algebraic Monoids

Let M be an affine variety over an algebraically closed field K together with the structure of a semigroup. We call M an *affine algebraic semigroup*, or simply *algebraic semigroup*, if the associative operation in M is a morphism of varieties. An affine algebraic monoid is an affine algebraic semigroup with an identity. The *unit group* of an algebraic monoid M is the set of elements of M with an inverse in M. We are concerned mainly with algebraic monoids, though we sometimes state some results on algebraic semigroups.

There are so many interesting examples of algebraic monoids. Every algebraic group is an algebraic monoid; every finite monoid is an algebraic monoid. Viewed as an affine space of dimension n^2, the set $\mathbf{M}_n$ of all $n \times n$ matrices over K is an algebraic monoid under matrix multiplication, called the *general linear monoid*. The unit group of $\mathbf{M}_n$ is the general linear group $\mathbf{GL}_n$. The monoid $\mathbf{D}_n$ of diagonal matrices is algebraic with the group $\mathbf{T}_n$ of invertible diagonal matrices as its unit group. Let $\overline{\mathbf{B}}_n$ be the monoid of all upper triangular matrices. Then $\overline{\mathbf{B}}_n$ is an algebraic monoid with unit group $\mathbf{B}_n$ consisting of all invertible upper triangular matrices.

A Zariski closed submonoid of $\mathbf{M}_n$ is called a *linear algebraic monoid*. The following theorem shows that every affine algebraic monoid is isomorphic to a linear algebraic monoid.

Theorem 1 ([11, II, §2, Theorem 3.3]; [51, Corollary 1.3]). *Every affine algebraic semigroup is isomorphic to a closed subsemigroup of some* $\mathbf{M}_n$. *In particular, every affine algebraic monoid is isomorphic to a closed submonoid of some* $\mathbf{M}_n$.

Just as the closed embedding of an algebraic group into some $\mathbf{GL}_n$ in algebraic group theory reduces the study of algebraic groups to that of closed subgroups in $\mathbf{GL}_n$, this theorem reduces the study of algebraic monoids to that of closed submonoids in $\mathbf{M}_n$. From now on, we identify an affine algebraic monoid with its closed embedding in $\mathbf{M}_n$, and simply refer to it as an *algebraic monoid*. Every algebraic monoid M has a dimension, which is the dimension of M as an algebraic variety [24]. If M is a point then its dimension is zero; if M is a curve then its dimension is one; if M is a surface then its dimension is two. Also $\dim \mathbf{M}_n = n^2$, $\dim \mathbf{D}_n = n$ and $\dim \overline{\mathbf{B}}_n = \frac{n(n+1)}{2}$.

The unit group of an algebraic monoid M determines the structure of M to some extent, and it has been of primary interest in finding connections between the structures of an algebraic monoid and its unit group [60]. Theorem 2 shows that the unit group is an open subgroup in the monoid and is equal to the intersection of the monoid with the general linear group.

Theorem 2 ([11, II, §2, Corollary 3.5]; [74, Corollary 2.2.3]). *Let* M *be an algebraic monoid. Then its unit group* $G = M \cap \mathbf{GL}_n$. *Furthermore,* G *is an algebraic group and there is a morphism* $\alpha : M \to K$ *such that* $G = \alpha^{-1}(K^*)$, *where* $K^* = K \setminus \{0\}$. *In particular,* G *is open in* M.

The set $E(M)$ of idempotents of M contains certain controlling structural information about M. This set carries the partial order

$$e \leq f \Leftrightarrow fe = e = ef.$$

In what follows, we assume that the partial order on any subset of $E(M)$ is inherited from this one.

Proposition 1 ([61, Corollary 3.26]; [82, Proposition 3.12]). *Let* M *be an algebraic monoid and* $e \in E(M)$. *Then* eMe *is an algebraic monoid; its unit group is precisely the* $\mathscr{H}$*-class of* e. *This unit group is an algebraic group and is open in* eMe.

The Zariski closures of subsets of M are fundamental in the theory of algebraic monoids. Lemma 1 below is useful technically in dealing with these closures. If X is a subset of M, we use $\overline{X}$ to denote the Zariski closure of X in M. In particular, if $M = \mathbf{M}_n$, then $\overline{\mathbf{GL}_n} = \mathbf{M}_n$, $\overline{\mathbf{T}}_n = \mathbf{D}_n$ and $\overline{\mathbf{B}}_n$ is the Zariski closure of $\mathbf{B}_n$.

Lemma 1 ([53, Lemma 1.2]). *Let* X *and* Y *be subsets of an algebraic monoid* M *with unit group* G. *Then*

(1) $\overline{XY} = \overline{\overline{X}\,\overline{Y}}$.
(2) If $a, b \in G$, *then* $\overline{aXb} = a\overline{X}b$.

How to construct algebraic monoids? It is an easy task, based on the obvious fact that a closed submonoid of an algebraic monoid is again an algebraic monoid. The following corollary provides us with a great deal of examples of algebraic monoids.

Corollary 1. *If S is a submonoid of $\mathbf{M}_n$, then $\overline{S}$ is an algebraic monoid.*

Indeed, it follows from Lemma 1 that $\overline{S}\,\overline{S} \subseteq \overline{\overline{S}\,\overline{S}} = \overline{S\,S} = \overline{S}$. Thus $\overline{S}$ is a closed submonoid of $\mathbf{M}_n$.

Corollary 2. *Let G be a subgroup of $\mathbf{M}_n$. Then $\overline{G} \subseteq \mathbf{M}_n$ is an algebraic monoid. If $G \subseteq \mathbf{GL}_n$, then the unit group of $\overline{G}$ is the Zariski closure of G in $\mathbf{GL}_n$. Furthermore, if G is an algebraic group then the unit group of $\overline{G}$ is G.*

Algebraic monoids are special semigroups, of which *Green relations* $\mathscr{J}, \mathscr{L}, \mathscr{R}$, and $\mathscr{H}$ are fundamental structure elements. Let S be a semigroup and $a, b \in S$. Then by definition

$$
\begin{aligned}
a \mathscr{J} b &\quad \text{if } S^1 a S^1 = S^1 b S^1; \\
a \mathscr{L} b &\quad \text{if } S^1 a = S^1 b; \\
a \mathscr{R} b &\quad \text{if } a S^1 = b S^1; \\
a \mathscr{H} b &\quad \text{if } a \mathscr{L} b \text{ and } a \mathscr{R} b.
\end{aligned}
$$

where $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$ with obvious multiplication if S is not a monoid. We use J_a and H_a to denote the $\mathscr{J}$-classes and $\mathscr{H}$-classes of a, respectively.

Algebraic semigroups are special kinds of strongly π-regular semigroups. A semigroup S is *strongly π-regular* if for any $a \in S$ there exists a positive integer k such that a^k lies in H_e for some idempotent $e \in E(S)$. A strongly π-regular semigroup is also refereed to as an *epigroup* or a *group-bound semigroup* in the literature of semigroup theory [14, 17, 27–30]. Every finite semigroup is strongly π-regular; so is the full matrix monoid consisting of all square matrices over a field. The concept of strongly π-regular captures the semigroup essence of algebraic semigroups [82]. Putcha [51, 52, 61] and Brion and Renner [5] in this proceedings contain more information on algebraic semigroups and strongly π-regular semigroups. Okninski [48] is a very comprehensive reference on strongly π-regular matrix semigroups.

A nonempty subset I of S is an *ideal* if $S^1 I S^1 \subseteq I$. Clearly S is an ideal of S. If S is strongly π-regular and S is its only ideal then we say that S is *completely simple*. The minimal ideal, if it exists, is called the kernel of S. The reader who is interested in kernel of linear algebraic monoids can find useful results in Huang [20–22]. The following two theorems confirm that every algebraic semigroup and hence algebraic monoid is strongly π-regular, and contains a closed completely simple kernel.

Theorem 3 ([51, Corollary 1.4]). *Let S be an algebraic semigroup. Then there exists a positive integer n such that a^n lies in a subgroup of S for all $a \in S$. In particular, every algebraic monoid is strongly π-regular.*

Theorem 4 ([51, Corollary 1.5]). *Every algebraic semigroup has a kernel which is closed and completely simple.*

2.1 Some Classical Monoids

We introduce some families of algebraic monoids, called classical monoids, which are closely related to classical groups. These monoids play an important role in the theory of algebraic monoids [31–33, 36]. The parameter l in each case is 1 less than the dimension of the closed subgroup of diagonal matrices in the unit group G of the monoid under discussion. This l is also the dimension of the Cartan subalgebra of the Lie algebra of G.

A_l: The *general linear monoid* $\mathbf{M}_n$ with $n = l + 1$: Let $G = K^*\mathbf{SL}_n$ where $\mathbf{SL}_n$ is the special linear group consisting of the matrices of determinant 1 in $\mathbf{GL}_n$. Then $G = \mathbf{GL}_n$, and $\mathbf{M}_n = \overline{G}$.

C_l: The *symplectic monoid* $\mathbf{MSp}_n$ with $n = 2l$: The symplectic group is

$$\mathbf{Sp}_n = \{A \in \mathbf{GL}_n \mid A^\top JA = J\}$$

where $J = \begin{pmatrix} 0 & J_l \\ -J_l & 0 \end{pmatrix}$ with $J_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. Let $G = K^*\mathbf{Sp}_n$. Then $G \subseteq \mathbf{GL}_n$. The monoid $\overline{G}$ is called the symplectic monoid which will be denoted by $\mathbf{MSp}_n$. It is usually hard to give a concrete algebraic description of the Zariski closure of a subset of an algebraic monoid. It follows, however, from Doty [12] that

$$\mathbf{MSp}_n = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}.$$

B_l: The *odd special orthogonal monoid* $\mathbf{MSO}_n$ with $n = 2l + 1$: If the characteristic of K is not 2, then the odd special orthogonal group is by definition

$$\mathbf{SO}_n = \{A \in \mathbf{SL}_n \mid A^\top JA = J\}$$

where $J = \begin{pmatrix} 0 & 0 & J_l \\ 0 & 1 & 0 \\ J_l & 0 & 0 \end{pmatrix}$. Let $G = K^*\mathbf{SO}_n \subseteq \mathbf{GL}_n$. The monoid $\overline{G}$ is called the odd special orthogonal monoid, denoted by $\mathbf{MSO}_n$. By [12] we have

$$\mathbf{MSO}_n = \{x \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}.$$

D_l: This is the *even special orthogonal monoid* $\mathbf{MSO}_n$ with $n = 2l$, defined by taking the Zariski closure of $K^*\mathbf{SO}_n$ in which $\mathbf{SO}_n$ is given by the same condition as B_l: $A^\top JA = J$, where the matrix J now is $\begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$ (if the characteristic of K is not 2). Notice that the set

$$M = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}$$

is an algebraic monoid. Naturally we ask: whether $\mathbf{MSO}_n$ equals M? Unfortunately, no. In fact, M is reducible, and $\mathbf{MSO}_n$ is its identity component. More information about this M will be provided in Sect. 2.3.

The symplectic and special orthogonal algebraic monoids arise geometrically as monoids of linear transformations that dilate certain skew-symmetric and symmetric bilinear forms, respectively.

2.2 *Monoids Induced from Representations*

To construct further examples of algebraic monoids, we start with rational representations of algebraic groups. A rational representation of an algebraic group G_0 is a group homomorphism $\rho : G_0 \to \mathbf{GL}_n$ which is also a morphism of varieties [1, 24]. The image $\rho(G_0)$ is an algebraic group. Let

$$G = K^*\rho(G_0) = \{c\rho(g) \mid c \in K^* \text{ and } g \in G_0\}.$$

Then G is an algebraic group by [24, Corollary 7.4]. However, G is not a closed subset of $\mathbf{M}_n$ since the zero matrix is in $\overline{G}$ but not in G. Write

$$M(\rho) = \overline{G}.$$

It follows from Corollary 2 that $M(\rho)$ is an algebraic monoid with unit group G. In addition, if G_0 is irreducible, so are $M(\rho)$, G and $\rho(G_0)$. Clearly, $\rho(G_0)$ is a subgroup of G.

Why do we multiple $\rho(G_0)$ by K^* and then take the Zariski closure of the product? We note that if $\rho(G_0)$ is closed in $\mathbf{M}_n$, then the monoid $\overline{\rho(G_0)} = \rho(G_0)$ is a group, nothing new. This is the case if G_0 is the *special linear group*. To make sure that $\overline{G}$ is a monoid which includes G properly, G must contain at least one matrix whose determinant is not 1. Renner [74, Theorem 3.3.6] and Waterhouse [91] provide conditions under which an algebraic group G can be embedded as the unit group of an algebraic monoid which are not a group. Huang [21, Theorem 5.1] refines the above result and states that under the same conditions, the group G may be embedded properly into a normal regular algebraic monoid. We refer to the above references for more details.

The classical monoids can be constructed via certain representations of classical groups. Let $V = K^n$, and G_0 be the special linear group, symplectic group (n is even), or special orthogonal group. Then G_0 acts naturally on V by their very definition, and we obtain the *natural representation* $\rho : G_0 \to \mathbf{GL}_n$ with $\rho(g) = g$. The monoid $M(\rho)$ is the general linear monoid, symplectic monoid, and special orthogonal monoid, respectively.

Let's explore two more examples obtained by representations. They are taken from [85] and the latter is a variant of Example 8.5 of [61].

Example 1. Let $G_0 = \mathbf{SL}_m$ and $V = K^m \otimes K^m$ with basis $\{v_i \otimes v_j \mid 1 \le i < j \le m\}$. Define a rational representation $\rho : G_0 \to \mathbf{GL}_n$ by $\rho(g)(v \otimes v') = gv \otimes gv'$, where $n = m^2$. The monoid $M(\rho) = \{a \otimes a \mid a \in \mathbf{M}_m\}$ is isomorphic to $\mathbf{M}_m$. In particular, $G = \{g \otimes g \mid g \in \mathbf{GL}_m\}$, isomorphic to $\mathbf{GL}_m$.

Example 2. Let $V = K^m \otimes K^m$ be as in Example 1 and let $G_0 = \mathbf{SL}_m$. Define a rational representation $\rho : G_0 \to \mathbf{GL}_n$ by $\rho(g)(v \otimes v') = gv \otimes (g^{-1})^\top v'$, where $n = m^2$. Though the monoid $M(\rho)$ is hard to describe algebraically, we however know that the unit group of $M(\rho)$ is closely related to $\mathbf{SL}_m$. But $M(\rho)$ is different dramatically from $\mathbf{M}_m$ since $E(M(\rho))$ and $E(\mathbf{M}_m)$ are not isomorphic.

2.3 *Irreducible Algebraic Monoids*

An algebraic monoid is *irreducible* if it is irreducible as an affine algebraic variety, that is, it is not a union of proper Zariski closed subsets. The monoids $\mathbf{M}_n$, $\mathbf{D}_n$, and $\overline{\mathbf{B}}_n$ are irreducible. The classical monoids of types A_l, B_l, C_l, and D_l above are all irreducible. The monoid in Example 8 is irreducible since it is isomorphic to the affine space K^{n+1} as varieties.

An algebraic monoid $M \subseteq \mathbf{M}_n$ is connected if it is connected as a subset of $\mathbf{M}_n$ in the Zariski topology. Irreducible algebraic monoids are connected, but not conversely. For example, the monoid

$$M = \{(a, b) \in K^2 \mid a^2 = b^2\}$$

is connected but not irreducible. The monoid in the following example is another instance of connected monoids even though not irreducible.

Example 3 ([12, Section 6]). Assume that the characteristic of K is not 2. Let

$$M = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}$$

where $J = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$. The unit group of M is

$$G = \{A \in \mathbf{M}_n \mid AJA^\top = cJ \text{ for some } c \in K^*\}.$$

The subgroup T of G consisting of invertible diagonal matrices in G is a maximal torus of G. The orthogonal group in $\mathbf{GL}_n$ is

$$O_n = \{A \in \mathbf{GL}_n \mid AJA^\top = J\}.$$

Let $O_n^+ = \{A \in O_n \mid \det A = 1\}$ and $O_n^- = \{A \in O_n \mid \det A = -1\}$. Then $O_n = O_n^+ \cup O_n^-$. Denote by G^+ the subgroup of G generated by T and O_n^+. Then $G^+ = K^* SO_n$ is a closed and connected subgroup of G. So the Zariski closure of G^+ in $\mathbf{M}_n$ is the even special orthogonal monoid $\mathbf{MSO}_n$ with unit group G^+. Therefore, $\mathbf{MSO}_n$ is the irreducible identity component of M.

If $n = 2$, then the monoid in Example 3 is

$$M = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \,\middle|\, a, b \in K \right\} \cup \left\{ \begin{pmatrix} & c \\ d & \end{pmatrix} \,\middle|\, c, d \in K \right\}.$$

This monoid has two irreducible components, and so is reducible, but is connected. However, its unit group G is not connected since G has two connected components.

Theorem 5 ([61, Proposition 6.1]). *Suppose that M is an irreducible algebraic monoid with unit group G, and $a, b \in M$. Then*

(1) $a \mathscr{J} b$ if and only $a \in GbG$.
(2) $a \mathscr{L} b$ if and only if $a \in Gb$.
(3) $a \mathscr{R} b$ if and only if $a \in bG$.

This theorem allows us to interpret $\mathscr{J}$, $\mathscr{L}$ and $\mathscr{R}$-classes in M using group actions of G on M. Each $\mathscr{J}$, $\mathscr{L}$, and $\mathscr{R}$-class is an orbit of a group action, which sometimes indicates connections with geometry such as orbits and closures.

Since $GM \subseteq M$ and $MG \subseteq M$, we have the *left action* of G given by $g \cdot a = ga$, and the *right action* given by $a \cdot g = ag^{-1}$. Theorem 5 shows that if M is irreducible, then a, b lie in the same $\mathscr{L}$-class if and only if they lie in the same left G orbit, and that a, b are in the same $\mathscr{R}$-class if and only if they are in the same right G orbit. The $\mathscr{L}$-class of a is thus the orbit Ga, and the $\mathscr{R}$-class of a is the orbit aG. The left and right G orbits are closely related to row and column echelon forms of M, respectively, which will be described in Sect. 4.4.

Consider the group action of $G \times G$ on M by $(g, h) \cdot a = gah^{-1}$ for $g, h \in G$ and $a \in M$. Let $G \backslash M / G$ denote the set of orbits GaG for this action. It follows from Theorem 5 that if M is irreducible, then a, b lie in the same $\mathscr{J}$-class if and only if they lie in the same $G \times G$ orbit. Moreover, the $\mathscr{J}$-class $J_a = GaG$. We give $G \backslash M / G$ the partial order

$$J_a \le J_b \Leftrightarrow MaM \subseteq MbM \Leftrightarrow GaG \subseteq \overline{GbG},$$

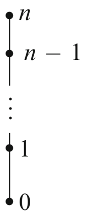
henceforth $(G \backslash M / G, \le)$ is a poset. We examine this poset for different irreducible monoids.

Example 4. Let $M = \mathbf{M}_n$. Then $G = \mathbf{GL}_n$. If $a, b \in M$ then $GaG = GbG$ if and only if a and b are of the same rank. There is a bijection of $G\backslash M/G$ onto $\{0, 1, \cdots, n\}$ given by $GaG \mapsto \text{rank } a$. The partial order is the natural linear order on $\{0, 1, \cdots, n\}$ as illustrated in the first figure below. Clearly, the number of $G \times G$ orbits in M is $n + 1$.

Example 5. Let $M = \mathbf{MSp}_n$ with unit group G and $n = 2l$. If $a, b \in M$ then $GaG = GbG$ if and only if rank a = rank b. There is a bijection of $G\backslash M/G$ onto $\{0, 1, \cdots, l, n\}$ given by $GaG \mapsto \text{rank } a$. The partial order is the natural linear order on $\{0, 1, \cdots, l, n\}$ as illustrated in the second figure. Note that there are no elements of rank greater than l but less than n in M. The number of $G \times G$ orbits in M is $l + 2$.

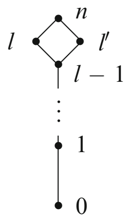
Example 6. The lattice of the $G \times G$ orbits of the odd special orthogonal monoid $\mathbf{MSO}_n$ with $n = 2l + 1$ is isomorphic to that of the symplectic monoid $\mathbf{MSp}_{2l}$.

Example 7. Let $M = \mathbf{MSO}_n$ with unit group G and $n = 2l$. If $a, b \in M$ then $GaG = GbG$ if and only if rank a = rank b = $0, 1, \cdots, l - 1, n$. However, there are two $G \times G$ orbits of rank l whose representatives are, respectively, $\text{diag}(1, \cdots, 1, 0, \cdots, 0)$ and $\text{diag}(1, \cdots, 1, 0, 1, \cdots, 0)$ each with l copies of 1. Let l' be a symbol. Then there is a bijection of $G\backslash M/G$ onto $\{0, 1, \cdots, l, l', n\}$ whose partial order is given in the third figure below. There are no elements of rank greater than l but less than n in M. The number of $G \times G$ orbits in M is $l + 3$.



$G \times G$ orbitsof $\mathbf{M}_n$

$G \times G$ orbitsof $\mathbf{MSp}_n$



$G \times G$ orbitsof $\mathbf{MSO}_n$

Example 8 ([54], Example 15). Let $M \subseteq \mathbf{M}_{n+1}$ consist of matrices

$$\begin{pmatrix} a & a_1 & a_2 & \cdots & a_n \\ & a & 0 & \cdots & 0 \\ & & a & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & a \end{pmatrix}$$

where $a, a_1, \ldots, a_n \in K$. The unit group of M consists of matrices in M whose diagonal element a is not zero. There are infinitely many $G \times G$ orbits if $n > 1$. In fact, if we denote by $(a, a_1, \cdots, a_n)$ the matrix above, then the $G \times G$ orbits are G, $\{0\}$, and orbits which contain matrices $(0, a_1, \cdots, a_n)$ with at least one a_i not zero. Moreover, for the latter we have that two elements $(0, a_1, \cdots, a_n)$ and $(0, b_1, \cdots, b_n)$ lie in the same orbit if and only if there is $c \in K^*$ such that $b_i = ca_i$ for all i. So these orbits are in bijection with points in $\mathbf{P}^{n-1}(K)$, the projective space of dimension $n-1$. More specifically, M has n orbits of form $(0, 0, \cdots, 0, a_i, 0, \cdots, 0)$ where $a_i \neq 0$ and $1 \leq i \leq n$, and M has infinitely many orbits $(0, a_1, \cdots, a_n)$ with at least two nonzero entries. Let $a_{i_1}, \cdots, a_{i_k}$ be all the nonzero entries in orbit $(0, a_1, \cdots, a_n)$. Then

$$(0, a_1, \cdots, a_n) \leq (0, b_1, \cdots, b_n) \text{ if and only if none of } b_{i_1}, \cdots, b_{i_k} \text{ is zero.}$$

Clearly, 0, 1 are idempotents of M. Check that they are the only idempotents of M. This leads to the following important definition.

Definition 1. Let M be an algebraic monoid. A $\mathscr{J}$-class J is regular if $E(J) \neq \emptyset$. Define

$$\mathscr{U}(M) = \{J \subseteq M \mid J \text{ is a regular } \mathscr{J}\text{-class}\}.$$

If M is irreducible, then $\mathscr{U}(M) = \{J \in G\backslash M/G \mid J \cap E(M) \neq \emptyset\}$ and is a finite lattice. A key result of [54] is that idempotents e, f are in the same $G \times G$ orbit if and only if they are conjugate under G. This result is useful throughout the theory of algebraic monoids. In particular, it plays a critical role in describing certain monoids with exactly one nonzero minimal $G \times G$ orbit.

Our intention below is to introduce height function on $E(M)$ and $\mathscr{U}(M)$ for irreducible algebraic monoids M. We begin by collecting results about idempotents of M.

Theorem 6 ([61, Corollaries 6.8 and 6.10 and Proposition 6.25]). *Let M be an irreducible algebraic monoid M with unit group G. Let T be a maximal torus, and W the Weyl group of G. Then*

(1) $E(M) = \cup_{g \in G}\, gE(\overline{T})g^{-1}$.
(2) Two elements $e, f \in E(\overline{T})$ are conjugate under G if and only if they are conjugate under W.

We observe from the previous theorem that there are as many G-orbits as W-orbits in $E(\overline{T})$, and that $E(M)$ is not only stable under the conjugation action of G on M

$$a \mapsto gag^{-1} \quad \text{for } a \in M \text{ and } g \in G,$$

but also completely determined by the G-orbits of the idempotents in $\overline{T}$. Theorem 7 below describes the lengths of chains of idempotents in $E(M)$. A *chain of idempotents* is a linearly ordered subset $\Gamma = \{e_0 < e_1 < e_2 < \cdots < e_k\}$ of the poset $E(M)$, and the *length* of Γ is k. A chain is *maximal* if it is properly contained in no other chain.

Theorem 7 ([61, Corollary 6.10 and Theorem 6.20]). *Let M be an irreducible algebraic monoid with unit group G. Then every chain of idempotents is contained in a maximal torus T of G. Furthermore, the lengths of the maximal chains in $E(\overline{T})$, $E(M)$, and $\mathscr{U}(M)$ are all the same. If M has a zero, then this number is equal to* $\dim T$.

We now define height function on $\mathscr{U}(M)$ and $E(M)$ for any irreducible algebraic monoid M with kernel J_0.

Definition 2. Define $\mathrm{ht}(J_0) = 0$ and $\mathrm{ht}(J) = \mathrm{ht}(J') + 1$ if $J, J' \in \mathscr{U}(M)$ and J covers J'. If $e \in J \in \mathscr{U}(M)$, then $\mathrm{ht}(e) = \mathrm{ht}(J)$. If $\mathrm{ht}(1) = p$, then $\mathrm{ht}(M) = \mathrm{ht}(E(M)) = p$.

This function is a powerful tool to prove and obtain useful results using induction on height of regular $\mathscr{J}$-classes of the monoid. This approach has been employed extensively in [61].

We can extend height function from $\mathscr{U}(M)$ to M if M is an irreducible regular algebraic monoid. A monoid M is *regular* if for each $a \in M$, there is $b \in M$ such that $a = aba$. A monoid M with unit group G is *unit regular* if for each $a \in M$, there is $b \in G$ such that $a = aba$.

Theorem 8 ([54, Theorem 1.3]; [85, Proposition 3.2]). *Suppose that M is an irreducible algebraic monoid with unit group G. The following are equivalent.*

(1) M is regular.
(2) M is unit regular.
(3) $M = GE(M)$.
(4) $G\backslash M/G = \mathscr{U}(M)$.

By Theorem 8 if M is an irreducible regular algebraic monoid then $\mathscr{U}(M)$ is equal to the set of all $\mathscr{J}$-classes. Thus height function on $\mathscr{U}(M)$ can be extended to M by

$$\mathrm{ht}(a) = \mathrm{ht}(J_a)$$

for all $a \in M$.

The height functions on classical monoids are consistent with the usual rank functions. If $a \in \mathbf{M}_n$, then $\mathrm{ht}(a) \in \{0, \cdots, n\}$. If M is a classical monoid of type B_l, C_l or D_l as defined in Sect. 2.1, then $\mathrm{ht}(a) \in \{0, \cdots, l, n\}$.

2.4 Putcha Lattice

The *Putcha lattice of cross sections*, for short *Putcha lattice*, of an irreducible algebraic monoid M with unit group G was initially introduced in [57]. Let T be a maximal torus of G.

Definition 3. A subset $\Lambda \subseteq E(\overline{T})$ is called a Putcha lattice of M if $|J \cap \Lambda| = 1$ for all $J \in \mathscr{U}(M)$, and for all $e, f \in \Lambda$, $e \leq f \Leftrightarrow J_e \leq J_f$.

A Putcha lattice Λ is indeed a sublattice of $E(\overline{T})$. We agree that Λ inherits the partial order on $E(\overline{T})$ which in turn inherits the partial order on $E(M)$. By definition Λ is a set of representatives for the $G \times G$ orbits. Thus M is a disjoint union of $G \times G$ orbits GeG with Λ as the index set

$$M = \bigsqcup_{e \in \Lambda} GeG,$$

and the bijection $\Lambda \to G\backslash M/G$ is order preserving. In addition, the lattice Λ is a set of representatives for the orbits of the conjugation action of W on $E(\overline{T})$. Thus $E(\overline{T}) = \bigsqcup_{e \in \Lambda} \{wew^{-1} \mid w \in W\}$.

Putcha lattices exist for irreducible algebraic monoids [57, Theorem 6.2]. The following theorem describes Putcha lattices making use of $\mathscr{R}$ relation and Borel subgroups of M with a zero.

Theorem 9 ([61, Theorem 9.3]). *Let M be an irreducible algebraic monoid with a zero and unit group G. Let B be a Borel subgroup of G containing a maximal torus T. Then*

$$\Lambda = \{e \in E(\overline{T}) \mid \text{for all } f \in E(M), \text{ if } e\mathscr{R}f \text{ then } f \in \overline{B}\}$$

is a Putcha lattice of M.

We describe Putcha lattices of classical algebraic monoids. Let e_i denote the diagonal matrix $\mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1 for $i = 0, \cdots, n$, and let E_{ij} be the matrix unit of size n whose (i, j)-entry is 1 and others are all 0. So, e_n is the identity matrix of $\mathbf{M}_n$. The Putcha lattice of $\mathbf{M}_n$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, n\}.$$

The Putcha lattice of $\mathbf{MSp}_n$ with $n = 2l$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, l, n\},$$

which is formally the Putcha lattice of $\mathbf{MSO}_n$ where $n = 2l + 1$. The Putcha lattice of $\mathbf{MSO}_n$ with $n = 2l$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, l, n\} \cup \{e_{l+1,l+1} - E_{l,l}\}.$$

2.5 Rook Monoids

Our objective here is to introduce the rook monoid and its relatives. These monoids are finite, and hence algebraic. They are vital in determining the structure of classical algebraic monoids.

2.5.1 The General Rook Monoid

A matrix of size n is a *rook matrix* if its entries are 0 or 1 and there is at most one 1 in each row and each column. Viewing each 1 as a rook, we can identify a rook matrix of rank r with an arrangement of r non-attacking rooks on an $n \times n$ chess board. Let

$$\mathbf{R}_n = \{A \in \mathbf{M}_n \mid A \text{ is a rook matrix}\}.$$

Then $\mathbf{R}_n$ is a monoid with respect to the multiplication of matrices. We call this monoid the *general rook monoid*, for short *rook monoid*. Its unit group is the *permutation group* P_n consisting of *permutation matrices* whose each row and each column have exactly one 1. The order of $\mathbf{R}_n$ is $|\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$. In particular,

$$\mathbf{R}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

A *partial injective transformation* σ of $\mathbf{n} = \{1, 2, \cdots, n\}$ is a one to one correspondence from a subset X of $\mathbf{n}$ onto a subset Y of $\mathbf{n}$. We call X the domain of σ, denoted by $I(\sigma)$, and Y the range of σ, denoted by $J(\sigma)$. Let $\mathbf{I}_n$ be the set of all injective partial transformations of $\mathbf{n}$. Then $\mathbf{I}_n$ is a monoid with respect to the composition of partial transformations, and is called *symmetric inverse semigroup*. The zero element of $\mathbf{I}_n$ is the empty function whose domain and range are the empty set. The unit group of $\mathbf{I}_n$ is the symmetric group S_n on n letters.

Let $A = (a_{ji}) \in \mathbf{R}_n$, and let $I(A)$ and $J(A)$ denote the sets of indices of nonzero columns and rows of A, respectively. Then A induces a partial injective transformation $\sigma_A : I(A) \to J(A)$ with $\sigma_A : i \mapsto j$, if $a_{ji} = 1$. It follows that the rook monoid is isomorphic to the symmetric inverse semigroup $\mathbf{I}_n$ via the isomorphism,

$$\zeta : \quad \mathbf{R}_n \to \mathbf{I}_n, \quad A \mapsto \sigma_A .$$

2.5.2 The Symplectic Rook Monoid

To introduce symplectic rook monoids we need some preparations. Define an involution θ of $\mathbf{n} = \{1, 2, \cdots, n\}$ by $\theta(i) = n+1-i$. A subset I of $\mathbf{n}$ is *admissible* if whenever $i \in I$, then $\theta(i) \notin I$. The empty set $\emptyset$ and the whole set $\mathbf{n}$ are considered admissible. A proper subset I of $\mathbf{n}$ is admissible if and only if $I \cap \theta(I) = \emptyset$ if and only if $\theta(I)$ is admissible. Write

$$\bar{i} = \theta(i).$$

Clearly $\{i, \bar{i}\}$ is not admissible. If $n = 4$, then the admissible subsets of $\mathbf{n}$ are

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{1,2\},\ \{1,3\},\ \{2,4\},\ \{3,4\},\ \{1,2,3,4\}.$$

Notice the difference of these admissible subsets from those for $n = 5$ below

$$\emptyset,\ \{1\},\ \{2\},\ \{4\},\ \{5\},\ \{1,2\},\ \{1,4\},\ \{2,5\},\ \{4,5\},\ \{1,2,3,4,5\}.$$

The centralizer C of θ in S_n consists of those elements $\sigma \in S_n$ that map any admissible subset of $\mathbf{n} = \{1, 2, \cdots, n\}$ to an admissible subset. Indeed, if $\sigma \in C$ and I is an admissible subset of $\mathbf{n}$, then for $i \in I$ we have $\overline{\sigma(i)} = \sigma(\bar{i}\,) \notin \sigma(I)$ since $\bar{i} \notin I$. Thus $\sigma(I)$ is admissible. Next, if $\sigma \in S_n$ and it maps all admissible subsets to admissible subsets, so is σ^{-1}. We show that $\theta\sigma = \sigma\theta$ by contradiction. Suppose that there is $i \in \mathbf{n}$ such that $\overline{\sigma(i)} \neq \sigma(\bar{i}\,)$. Then $\{\sigma(i), \sigma(\bar{i}\,)\}$ is admissible. But then $\sigma^{-1}\{\sigma(i), \sigma(\bar{i}\,)\} = \{i, \bar{i}\}$ is admissible, which is a contradiction.

Thus, C acts on the set of all admissible subsets of $\mathbf{n}$. From [36, Theorem 3.1.7] it follows that the orbits of this action are

$$\emptyset,\ \{1, \cdots, i\},\quad \text{where } i = 1, \cdots, l, n.$$

Next, let $n = 2l$ and W the preimage of C under ζ. Then W is a subgroup of $\mathbf{R}_n$, and is referred to as the *symplectic rook group*. A rook matrix A is *symplectic* if both $I(A)$ and $J(A)$ are proper admissible subsets of $\mathbf{n}$, or if $A \in W$.

The set of all symplectic rook matrices is a submonoid of $\mathbf{R}_n$, called the *symplectic rook monoid*, and will be denoted by $\mathbf{RSp}_n$. The unit group of $\mathbf{RSp}_n$ is W. The zero element of $\mathbf{RSp}_n$ is the zero matrix of size n.

Theorem 10 ([36, Corollary 3.1.9 and Theorem 3.1.10]; [39, Corollary 2.3]). *The symplectic rook monoid is*

$$\begin{aligned}
\mathbf{RSp}_n &= \Big\{A \in \mathbf{R}_n \;\Big|\; A = \sum_{i \in I,\, w \in W}^{n} E_{wi,i} \text{ where } I \text{ is admissible}\Big\} \\
&= \Big\{A \in \mathbf{R}_n \;\Big|\; A \text{ is singular and } I(A) \text{ and } J(A) \text{ are admissible}\Big\} \cup W \\
&\simeq \{A \in \mathbf{R}_n \mid AJA^\top = A^\top JA = 0 \text{ or } J\}.
\end{aligned}$$

where J is as in the definition of $\mathbf{MSp}_n$ *for $n = 2l$.*

2.5.3 The Even Special Orthogonal Rook Monoid

Let $n = 2l \geq 2$. An admissible subset is referred to as *r-admissible* if its cardinality is r. There are no r-admissible subsets for $r > l$ except the whole set $\mathbf{n}$. A subset I of $\mathbf{n}$ is r-admissible if and only if $\theta(I)$ is r-admissible. Let C be the centralizer of θ in S_n. Denote by C_1 the subgroup of C generated by

$$(1\bar{1})(2\bar{2}), (2\bar{2})(3\bar{3}), \cdots, (l-1\ \overline{l-1})(l\bar{l}),$$

and let

$$C_2 = \{\, \sigma \in S_n \mid \sigma \text{ stablizes } \{1, \ldots, l\} \text{ and } \sigma(\,\bar{i}\,) = \overline{\sigma(i)}\,\}.$$

Then $C' = C_1C_2$ is a subgroup of C. It follows from [32, Lemmas 5.2 and 5.4] that the orbits of the restriction to C' of the action of C on the set of all admissible subsets of $\mathbf{n}$ are

$$\emptyset,\ \{1, \cdots, l-1, l+1\}, \text{ and } \{1, \cdots, i\}, \text{ where } i = 1, \cdots, l, n.$$

An admissible subset I is called **type I** if there exists w in W such that $wI = \{1, \cdots, l-1, l\}$; **type II** if $wI = \{1, \cdots, l-1, l+1\}$. Such admissible sets contain l elements.

Let $W = \zeta^{-1}(C')$. Then W is a subgroup of $\mathbf{R}_n \cap \mathbf{SO}_n$ and is isomorphic to $(Z_2)^{l-1} \rtimes S_l$. In addition, $|W| = 2^{l-1}l!$. We call W the *even special orthogonal rook group*. A rook matrix A is *even special orthogonal* if $I(A)$ is admissible and there is $w \in C'$ such that $J(A) = w(I(A))$, or if $A \in W$. The set of all even special orthogonal rook matrices is a submonoid of $\mathbf{R}_n$, called the *even special orthogonal rook monoid*, and will be denoted by $\mathbf{RSO}_n$. The unit group of $\mathbf{RSO}_n$ is W.

Theorem 11 ([32, Corollary 5.8 and Theorem 5.9]). *The even special orthogonal rook monoid is*

$$\begin{aligned}\mathbf{RSO}_n &= \left\{ A \in \mathbf{R}_n \;\middle|\; A = \sum_{i \in I, w \in W}^{n} E_{wi,i} \text{ where } I \text{ is admissible,} \right\} \\ &= \left\{ A \in \mathbf{R}_n \;\middle|\; \begin{array}{l} A \text{ is singular, } I(A) \text{ and } J(A) \text{ are admissible} \\ \text{and of the same type if } |I(x)| = |J(x)| = l \end{array} \right\} \cup W \\ &= \{A \in \mathbf{R}_n \mid AJA^\top = A^\top JA = 0 \text{ or } J\}.\end{aligned}$$

where J is as in the definition of $\mathbf{MSO}_n$ *for $n = 2l$.*

2.5.4 The Odd Special Orthogonal Rook Monoid

Let $n = 2l+1 \geq 3$ and W the preimage of C under ζ, where C is the centralizer of θ in S_n. Then W is a subgroup of $\mathbf{R}_n$, and is referred to as the *odd special orthogonal* *rook group*. A rook matrix A is *odd special orthogonal* if both $I(A)$ and $J(A)$ are proper admissible subsets of $\mathbf{n}$, or if $A \in W$. The set of all odd special orthogonal rook matrices is a submonoid of $\mathbf{R}_n$, called the *odd special orthogonal rook monoid*, and will be denoted by $\mathbf{RSO}_n$. The unit group of $\mathbf{RSO}_n$ is W. Combining [33, Theorem 3.10] and Theorem 10, we have the following conclusion.

Theorem 12. *The odd special orthogonal rook monoid* $\mathbf{RSO}_n$ *is isomorphic to the symplectic rook monoid* $\mathbf{RSp}_{2l}$*, where* $n = 2l + 1 \geq 3$.

3 Reductive Monoids

An irreducible algebraic monoid is *reductive* if its unit group is a reductive algebraic group. The monoids $\mathbf{M}_n$ and $\mathbf{D}_n$ are reductive, but $\overline{\mathbf{B}}_n$ is not for $n \geq 2$. The classical monoids of types A_l, B_l, C_l, and D_l are all reductive. The monoid in Example 3 is not reductive if $n \geq 2$, since its unit group is not connected and so not reductive. The monoid in Example 8 is not reductive for $n \geq 1$ because the unipotent radical of its unit group is

$$\{(1, a_1, \cdots, a_n) \mid a_i \in K \text{ for } i = 1, \cdots, n\}.$$

Reductive monoids are central to the theory of algebraic monoids; regular semigroups form an eminent class in semigroup theory. At a glance, reductive monoids have nothing to do with regular semigroups. But, the two notions are connected very closely. The following result is a summary of [57, Theorem 2.11], [58, Theorem 2.4], [59, Theorem 1.1], [74, Theorem 4.4.15], and [76, Theorem 3.1].

Theorem 13. *Every reductive algebraic monoid is regular. Moreover, an irreducible algebraic monoid with a zero is reductive if and only if it is regular.*

It follows from Theorems 8 and 13 that every reductive monoid is unit regular. A complete description of the reductivity of an irreducible algebraic monoid is given in [19].

Theorem 14 ([19, Theorem 2.1]). *Suppose that* M *is an irreducible algebraic monoid. Then* M *is reductive if and only if* M *is regular and the semigroup kernel of* M *is a reductive group.*

Reductive monoids are regular and unit dense monoids, which are distinguished from irreducible algebraic monoids in that they have finite number of $G \times G$ orbits, and each $G \times G$ orbit contains an idempotent. This, however, is not the case for all irreducible algebraic monoids. If $n > 1$, then the monoid in Example 8, again, is not reductive since it has infinitely many $G \times G$ orbits, but only two orbits $\{0\}$ and G have idempotents 0 and 1, respectively.

3.1 Jordan Decomposition

Every element x in an algebraic group G has its *Jordan decomposition*

$$x = su = us$$

where s is semisimple (diagonalizable) and u is unipotent (sole eigenvalue 1). This decomposition is unique. Is there an analogue of such decomposition in algebraic monoids? Putcha [71] shows that each element in a reductive monoid M is a product of a semisimple element and a quasi-unipotent element.

The unit group H_e of eMe for $e \in E(M)$ is an algebraic group. If $a \in M$, it follows from Theorem 3 that there is a positive integer k such that $a^k \in H_e$ for some $e \in E(M)$. Such e is uniquely determined by a, since if $a^k \in H_f$ for some $f \in E(M)$ then $e\mathscr{H}f$ and hence $e = ef = f$. By [27, Corollary 1] we have $ae = ea \in H_e$. The element ae is called the *invertible part* of a. An element $a \in M$ is *completely regular* if $a \in H_e$ for some $e \in E(M)$, and H_e is called the *bubble group* of a. Clearly, for any $a \in M$, the invertible part of a is always completely regular.

An element $s \in M$ is *semisimple* if s is completely regular and is semisimple in its bubble group. If $s \in M = \mathbf{M}_n$ is semisimple then s is diagonalizable. The set of all semisimple elements of M is denoted by M_s. An element u of M is *quasi-unipotent* if its invertible part ue is unipotent in its bubble group H_e. If M has a zero, then every nilpotent element is quasi-unipotent. The set of all quasi-unipotent elements will be denoted by M_u. Then

$$M_s \cap M_u = E(M).$$

If M is a closed monoid of $\mathbf{M}_n$, then M_u is the zero set of the polynomial $X^n(X - I)^n$. Renner studies the conjugacy classes of semisimple elements in algebraic monoids [78]; Winter investigates quasi-unipotent elements in a different name in [92]. Theorem 15 below shows that Jordan decomposition exists for reductive monoids.

Theorem 15 ([71, Theorem 2.2]). *Let M be a reductive monoid and $a \in M$. Then $a = su = us$ for some invertible semisimple element s and quasi-unipotent element u.*

Such decomposition is not unique. For example (cf. Example 2.3 of [71]), in $\mathbf{M}_2$, for any $b \in K$,

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & b/\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b/\alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

where $\alpha \in K^*$.

Our next objective is to study the structure of a reductive monoid M in terms of root semigroups, which are analogues of root groups U_α for reductive groups. We fix notation. Let G be the unit group of M, let T be a maximal torus of G and B a Borel subgroup containing T, and let Φ be the roots of G relative to T. Denote by B^- the unique Borel subgroup such that $B \cap B^- = T$. Then G is generated by the root groups U_α along with T where $\alpha \in \Phi$ [24, Theorem 26.3 d)].

Is there a monoid analogue of this result for M? Putcha confirms this matter in [71]. The key is to find a monoid analogue $\widetilde{U_\alpha}$ of the one-dimensional root subgroup U_α associated with a root $\alpha \in \Phi$. Let $\widetilde{U_\alpha} = (\overline{TU_\alpha})_u$, the set of quasi-unipotent elements of $\overline{TU_\alpha}$. Then $\widetilde{U_\alpha}$ is referred to as the *root semigroup* associated with α. It is easy to see that $U_\alpha \subseteq \widetilde{U_\alpha}$. Denote by $\tilde{U}$ the set of quasi-unipotents of $\overline{B}$. Imbedding M into $\mathbf{M}_n$ in such a way that every element of B is upper triangular and every element of B^- is lower triangular, we can define a map $\phi : \overline{B} \to \overline{T}$ such that $\phi(b)$ is the diagonal matrix of the diagonal of $b \in \overline{B}$. Then ϕ is an epimorphism and $\phi|_{\overline{T}}$ is the identity.

Theorem 16 ([71, Theorem 2.6 and Corollary 4.4]). *Let M be a reductive monoid and let Φ^+ be the set of positive roots. Then*

(1) *$\tilde{U}$ is an algebraic monoid and equal to $\phi^{-1}(E(\overline{T}))$.*
(2) *$\overline{B}$ is generated by T and $\widetilde{U_\alpha}$ for $\alpha \in \Phi^+$, and $\overline{B} = T\tilde{U} = \tilde{U}T$.*
(3) *$M_u = \bigcup_{g \in G} g\tilde{U}g^{-1}$.*

The following corollary is from [61, Proposition 6.3] and Theorem 16.

Corollary 3. *Let M be a reductive monoid, T a maximal torus of the unit group of M, and Φ the set of roots relative to T. Then M is generated by T and $\tilde{U}_\alpha$, $\alpha \in \Phi$.*

3.2 Parabolic Subgroups

The aim here is to describe parabolic subgroups of G in terms of idempotents of a reductive monoid M. When M has a zero, these subgroups are completely determined by the chains in $E(M)$ [57, 60]. Recall that a chain of idempotents is a linearly ordered subset $\Gamma = \{e_0 < e_1 < e_2 < \cdots < e_k\}$ of the poset $E(M)$. In view of [61, Corollary 6.10], every chain of idempotents is contained in a maximal torus T of G. If $\Gamma \subseteq E(M)$, define the *left centralizer* and the *right centralizer* of Γ by

$$P(\Gamma) = \{x \in G \mid xe = exe\} \quad \text{and} \quad P^-(\Gamma) = \{x \in G \mid ex = exe\}.$$

As Brion did in [3], we switched Putcha's notation for left and right centralizers to comply with standard conventions in algebraic geometry and algebraic groups. The *centralizer* of Γ is by definition

$$C_G(\Gamma) = \{x \in G \mid xe = ex\}.$$

More information on local structures such as stabilizers, centralizers, and kernels of algebraic monoids can be found in [3, 20, 22, 61, 70, 82].

Theorem 17 ([57, Theorem 4.6]; [60, Theorem 2.7]). *Let M be a reductive monoid and let Γ be a chain in $E(M)$. Then $P(\Gamma)$ and $P^{-}(\Gamma)$ are a pair of opposite parabolic subgroups with common Levi factor $C_G(\Gamma)$. Furthermore, if M has a zero, then every parabolic subgroup P of G is of the form $P = P(\Gamma)$ for some chain $\Gamma \subseteq \Lambda$, where Λ is a Putcha lattice of M.*

When the chain Γ in the above theorem is maximal, its left and right centralizers are Borel subgroups as described below.

Theorem 18 ([57, Theorem 4.5]; [61, Theorem 7.1]). *Let M be a reductive monoid with a zero and let Γ be a maximal chain of $E(M)$. Then*

(1) $P(\Gamma)$ is a Borel subgroup of G whose opposite Borel subgroup is $P^{-}(\Gamma)$. Moreover, every Borel subgroup of G can be obtained this way.
(2) $C_G(\Gamma)$ is a maximal torus of G and every maximal torus of G is obtainable in this manner.

The set of Borel subgroups containing a maximal torus T is in one to one correspondence with the set of Putcha lattices in $E(\overline{T})$.

Theorem 19 ([60, Lemma 1.1]; [61, Theorem 7.1]). *Let M be a reductive algebraic monoid with a zero and unit group G. Let B be a Borel subgroup of G containing a maximal torus T. Then*

$$\Lambda(B) = \{e \in E(\overline{T}) \mid Be = eBe\}$$

is a Putcha lattice of M. Moreover, the map $B \mapsto \Lambda(B)$ is a bijection from the set of all Borel subgroups containing T onto the set of Putcha lattices in $E(\overline{T})$.

3.3 The Type Map

Let M be a reductive monoid with unit group G and let $W = N_G(T)/T$ be the Weyl group. Denote by Δ the set of simple roots relative to T and B, and by $S = \{s_\alpha \mid \alpha \in \Delta\}$ the set of simple reflections that generate the Weyl group. Let Λ be the cross-section lattice of M.

Definition 4. The type map of M is defined by

$$\lambda : \Lambda \to 2^{\Delta}; \quad \lambda(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha\}.$$

As Renner mentions in his book [82], the type map is the most important combinatorial invariant in the structure theory of reductive monoids. In some sense, it is the monoid analogue of the Coxeter-Dynkin diagram. Especially, for $\mathscr{J}$-irreducible monoids, Putcha and Renner [72] give a very precise recipe to completely determine the type map using the Coxeter-Dynkin diagram associated with the monoids. We consider the type maps of classical algebraic monoids, and refer the reader to [34, 35, 72, 82] for further details about type maps of reductive monoids.

Example 9. The type maps of $\mathbf{M}_n$ with $n = l + 1$, $\mathbf{MSp}_n$ with $n = 2l$, and $\mathbf{MSO}_n$ with $n = 2l + 1$. Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of type A_l, B_l and C_l, and let $e_i = \text{diag }(1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1, for $i = 1, \cdots, l$. Then $\Lambda = \{0, e_1, \cdots, e_l, 1\}$, and the type map is determined by $\lambda(0) = \lambda(1) = \Delta$, $\lambda(e_1) = \{\alpha_2, \cdots, \alpha_l\}$, and for $2 \le i \le l$,

$$\lambda(e_i) = \{\alpha_1, \cdots, \alpha_{i-1}\} \cup \{\alpha_{i+1}, \cdots, \alpha_l\}.$$

Example 10. The type map of $\mathbf{MSO}_n$ with $n = 2l$. Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of SO_n, and let $e_i = \text{diag }(1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1, for $i = 1, \cdots, l$. Let $e_l' = \text{diag }(1, \cdots, 1, 0, 1, \cdots, 0)$ with l-copies of 1. Then $\Lambda = \{0, e_1, \cdots, e_l, e_l', 1\}$, and the type map is determined by $\lambda(0) = \lambda(1) = \Delta$, $\lambda(e_1) = \{\alpha_2, \cdots, \alpha_l\}$, $\lambda(e_{l-1}) = \{\alpha_1, \cdots, \alpha_{l-2}\}$, $\lambda(e_l) = \{\alpha_1, \cdots, \alpha_{l-1}\}$, $\lambda(e_l') = \{\alpha_1, \cdots, \alpha_{l-2}, \alpha_l\}$, and for $2 \le i \le l - 2$ with $l \ge 4$,

$$\lambda(e_i) = \{\alpha_1, \cdots, \alpha_{i-1}\} \cup \{\alpha_{i+1}, \cdots, \alpha_l\}.$$

In general, associated with the type map of a reductive monoid are some parabolic subgroups of the Weyl group. Let $\lambda^*(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha \ne e\}$ and $\lambda_*(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha = e\}$. Then $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$. Denote by $W(e) = W_{\lambda(e)}$, $W^*(e) = W_{\lambda^*(e)}$ and $W_*(e) = W_{\lambda_*(e)}$ the parabolic subgroups of W associated with $\lambda(e), \lambda^*(e)$ and $\lambda_*(e)$, respectively. These subgroups are useful in determining the orders, conjugacy classes, and representations of Renner monoids [37, 40, 41]. Descriptions and applications of these subgroups can be found in the books [61, 82] and the references there.

Proposition 2. *Let e be an element of the Putcha lattice of a reductive monoid M. Then*

(1) $W(e) = \{w \in W \mid we = ew\}$
(2) $W^*(e) = \cap_{f \ge e} W(f)$.
(3) $W_*(e) = \cap_{f \le e} W(f) = \{w \in W \mid we = ew = e\}$.
(4) $W(e) = W^*(e) \times W_*(e)$.

3.4 $\mathscr{J}$-Irreducible Monoids

Renner introduces the concept of $\mathscr{J}$-irreducible monoids in his work on the classification of semisimple algebraic monoids in [75]. A reductive monoid M with a zero is *$\mathscr{J}$-irreducible* if its Putcha lattice has a unique minimal nonzero idempotent. A reductive monoid M with a zero and unit group G is *semisimple* if the dimension of the center $C(G)$ is one. In view of [75, Lemma 8.3.2], each $\mathscr{J}$-irreducible algebraic monoid is semisimple. The classical monoids defined in Sect. 2.1 are $\mathscr{J}$-irreducible and hence semisimple. The following results give alterative descriptions of $\mathscr{J}$-irreducible monoids.

Theorem 20 ([61, Corollary 15.3]; [75, Corollary 8.3.3]). *Let M be a reductive monoid with a zero and let W be the Weyl group of the unit group of M. Then the following are equivalent.*

(a) M is $\mathscr{J}$-irreducible.
(b) W acts transitively on the set of minimal nonzero idempotents of $E(\overline{T})$.
(c) There is an irreducible rational representation $\rho : M \to \mathbf{M}_n$ which is finite as a morphism of algebraic varieties.

Our intention next is to confirm that all $\mathscr{J}$-irreducible algebraic monoids can be obtained, up to finite morphism, from irreducible representations of semisimple algebraic groups. This is a known result given in Renner [83].

Theorem 21. *Let G be a semisimple algebraic group and ρ be an irreducible rational representation of G. Then $M(\rho) = \overline{K^*\rho(G)}$ is a $\mathscr{J}$-irreducible algebraic monoid. Furthermore, one can construct, up to finite morphism, all $\mathscr{J}$-irreducible algebraic monoids from irreducible representations of a semisimple algebraic group.*

Recall that $M(\rho)$ is the Zariski closure of $K^*\rho(G)$. Suppose that ρ is an irreducible representation of a semisimple group G, then the inclusion map $M(\rho) \to \mathbf{M}_n$ is a faithful representation of $M(\rho)$. Thus $M(\rho)$ is $\mathscr{J}$-irreducible. Now suppose that M is $\mathscr{J}$-irreducible and let H be the unit group of M. Then $M = \overline{H}$ since M is irreducible. The radical $R(H)$ of H is the identity component of the center $C(H)$ of H, and $\dim R(H) = 1$, since $C(H)$ is one dimensional. Thanks to [87, Proposition 6.15] and [61, Theorem 4.32], we have $H = R(H)G$ where G is the semisimple commutator group of H. By [61, Corollary 10.13], there exists a finite morphism $\rho : M \to \mathbf{M}_n$ of algebraic varieties such that $\rho(R(H)) = K^*$. We obtain that $\rho(H) = K^*\rho(G)$, and hence $\rho(M) = \rho(\overline{H}) \subseteq \overline{\rho(H)} = \overline{K^*\rho(G)}$. On the other hand, it is clear that $\overline{K^*\rho(G)} \subseteq \rho(M)$. Therefore, $\rho(M) = \overline{K^*\rho(G)}$ is $\mathscr{J}$-irreducible.

The Putcha lattice of a $\mathscr{J}$-irreducible monoid is completely determined by its type $J_0 = \lambda(e_0)$ where e_0 is the unique nonzero minimal element of Λ. Putcha and Renner determine the Putcha lattices of $\mathscr{J}$-irreducible monoids associated with an arbitrary dominant weight by using the following theorem, which is a summary of [72, Corollary 4.11 and Theorem 4.16].

Theorem 22. *Let M be a $\mathscr{J}$-irreducible monoid associated with a dominant weight μ and $J_0 = \{\alpha \in \Delta \mid \langle \mu, \alpha \rangle = 0\}$ where $\langle\ ,\ \rangle$ is defined as in ([23], p42). Then*

(1) $\lambda^*(\Lambda \setminus \{0\}) = \{X \subseteq \Delta \mid X$ has no connected component lying in $J_0\}$.
(2) $\lambda_*(e) = \{\alpha \in J_0 \setminus \lambda^*(e) \mid s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)\}$, for $e \in \Lambda \setminus \{0\}$.

4 Renner Monoids

The Bruhat decomposition and Tits system are among the gems in the structure theory of reductive algebraic groups G. This makes it possible to reduce many questions about G to questions about the Weyl group. Renner [77, 80] finds an analogue of such decomposition for reductive algebraic monoids with many useful consequences, resulting in the Bruhat-Renner decomposition. This decomposition is now central in the structure theory of reductive monoids.

Let M be a reductive monoid with unit group G, $B \subseteq G$ a Borel subgroup, and $T \subseteq B$ a maximal torus of G. Denote by N the normalizer of T in G and $\overline{N}$ the Zariski closure of N in M. Thus $\overline{N}$ is an algebraic monoid and has N as its unit group, and $\overline{T}$ is an algebraic monoid with unit group T. The Weyl group $W = N/T$ is a finite reflection group.

Recall that an *inverse monoid* is a monoid M such that for $a \in M$, there is a unique $b \in M$ that satisfies $a = aba$ and $b = bab$. A regular monoid with commutative idempotents is an inverse monoid. An irreducible regular monoid M is inverse if and only if M have finitely many idempotents. In particular, by [18, Theorem 3.1] a regular irreducible algebraic monoid with nilpotent unit group is an inverse monoid.

Lemma 2 ([61, Proposition 11.1]; [77, Proposition 3.2.1]). *$\overline{N} = N\overline{T}$ is a unit regular inverse monoid with unit group N and idempotent set $E(\overline{T})$. Furthermore, $\overline{N} = NE(\overline{T})$.*

To show that $\overline{N} = N\overline{T}$, note that W is finite. Let $k = |W|$. Then there exists $y_i \in N$ such that $N = \bigcup_{i=1}^{k} y_i T$. It follows from Lemma 1 that

$$\overline{N} = \cup_{i=1}^{k} \overline{y_i T} = \cup_{i=1}^{k} y_i \overline{T} \subseteq N\overline{T}.$$

By Corollary 2, the unit group of $\overline{N}$ is N.

Next, we show that an idempotent of $\overline{N}$ is in $E(\overline{T})$. Let $x \in \overline{N}$. Then $x \in y\overline{T}$ for some $y \in N$. Since $yT = Ty$, we obtain that $y\overline{T} = \overline{T}y$ by Lemma 1. As $y^k \in T$, we have

$$x^k \in (y\overline{T})^k = y^k(\overline{T})^k \subseteq T\overline{T} \subseteq \overline{T}.$$

If x is an idempotent in $\overline{N}$, then $x = x^2 = x^k \in \overline{T}$, that is, $x \in E(\overline{T})$.

Finally, in view of $\overline{T} = TE(\overline{T})$, we have $\overline{N} = NE(\overline{T})$, which shows that $\overline{N}$ is unit regular. Since $E(\overline{T})$ is commutative, $\overline{N}$ is an inverse monoid.

Lemma 3. *Let $\sim$ be the relation on $\overline{N}$ given by*

$$x \sim y \text{ if and only if } x \in yT.$$

Then $\sim$ is a congruence, and the quotient set $R = \overline{N}/\sim$ is a monoid.

It is straightforward that $\sim$ is an equivalent relation. It suffices to show that for $x, y, u, v \in \overline{N}$, if $x \sim y$ and $u \sim v$ then $xu \sim yv$. Assume that $x = yt_1$, $u = vt_2$ where $t_1, t_2 \in T$. Then $xu = yt_1vt_2$. By Lemma 2, we have $v = nt_3$ for some $n \in N$ and $t_3 \in \overline{T}$. Then

$$xu = yt_1nt_3t_2 = yn(n^{-1}t_1n)t_3t_2.$$

But $n^{-1}t_1n \in T$. Hence

$$xu = ynt_3(n^{-1}t_1n)t_2 = yv(n^{-1}t_1n)t_2 \in yvT.$$

Therefore, $xu \sim yv$. Write $R = \overline{N}/T$.

Definition 5. The monoid R is called the *Renner monoid* of M, and an element of R is called a Renner element.

4.1 Classical Renner Monoids

The Renner monoids of classical algebraic monoids are called *classical Renner monoids*. More specifically, the Renner monoids of the general, symplectic, and special orthogonal algebraic monoids are referred to as *general, symplectic* and *special orthogonal* Renner monoids, respectively. We describe these monoids below.

Example 11. The **general Renner monoid.** In this case $M = \mathbf{M}_n$. Then $T = \mathbf{T}_n$ and $\overline{N}$ consists of matrices with *at most* one nonzero entry in each row and each column. The unit group of $\overline{N}$ comprises matrices which have *exactly* one nonzero entry in each row and each column. Let E_{ji} for $1 \leq i, j \leq n$ be the matrix units whose (j, i) entry is 1 and the rest are all 0. Thus

$$\overline{N} = \{\Sigma_{i=1}^{n} t_i E_{\sigma i,i} \mid t_i \in K \text{ and } \sigma \in S_n\}$$

and

$$N = \{\Sigma_{i=1}^{n} t_i E_{\sigma i,i} \mid t_i \in K^* \text{ and } \sigma \in S_n\}.$$

The map $\sum_{i=1}^{n} t_i E_{\sigma i,i} \mapsto \sum_{i=1}^{n} b_i E_{\sigma i,i}$ is an epimorphism from $\overline{N}$ onto $\mathbf{R}_n$ with kernel T, where $b_i = 0$ if $t_i = 0$, and $b_i = 1$ if $t_i \neq 0$. Thus we have proved the following result.

Theorem 23 ([77, Section 7]). *The general Renner monoid $R = \overline{N}/T$ is isomorphic to the general rook monoid $\mathbf{R}_n$, and its unit group is isomorphic to the symmetric group S_n. The order of R is $|\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$.*

In what follows we identify the general Renner monoid with the general rook monoid $\mathbf{R}_n$.

Example 12. The **symplectic Renner monoid.** Here $M = \mathbf{MSp}_n$ where $n = 2l$ and $l \geq 1$. Recall that

$$\mathbf{MSp}_n = \bigsqcup_{c \in K} M_c$$

where $M_c = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ\}$ with $J = \begin{pmatrix} 0 & J_l \\ -J_l & 0 \end{pmatrix}$ and $J_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. We are led to the following map

$$\chi : \mathbf{MSp}_n \to K, \quad A \mapsto c \quad \text{if} \quad A \in M_c.$$

By [12], $T = \{t = \sum t_i E_{i\,i} \mid t_i \in K^* \text{ and } t_i t_{\bar{i}} = \chi(t)\}$. From [7] it follows that $\overline{N} = N \cup N'$ in which

$$N = \left\{ \omega = \sum_{i=1}^{n} t_i E_{\sigma i,i} \mid \sigma \in C, t_i \in K^* \text{ and } t_i t_{\bar{i}} = \varepsilon_i \varepsilon_{\sigma i} \chi(\omega) \right\},$$

where C is as in Sect. 2.5.2, and N' consists of matrices of the form

$$\omega' = \sum_{i=1}^{l} a_i E_{j_i,k_i}$$

where $a_i \in K$, $1 \leq i \leq l$, and $\{j_1, \cdots, j_l\}$ and $\{k_1, \cdots, k_l\}$ are admissible. The map of $\overline{N}$ onto the symplectic rook monoid $\mathbf{RSp}_n$, defined by

$$\omega = \sum_{i=1}^{n} t_i E_{\sigma i,i} \mapsto \sum_{i=1}^{n} E_{\sigma i,i} \text{ with } \sigma \in C, \text{ and}$$

$$\omega' = \sum_{i=1}^{l} a_i E_{j_i,k_i} \mapsto \sum_{i=1}^{l} b_i E_{j_i,k_i},$$

where $b_i = 0$ if $a_i = 0$, and $b_i = 1$ if $a_i \neq 0$, is a homomorphism of monoids with kernel T. We conclude:

Theorem 24 ([7, Proposition 2.3]; [36, Corollary 3.1.9]). *The symplectic Renner monoid $R = \overline{N}/T$ is isomorphic to the symplectic rook monoid $\mathbf{RSp}_n$. Its unit group is isomorphic to the symplectic rook group. The order of R is*

$$|\mathbf{RSp}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + 2^l l!.$$

In what follows we identify the symplectic Renner monoid with the symplectic rook monoid, and denote them by $\mathbf{RSp}_n$.

Example 13. The **odd special orthogonal Renner monoid** is the Renner monoid of the odd special orthogonal algebraic monoid $\mathbf{MSO}_n$ with $n = 2l + 1 \geq 3$. A similar discussion to that of Example 12 gives rise to the following result.

Theorem 25 ([13, Theorem 4.2]; [33, Corollary 3.12]). *The odd special orthogonal Renner monoid R is isomorphic to the odd special orthogonal rook monoid $\mathbf{RSO}_n$ where $n = 2l + 1$; its unit group is isomorphic to the odd special orthogonal rook group. The order of R is*

$$|\mathbf{RSO}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + 2^l l!.$$

Example 14. The **even special orthogonal Renner monoid** is the Renner monoid of $\mathbf{MSO}_n$ where $n = 2l$ with $l \geq 1$. Recall that

$$\mathbf{MSO}_n = \bigsqcup_{c \in K} M_c,$$

where $M_c = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ\}$ with $J = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$. We have the following homomorphism of algebraic monoids

$$\chi : \mathbf{MSp}_n \to K, \quad A \mapsto c \quad \text{if} \quad A \in M_c.$$

By [12], $T = \{t = \sum t_i E_{i\,i} \mid t_i \in K^* \text{ and } t_i t_{\bar{i}} = \chi(t)\}$. From [13] we obtain that

$$\overline{N} = \bigcup_{\sigma \in A_n} M_\sigma,$$

where $M_\sigma = \bigcup_{c \in K} \{a_i E_{i,\sigma i} \mid a_i \in K \text{ and } a_i a_{\bar{i}} = c\}$ and A_n is the *alternating group* on n letters. We have the result below.

Theorem 26 ([13, Theorem 4.4]; [32, Corollary 5.12]). *The even special orthogonal Renner monoid R is isomorphic to the even special orthogonal rook monoid $\mathbf{RSO}_n$; its unit group is isomorphic to the even special orthogonal rook group. The order of R is*

$$|\mathbf{RSO}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + (1 - 2^l)2^{l-1} l!$$

We will not distinguish the even special orthogonal Renner monoid from the even special orthogonal rook monoid, and will use $\mathbf{RSO}_n$ to denote them.

4.2 Basic Properties

Now return to the general theory of a Renner monoid R. Summarizing some primary properties of R from [77], we first describe the unit group, the idempotent set $E(R)$, relations with Putcha lattices, and the Bruhat-Renner decomposition.

Proposition 3 ([77, Proposition 3.2.1, Theorem 5.7 and Corollary 5.8]). *Let M be a reductive monoid with unit group G. Let $T \subseteq G$ be a maximal torus and $\Lambda \subseteq E(\overline{T})$ be a Putcha lattice. Then*

(1) R is a finite inverse monoid.
(2) The unit group of R is the Weyl group W, and $R = WE(R)$. So R is unit regular.
(3) The idempotent set $E(R) \cong E(\overline{T}) = \bigcup_{w \in W} w\Lambda w^{-1}$.
(4) $R = \bigsqcup_{e \in \Lambda} WeW$, and $WeW = WfW \Rightarrow e = f$.
(5) $M = \bigsqcup_{\sigma \in R} B\sigma B$, and $B\sigma B = B\tau B \Rightarrow \sigma = \tau$.
(6) If $s \in S$ is a Coxeter generator then $BsB \cdot B\sigma B \subseteq Bs\sigma B \cup B\sigma B$.

We observe from (1) and (2) of Proposition 3 that Renner monoids form a special class of inverse monoids and they are closely connected to the Weyl group, indicating that Renner monoids are by themselves extremely important discrete invariants for reductive monoids. The results (3) and (4) of Proposition 3 show that R is a disjoint union of $W \times W$ double cosets with a Putcha lattice as its index set, and that the idempotent set $E(\overline{T})$ of R is completely determined by the conjugation action of W on the Putcha lattice. From (5) and (6), the Renner monoid plays the same role for reductive monoids that the Weyl group does for reductive groups. Many questions about M may be reduced to questions about R.

The idempotent set $E(\overline{T})$ is closely connected to convex geometry and torus embeddings. We characterize this connection in Proposition 4. Solomon [85] elaborates on the connection in detail by using many interesting examples. Putcha and Renner [55, 61, 75] have more conclusions and further examples. The theory of torus embeddings can be found in [26].

Proposition 4 ([26, Theorem 2]; [56, Theorem 3.6]; [61, Theorem 8.7]). *Let M be a reductive monoid with unit group G. Suppose that T is a maximal torus of G. Then there is a rational convex polytope whose face lattice is isomorphic to $E(\overline{T})$.*

4.3 Standard Form

Let $D(e)$ be the set of minimal length representatives of left cosets $wW(e)$ and $D_*(e)$ be the set of minimal length representatives of left cosets $wW_*(e)$ where $e \in \Lambda$. Then $D(e)^{-1} = \{u^{-1} \mid u \in D(e)\}$ is the set of minimal length representatives of right cosets $W(e)w$. Now $R = W\Lambda W$ with $W = \bigsqcup_{e \in \Lambda} D_*(e)W_*(e)$ and $W = \bigsqcup_{e \in \Lambda} W(e)D(e)^{-1}$. Each element $\sigma \in R$ can be uniquely written as

$$\sigma = xey, \quad x \in D_*(e),\ e \in \Lambda, \text{ and } y \in D(e)^{-1}. \tag{1}$$

We call (1) the *standard form* of the Renner element σ.

The standard form of Renner elements is useful to determine R^+, the index set of the decomposition of $\overline{B}$ into double cosets $B\sigma B$, that is,

$$\overline{B} = \bigsqcup_{\sigma \in R^+} B\sigma B.$$

The set R^+ is a submonoid of R, and by [70]

$$R^+ = \{\sigma \in R \mid \sigma = xey \text{ with } x \leq y^{-1}\}.$$

If R is the general rook monoid, then R^+ consists of upper triangular rook matrices.

The standard form of Renner elements plays a role in describing parabolic submonoids obtained by taking the Zariski closures of parabolic subgroups of G. Let S be the set of simple reflections that generate the Weyl group W. For $I \subseteq S$, denote by W_I the subgroup of W generated by I, and call $P_I = BW_IB$ and $P_I^- = B^-W_IB^-$ opposite parabolic subgroups of G with common Levi factors $L_I = P_I \cap P_I^-$. Define

$$R_I^+ = \{xey \mid e \in \Lambda,\ x \in W,\ y \in D(e)^{-1},\ ux \leq y^{-1} \text{ for some } u \in W_I\},$$

$$R_I^- = \{xey \mid e \in \Lambda,\ x \in D(e),\ y \in W,\ yu \leq x^{-1} \text{ for some } u \in W_I\},$$

$$\Lambda_I^+ = \{xex^{-1} \mid e \in \Lambda,\ x \in D_I^{-1}\} = \{xex^{-1} \mid e \in \Lambda,\ x \in D_I^{-1} \cap D(e)\}.$$

Theorem 27 ([70, Theorem 2.3]). *Let I be a subset of S. Then*

(1) R_I^+ and R_I^- are submonoids of R.
(2) $\overline{P_I} = BR_I^+B$ and $\overline{P_I^-} = B^-R_I^+B^-$. In particular, $\overline{P_S} = \overline{P_S^-} = M$.
(3) $\overline{L_I} = \overline{P_I} \cap \overline{P_I^-} = L_I\Lambda_IL_I$ is a reductive group.
(4) Λ_I is the Putcha lattice of $\overline{L_I}$ and $R_I = R_I^+ \cap R_I^- = W_I\Lambda_IW_I$ is the Renner monoid of $\overline{L_I}$.

The standard form of elements in R can be used to describe the Bruhat-Chevalley order on R.

Definition 6. Let $\sigma, \tau \in R$. We say that $\sigma \leq \tau$ if $B\sigma B \subseteq \overline{B\tau B}$.

The Renner monoid is a poset with this partial order. This poset is characterized in Theorem 28 below using $(\Lambda, \leq)$ and $(W, \leq)$, where

$$e \leq f \text{ in } \Lambda \quad \Leftrightarrow \quad fe = e = ef,$$

and

$$u \leq v \text{ in } W \quad \Leftrightarrow \quad BuB \subseteq \overline{BvB}.$$

Theorem 28 ([50, Corollary 1.5]; [82, Corollary 8.35]). *Let $\sigma = xey$ and $\tau = ufv$ be in standard form. Then $\sigma \le \tau$ if and only if $e \le f$ and there is $w \in W(f)W_*(e)$ such that $x \le uw$ and $w^{-1}v \le y$.*

4.4 Reduced Row Echelon Form

Any matrix A over K may be changed to a matrix in *reduced row echelon form* by the Gauss-Jordan procedure, a finite sequence of elementary row operations. The set of all reduced row echelon forms of matrices in $\mathbf{M}_n$ is the set of well chosen representatives of the orbits of the left multiplication action of $\mathbf{GL}_n$ on $\mathbf{M}_n$. Row reduced echelon form in linear algebra can be generalized to any reductive monoid M with unit group G. This generalization [77] solves the orbit classification problem of the left multiplication action of G on M

$$G \times M \to M$$
$$(g, x) \mapsto gx.$$

We wish to describe the Gauss-Jordan elements of M. We begin by defining the Gauss-Jordan elements of R. The set

$$\mathscr{GJ} = \{\sigma \in R \mid B\sigma \subseteq \sigma B\}$$

is called the set of Gauss-Jordan elements of R. The Gauss-Jordan elements of R are useful to index the orbits in the conjugacy decomposition of M [69, 90]. Putcha gives a description of $\mathscr{GJ}$ using the standard form of Renner elements.

Theorem 29 ([70, Lemma 3.1]). $\mathscr{GJ} = \{ey \in R \mid e \in \Lambda,\ y \in D(e)^{-1}\}$.

The set $\mathscr{GJ}$ is a poset with respect to the following partial order

$$\sigma \le \tau \quad \text{if and only if} \quad \sigma B \subseteq \overline{\tau B}.$$

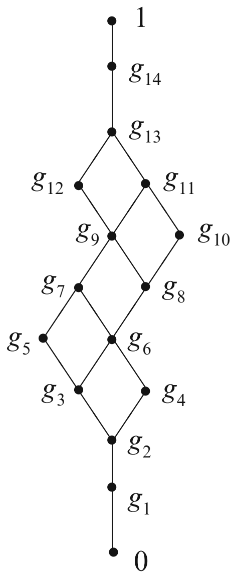
Combining [77, Theorem 9.6] and [82, Proposition 8.9], we conclude that the Renner monoid is the product of its unit group and the Gauss-Jordan elements, and that for each $\sigma \in R$ the orbit $W\sigma$ intersects the set of Gauss-Jordan elements at exactly one element g_σ. On the other hand, the orbit $W\sigma$ contains exactly one idempotent $e_\sigma \in E(\overline{T})$ (cf. [40, Lemma 3.2]). We obtain a one to one correspondence between $\mathscr{GJ}$ and $E(\overline{T})$

$$g_\sigma \mapsto e_\sigma.$$

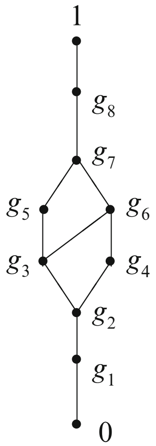
The Gauss-Jordan elements of $\mathbf{R}_n$ are the usual reduced row echelon form. If $n = 4$, we have

$$\mathscr{GJ} = \{0,\ g_1,\ \cdots,\ g_{14},\ 1\}$$

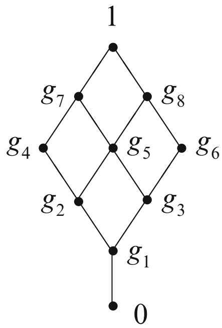
where $g_1 = E_{14}$, $g_2 = E_{13}$, $g_3 = E_{12}$, $g_4 = E_{13} + E_{24}$, $g_5 = E_{11}$, $g_6 = E_{12} + E_{24}$, $g_7 = E_{11} + E_{24}$, $g_8 = E_{12} + E_{23}$, $g_9 = E_{11} + E_{23}$, $g_{10} = E_{12}+E_{23}+E_{34}$, $g_{11} = E_{11}+E_{23}+E_{34}$, $g_{12} = E_{11}+E_{22}$, $g_{13} = E_{11}+E_{22}+E_{34}$, $g_{14} = E_{11} + E_{22} + E_{33}$. The poset structure of these elements is shown in the first figure below. The idempotent e_i corresponding to g_i can be obtained by positioning the 1 in each column of g_i to the diagonal for $1 \le i \le 14$.



$(\mathscr{GJ}, \le)$ of R_4



$(\mathscr{GJ}, \le)$ of RSp_4



$(\mathscr{GJ}, \le)$ of RSO_4

The Gauss-Jordan elements of the symplectic Renner monoid $\mathbf{RSp}_n$ with $n = 2l$ are the usual reduced row echelon form. There are, however, no reduced row echelon form of symplectic matrices of rank i for $l < i < n$. The Hasse diagram for poset $(\mathscr{GJ}, \le)$ of RSp_4 is given in the middle above. Note that $B_0 = \mathbf{B}_n \cap \mathbf{Sp}_n$ is a Borel subgroup of $\mathbf{Sp}_n$, and $B = K^* B_0$ is a Borel subgroup of the unit group of $\mathbf{MSp}_n$. If $n = 4$, then B_0 consists of the invertible upper triangular matrices

$$\begin{pmatrix} a & c & d & e \\ & b & f & \frac{bd-cf}{a} \\ & & \frac{1}{b} & -\frac{c}{ab} \\ & & & \frac{1}{a} \end{pmatrix}$$

where $a, b \in K^*$ and $c, d, e, f \in K$. A simple calculation yields that

$$\mathscr{GJ} = \{0, g_1, \dots, g_8, 1\}$$

where $g_0 = 0$, $g_1 = E_{14}$, $g_2 = E_{13}$, $g_3 = E_{12}$, $g_4 = E_{13} + E_{24}$, $g_5 = E_{11}$, $g_6 = E_{12} + E_{24}$, $g_7 = E_{11} + E_{23}$, and $g_8 = E_{11} + E_{22}$.

The third diagram above illustrates the poset $(\mathscr{GJ}, \leq)$ of RSO_4. Let $B_0 = \mathbf{B}_n \cap \mathbf{SO}_n$ be a Borel subgroup of $\mathbf{SO}_n$. Then $B = K^* B_0$ is a Borel subgroup of the unit group of $\mathbf{MSO}_n$. If $n = 4$, then B_0 consists of the following invertible upper triangular matrices

$$\begin{pmatrix} a & c & d & -\frac{cd}{a} \\ & b & 0 & -\frac{bd}{a} \\ & & \frac{1}{b} & -\frac{c}{ab} \\ & & & \frac{1}{a} \end{pmatrix}$$

where $a, b \in K^*$ and $c, d \in K$. Thus

$$\mathscr{GJ} = \{0, g_1, \cdots, g_8, 1\}$$

where $g_0 = 0$, $g_1 = E_{14}$, $g_2 = E_{13}$, $g_3 = E_{12}$, $g_4 = E_{13} + E_{24}$, $g_5 = E_{11}$, $g_6 = E_{12} + E_{34}$, $g_7 = E_{11} + E_{22}$, and $g_8 = E_{11} + E_{33}$.

The Gauss-Jordan elements of the even special orthogonal Renner monoid are not the usual reduced row echelon form. For instance,

$$g_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now describe the Gauss-Jordan elements of M. An element x of M is in *reduced form* if $x \in \sigma B$ and $x\sigma^{-1} \in \Lambda$ for some $\sigma \in \mathscr{GJ}$. The requirement $x \in \sigma B$ tells us that x is in row echelon form; the condition $x\sigma^{-1} \in \Lambda$ means roughly that it is reduced.

Theorem 30 ([82, Theorem 8.13]). *Let $x \in M$. Then $Gx \cap \sigma B \neq \emptyset$ for some unique $\sigma \in \mathscr{GJ}$. Moreover, there is a unique T-orbit in $Gx \cap \sigma B$ such that each element of the orbit is in reduced form.*

4.5 The Length Function on R

Identifying successfully the elements of length 0, Renner [80] introduces a length function on the Renner monoid of a reductive monoid. Each WeW has a unique element of length 0. Since $R = \bigsqcup_{e \in \Lambda} WeW$, there are totally $|\Lambda|$ such elements in R.

Theorem 31 ([80, Proposition 1.2]). *There is a unique element $\nu \in WeW$ such that $B\nu = \nu B$.*

Definition 7. Define the length function $l : R \to \mathbb{N}$ by $l(\sigma) = \dim(B\sigma B) - \dim(B\nu B)$ where $\nu \in W\sigma W$ with $B\nu = \nu B$.

Thus $l(\sigma) = 0$ if and only if $\sigma B = B\sigma$ if and only if $\sigma = \nu$ by Theorem 31. If $s \in S$ and $\sigma \in W$, then $l(s\sigma) = l(\sigma) \pm 1$ ([24, 29.3, Lemma A]). If $\sigma \in R$, there is a possibility that $l(s\sigma) = l(\sigma)$. By [65], if $\sigma, \tau \in WeW$, then $\sigma \leq \tau$ implies $l(\sigma) \leq l(\tau)$.

There is another description of this length function using the standard form of elements in R. If w_0, v_0 are respectively the longest elements of W and $W(e)$, then $w_0 v_0$ is the longest element of $D(e)$. It is shown in [67] that

$$l(e) = l(w_0 v_0),$$

and for $\sigma = xey$ in standard form,

$$l(\sigma) = l(x) + l(e) - l(y).$$

The length function is useful in many different topics of algebraic monoids. First, we show that it is useful to study the decomposition of nonidempotents of R^+ into positive root elements, where

$$R^+ = \{\sigma \in R \mid \sigma = xey \text{ with } x \leq y^{-1}\}.$$

For $\alpha \in \Phi^+$ where Φ^+ is the set of positive roots, let

$$R_\alpha = \{es \mid e \in \Lambda, es \neq se\}, \quad R_{-\alpha} = \{se \mid e \in \Lambda, es \neq se\}.$$

We call the elements of R_α *positive root elements*, and the elements of $R_{-\alpha}$ *negative root elements* of R.

Theorem 32 ([71, Theorem 4.2]). *Let $\sigma = xey \in R^+ \setminus E(R)$ be in standard form. Then σ is a product of $l(y) - l(x)$ positive root elements in WeW.*

Next, we characterize the product of $B \times B$ orbits of M using the length function.

Theorem 33 ([80, Theorem 1.4]).

$$BsB\sigma B = \begin{cases} B\sigma B, & \text{if } l(s\sigma) = l(\sigma) \\ Bs\sigma B, & \text{if } l(s\sigma) = l(\sigma) + 1 \\ Bs\sigma B \cup B\sigma B, & \text{if } l(s\sigma) = l(\sigma) - 1. \end{cases}$$

Our aim below is to introduce finite monoids of Lie type, and then show that the length function can also be used to describe the Iwahori-Hecke algebras associated with these monoids. Let G be a finite group of Lie type defined over F_q, a finite field with q elements. A finite regular monoid M with unit group G is a *monoid of Lie type* [64] if M is generated by $E(M)$ and G, and

1. For $e \in E(M)$, the *left centralizer* $P = \{x \in G \mid xe = exe\}$ and the *right centralizer* $P^- = \{x \in G \mid ex = exe\}$ of e in G are opposite parabolic subgroups of G, and $eP_u^- = P_u e = \{e\}$.
2. For $e \in E(M)$, if $e \mathscr{L} f$ or $e \mathscr{R} f$ then xex^{-1} for some $x \in G$.

Finite monoids of Lie type are a large class of finite regular monoids, and there are many examples of such monoids. For instance, the finite reductive monoids introduced by Renner [79] are finite monoids of Lie type [82, Section 10.5]. We elaborate briefly on finite reductive monoids now. Let $\mathbf{M}_n$ be the monoid of all $n \times n$ matrices over the algebraic closure of F_q, and let $\sigma : \mathbf{M}_n \to \mathbf{M}_n$ be the Frobenius map defined by $\sigma : [a_{ij}] \mapsto [a_{ij}^q]$. If $\underline{M} \subseteq \mathbf{M}_n$ is a reductive monoid with a zero and is stable under σ, then

$$M = \{a \in \underline{M} \mid \sigma(a) = a\}$$

is a finite monoid of fixed points, and is called a *finite reductive monoid*. For example, if $\underline{M} = \mathbf{M}_n$, then $M = \mathbf{M}_n(F_q)$. If $\underline{M} = \mathbf{MSp}_n$, then $M = \mathbf{MSp}_n(F_q)$. If $\underline{M} = \mathbf{MSO}_n$, then $M = \mathbf{MSO}_n(F_q)$. If $\underline{M} = \mathbf{D}_n$, then M is the monoid of diagonal matrices with coefficients in F_q. If $\underline{M} = \overline{\mathbf{B}}_n$, then M is the monoid of upper triangular matrices with coefficients in F_q.

Iwahori [25] initiates the study of the Iwahori-Hecke algebra associated with a Chevalley group G. Let B be a Borel subgroup of G, and W the Weyl group of G with generating set S of simple reflections. Let

$$\epsilon = \frac{1}{|B|}\Sigma_{b \in B} b \in \mathbb{C}[G].$$

The Iwahori-Hecke algebra

$$H_{\mathbb{C}}(G) = H_{\mathbb{C}}(G, B) = \epsilon\mathbb{C}[G]\epsilon.$$

is semisimple and is isomorphic to $\mathbb{C}[W]$ [9, 10]. The set $\{A_w = \epsilon w \epsilon \mid w \in W\}$ is a basis of $H_{\mathbb{C}}(G, B)$, which is normalized as $\{T_w = q^{l(\sigma)} A_w \mid w \in W\}$. With respect to this base, Iwahori found that the structure constants are integer polynomials in q, depending only on W.

Using the length function, Putcha [67] studies the monoid Iwahori-Hecke algebra of a finite monoid M of Lie type. He introduces a Putcha lattice Λ for the $G \times G$ orbits, and an analogue of the Renner monoid $R = \langle W, \Lambda \rangle$ such that

$$M = \bigsqcup_{\sigma \in R} B\sigma B.$$

The complex monoid algebra $\mathbb{C}[M]$ of M is semisimple [49]. Let

$$\epsilon = \frac{1}{|B|}\Sigma_{b \in B} b \in \mathbb{C}[G].$$

The monoid Hecke algebra of M is by definition

$$H_{\mathbb{C}}(M) = H_{\mathbb{C}}(M, B) = \epsilon\mathbb{C}[M]\epsilon.$$

It is a semisimple algebra with a natural basis

$$A_\sigma = \epsilon\sigma\epsilon, \quad \sigma \in R.$$

This basis can be normalized as

$$T_\sigma = q^{l(\sigma)} A_\sigma, \quad \sigma \in R.$$

Theorem 34 ([67, Theorem 2.1]). *The structure constants of $H_{\mathbb{C}}(M, B)$ with respect to the basis $\{A_\sigma \mid \sigma \in R\}$, and hence with respect to the normalized basis $\{T_\sigma \mid \sigma \in R\}$ are integer Laurent polynomials in q, depending only on R.*

Using Kazhdan-Lusztig polynomials and "R-polynomials", Putcha obtains the following result.

Theorem 35 ([63, Theorem 4.1]). *The Iwahori-Hecke algebra $H_{\mathbb{C}}(M, B)$ is isomorphic to the complex monoid algebra $\mathbb{C}[R]$ of the Renner monoid.*

Here are some historical notes on the length function and Iwahori-Hecke algebra. Solomon [84] first finds Theorems 31 and 33 for the Renner monoid $\mathbf{R}_n$ of $M = \mathbf{M}_n(F_q)$. He defines a length function on $\mathbf{R}_n$ in a different approach, but it agrees with Definition 7. Furthermore, he introduces the Iwahori-Hecke algebra associated with this M

$$H(M, B) = \bigoplus_{x \in \mathbf{R}_n} \mathbb{Z} \cdot T_x$$

with multiplication defined by

$$T_s T_x = \begin{cases} qT_x, & \text{if} \quad l(sx) = l(x) \\ T_{sx}, & \text{if} \quad l(sx) = l(x) + 1 \\ qT_{sx} + (q-1)T_x, & \text{if} \quad l(sx) = l(x) - 1. \end{cases}$$

$$T_x T_s = \begin{cases} qT_x, & \text{if} \quad l(xs) = l(x) \\ T_{xs}, & \text{if} \quad l(xs) = l(x) + 1 \\ qT_{xs} + (q-1)T_x, & \text{if} \quad l(xs) = l(x) - 1. \end{cases}$$

$$T_\nu T_x = q^{l(x) - l(\nu x)} T_{\nu x}$$

$$T_x T_\nu = q^{l(x) - l(x\nu)} T_{x\nu}$$

where

$$\nu = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

4.6 Presentation of R

Let G be the unit group of a reductive monoid M. Then the commutator group (G, G) is semisimple. The root system Φ and the Weyl group W of (G, G) may be identified with those of G [24, 27.1]. Since each semisimple algebraic group is a product of simple algebraic groups corresponding to the decomposition of Φ into its irreducible components [24, 27.5], without loss of generality, we may assume that G is a simple algebraic group. Denote by $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ a base of Φ and let $A = (a_{ij})$ be the Cartan matrix associated with Δ. Then W is generated by $S = \{s_1, \cdots, s_l\}$ with defining relations

$$s_i^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1, \quad i, j = 1, \cdots, l,$$

where $m_{ij} = 2, 3, 4$ or 6 according to $a_{ij}a_{ji} = 0, 1, 2$ or 3, respectively. Let $\mathscr{E} = \{(s_i, s_j, m_{ij}) \mid i, j = 1, \cdots, l\}$. For $(s, t, m) \in \mathscr{E}(\Gamma)$, denote by $|s, t\rangle^m$ the word $sts \cdots st$ of length m or the word $sts \cdots s$ of length m.

Let $e, f \in \Lambda_0 = \Lambda \setminus \{1\}$ and $w \in D(e)^{-1} \cap D(f)$. Thanks to [15, Proposition 1.21], there exist a unique $h \in \Lambda_0$, and $w \in W_*(h)$ such that $h \le e \wedge f$ and $ewf = hw = h$; this unique element h will be denoted by $e \wedge_w f$. We fix a reduced word representative $\underline{w}$ for each $w \in W$.

Theorem 36 ([15, Proposition 1.24]). *The Renner monoid has the following monoid presentation with generating set $S \cup \Lambda_0$ and defining relations*

$$\begin{aligned}
&s^2 = 1, && s \in S;\\
&|s, t\rangle^m = |t, s\rangle^m, && (s, t, m) \in \mathscr{E};\\
&se = es, && e \in \Lambda_0, s \in \lambda^*(e);\\
&se = es = e, && e \in \Lambda_0, s \in \lambda_*(e);\\
&e\underline{w} f = e \wedge_w f, && e, f \in \Lambda_0, w \in D(e)^{-1} \cap D(f).
\end{aligned}$$

4.7 Orders of Renner Monoids

The orders of Renner monoids provide numerical information about their structures. The information can sometimes be used to study the generating functions associated with the orders, indicating connections between Renner monoids and combinatorics.

Theorem 37 ([38, Theorem 2.1]). *The order of the Renner monoid R of a reductive monoid is*

$$|R| = \sum_{e \in \Lambda} \frac{|W|^2}{|W_{\lambda(e)}| \times |W_{\lambda_*(e)}|} = \sum_{e \in \Lambda} \frac{|W|^2}{|W_{\lambda^*(e)}| \times |W_{\lambda_*(e)}|^2}.$$

Consider the action of $W \times W$ on R defined by $(w_1, w_2)r = w_1 r w_2^{-1}$. The isotropic group of $e \in \Lambda$ is

$$(W \times W)_e = \{(w, ww_*) \in W \times W \mid w \in W_{\lambda(e)} \text{ and } w_* \in W_{\lambda_*(e)}\}.$$

Thus $|WeW| = |W|^2/(|W_{\lambda(e)}| \times |W_{\lambda_*(e)}|)$, and the theorem follows.

4.8 Group Conjugacy Classes

Two elements σ, τ in a Renner monoid R are *group conjugate*, denoted by $\sigma \sim \tau$, if $\tau = w\sigma w^{-1}$ for some $w \in W$. Let $W/W_*(e)$ be the set of left cosets of $W_*(e)$ in W and let $We = \{we \mid w \in W\}$.

Lemma 4 ([41, Lemmas 3.1 and 3.3]). *Each element in a Renner monoid R is group conjugate to an element in $\{we \mid w \in D_*(e)\} \subseteq We$ for some $e \in \Lambda$. Furthermore, if $f \in \Lambda$ and $f \neq e$, then no element of Wf is group conjugate to an element of We.*

Let $W(e)$ act on $W/W_*(e)$ by conjugation

$$w \cdot uW_*(e) = wuw^{-1}W_*(e),$$

where $w \in W(e)$ and $u \in W$. The normality of $W_*(e)$ in $W(e)$ shows that the action is well defined. The following theorem gives a necessary and sufficient condition for two elements to be group conjugate.

Theorem 38 ([41, Theorem 3.4]). *Let $e \in \Lambda$. Two elements ue, ve in We are group conjugate if and only if the two cosets $uW_*(e)$ and $vW_*(e)$ lie in the same $W(e)$-orbit of $W/W_*(e)$.*

Thus, there is a one-to-one correspondence between the group conjugacy classes of a Renner monoid and the orbits of the conjugation action of $W(e)$ on $W/W_*(e)$

for $e \in \Lambda$. Let n_e be the number of $W(e)$-orbits in $W/W_*(e)$. Then the number of the group conjugacy classes in a Renner monoid is $\sum_{e\in\Lambda} n_e$.

From now on, we identify the general rook monoid $\mathbf{R}_n$ with the symmetric inverse semigroup $\mathbf{I}_n$. Our purpose is to describe the group conjugacy classes of classical Renner monoids. First we collect some standard results about the conjugacy classes of the rook monoid.

Theorem 39 ([47, Theorem 1.1]). *Every injective partial transformation in the rook monoid $\mathbf{R}_n$ may be expressed uniquely as a join of disjoint cycles and links up to the order of cycles and links, where cycles and links of length 1 cannot be omitted.*

We explain the concepts used in the theorem. A cycle $(i_1 i_2 \ldots i_m)$ of *length m* is an injective partial transformation with domain and range $\{i_1, i_2, \ldots, i_m\}$ given by $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$. This is different from the usual meaning of $(i_1 i_2 \ldots i_m)$ in S_n whose domain and range are $\mathbf{n}$. A link $[j_1 j_2 \ldots j_m]$ of *length m* is an injective partial transformation determined by $j_1 \mapsto j_2 \mapsto \cdots \mapsto j_m$ with j_m going to nowhere; its domain is $\{j_1, \ldots, j_{m-1}\}$ and range is $\{j_2, \ldots, j_m\}$. Note that a cycle (i_1) of length 1 means i_1 is mapped to itself, and a link $[j_1]$ of length 1 means that j_1 is neither in its domain nor in its range, i.e., $[j_1]$ is the zero element of $\mathbf{R}_n$. A cycle of length m has m distinct expressions: $(i_1 \ldots i_m) = (i_2 \ldots i_m i_1) = \cdots = (i_m i_1 \ldots i_{l-1})$; A link of length m has only one expression $[j_1 \ldots j_m]$ since the starting point j_1 and the terminal point j_l are fixed.

Two elements $\sigma, \tau \in \mathbf{R}_n$ are *disjoint* if $(I(\sigma) \cup J(\sigma)) \cap (I(\tau) \cup J(\tau)) = \emptyset$. If $\sigma, \tau \in \mathbf{R}_n$ are disjoint, then the *join* of σ and τ is defined to be the map $\eta : I(\sigma) \cup I(\tau) \to J(\sigma) \cup J(\tau)$ given by

$$\eta(i) = \begin{cases} \sigma(i) \text{ if } & i \in I(\sigma), \\ \tau(i) \text{ if } & i \in I(\tau). \end{cases}$$

This join is denoted by $\eta = \sigma\tau$. It is clear that $\sigma\tau = \tau\sigma$.

A *signed partition* of a positive integer n is a tuple of positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_s \,|\, \mu_1, \ldots, \mu_t),$$

where $\sum_{i=1}^{s} \lambda_i + \sum_{j=1}^{t} \mu_j = n$ with $\lambda_1 \geq \cdots \geq \lambda_s$ and $\mu_1 \geq \cdots \geq \mu_t$. Let $\sigma \in \mathbf{R}_n$ be the join of s cycles of lengths $\lambda_1, \ldots, \lambda_s$ with $\lambda_1 \geq \cdots \geq \lambda_s$ and t links of lengths $\mu_1, \ldots, \mu_t$ with $\mu_1 \geq \cdots \geq \mu_t$. Then σ corresponds uniquely to a signed partition of n

$$(\lambda_1, \ldots, \lambda_s \,|\, \mu_1, \ldots, \mu_t).$$

This partition is called the *cycle-link type* of σ.

Theorem 40 ([42, Theorem 63.5]). *Two partial injective transformations are group conjugate if and only if their cycle-link types are the same. Moreover, the number of conjugacy classes in $\mathbf{R}_n$ is*

$$\sum_{0 \le k \le n} p(k)p(n-k)$$

where $p(k)$ is the number of usual partitions of k.

The orders of conjugacy classes in $\mathbf{R}_n$ are given in [6]. Writing the cycle-link type of $\sigma \in \mathbf{R}_n$ as

$$(\lambda_1^{p_1}, \ldots, \lambda_u^{p_u} \mid \mu_1^{q_1}, \ldots, \mu_v^{q_v}), \tag{2}$$

where $\lambda_1, \ldots, \lambda_u$ are distinct positive integers and so are $\mu_1, \ldots, \mu_v$, we have the following

Theorem 41 ([6, Proposition 2.4]). *The order of the conjugacy class of σ is equal to*

$$\frac{n!}{p_1! \ldots p_u! q_1! \ldots q_v! \lambda_1^{p_1} \ldots \lambda_u^{p_u}}.$$

The conjugacy class of an injective partial transformation in $\mathbf{R}_n$ corresponds to a unique signed partition of n. Is there a similar result for the conjugacy classes of symplectic transformations? A result of [6] answers this question affirmatively. We need some preparation to state the result.

Strictly disjoint symplectic transformations are introduced in [6]. Let $\hat{I} = I \cup \bar{I}$ where $\bar{I} = \{\bar{i} \mid i \in I\}$ for $I \subseteq \mathbf{n}$. It is clear that if $I, J \subseteq \mathbf{n}$, then

$$I \cap \hat{J} = \emptyset \quad \Leftrightarrow \quad \hat{I} \cap J = \emptyset \quad \Leftrightarrow \quad \hat{I} \cap \hat{J} = \emptyset.$$

Two symplectic transformations $\sigma, \tau \in \mathbf{RSp}_n$ are *strictly disjoint* if

$$\Big(I(\sigma) \cup J(\sigma)\Big) \bigcap \Big(\widehat{I(\tau)} \cup \widehat{J(\tau)}\Big) = \emptyset.$$

Strictly disjoint symplectic transformations are disjoint, but not the other way around. Let $V = \mathbf{RSp}_n \setminus W$ be the submonoid of all singular symplectic transformations. We will describe conjugacy classes in V first and then those in W.

There is an epimorphism $\varphi : \mathbf{RSp}_n \to \mathbf{R}_l$ with $\varphi : \sigma \mapsto \tilde{\sigma}$ defined by

$$\tilde{\sigma}(|i|) = |\sigma(i)| \quad \text{for} \quad i \in I(\sigma),$$

where $|i| = i$ if $1 \le i \le l$, and $|i| = n + 1 - i$ if $l < i \le n$. The following definition will be used in Theorem 42.

Definition 8. (a) The join $\varsigma = [i_{11} \ldots i_{1t_1}][i_{21} \ldots i_{2t_2}] \ldots [i_{u1} \ldots i_{ut_u}]$ of disjoint links in V is called a *string* if $t_i \ge 2$ for $1 \le i \le u$ and

$$\bar{i}_{1t_1} = i_{21},\ \bar{i}_{2t_2} = i_{31},\ \ldots,\ \bar{i}_{u-1,t_{u-1}} = i_{u1}.$$

(b) If $\bar{i}_{ut_u} = i_{11}$, then $\tilde{\varsigma}$ is a cycle in $\mathbf{R}_l$ and ς is referred to as *positive*, otherwise, $\tilde{\varsigma}$ is a link in $\mathbf{R}_l$ and ς is called *negative*.
(c) The *length* of ς is the length of $\tilde{\varsigma}$ in $\mathbf{R}_l$.
(d) The link $[j]$ of length one is considered a negative string.

Theorem 42 ([6, Theorem 3.5]). *Every symplectic transformation $\sigma \in V$ can be expressed uniquely as a join of strictly disjoint cycles and strings up to the order in which they occur.*

For clarity, we now state the main result of [6], and provide necessary concepts needed after it.

Theorem 43 ([6, Theorem 4.5]). *There is a one-to-one correspondence between conjugacy classes in V and symplectic partitions of l.*

What is a symplectic partition of a positive integer l? Well, the story is quite long. Let m be a positive integer. A *composition* of m of *length s* is an ordered sequence of s positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $\sum_{i=1}^{s} \lambda_i = m$. We agree that 0 has one composition, the empty sequence. It is also regarded as the only partition of 0. Define an equivalence relation on the set of compositions of m of length s: λ and λ' are equivalent if λ' is a cycle-permutation of λ. For instance, $(1, 3, 5)$ and $(3, 5, 1)$ are equivalent, but $(1, 3, 5)$ and $(1, 5, 3)$ are not equivalent. The equivalence class of a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ in the set of compositions of m of length s is called a *positive composition*, and will be denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ if there is no confusion. A *negative composition* is a composition itself.

A weak composition of a positive integer m is similar to a composition of m, but allowing parts of the sequence to be zero. For example, $(4, 0, 2)$ is a weak composition of 6 of length 3.

We can now define the symplectic partition of a positive integer l. Let (m_1, m_2, m_3) be a weak composition of l of length 3. If $m_1 > 0$, let $(\lambda_1, \ldots, \lambda_s)$ be a partition of m_1 where $\lambda_1 \geq \cdots, \geq \lambda_s \geq 1$. If $m_2 > 0$, let $(f_1, \cdots, f_t)$ with $f_1 \geq \cdots, \geq f_t \geq 1$ be a partition of m_2. If $m_3 > 0$, let $(g_1, \cdots, g_u, 1^{(v)})$ be a partition of m_3 where $v \geq 0$, and $g_1 \geq \cdots, \geq g_u \geq 2$ if $u \geq 1$. A *symplectic partition* of l is a set of non-negative integers

$$\left(\lambda_1, \ldots, \lambda_s \,\middle|\, \mu_{11}, \ldots, \mu_{1p_1}; \ldots; \mu_{t1}, \ldots, \mu_{tp_t} \,\middle|\, \nu_{11}, \ldots, \nu_{1q_1}; \ldots; \nu_{u1}, \ldots, \nu_{uq_u}; 1^{(v)}\right) \quad (3)$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of m_1, $(\mu_{j1}, \mu_{j2}, \ldots, \mu_{jp_j})$ is a positive composition of f_j for $1 \leq j \leq t$, $(\nu_{k1}, \nu_{k2}, \ldots, \nu_{kq_k})$ with $\nu_{kq_k} \geq 2$ is a negative composition of g_k for $1 \leq k \leq u$, and $1^{(v)}$ is v negative compositions of length one, such that $m_1 + m_2 + m_3 = l$. We agree that if $m_i = 0$ for $i = 1, 2, 3$ then the corresponding part in the symplectic partition is empty.

The concept of the cycle-link type of a symplectic transformation $\sigma \in V$ plays a crucial role in determining conjugacy classes.

Definition 9. Let $\varsigma = [i_{11} \ldots i_{1t_1}][i_{21} \ldots i_{2t_2}] \ldots [i_{u1} \ldots i_{ut_u}]$ be a string. If ς is positive, the positive composition $(t_1 - 1, t_2 - 1, \cdots, t_{u-1} - 1, t_u - 1)$ is referred to as the *string type* of ς. If ς is negative, the negative composition $(t_1 - 1, t_2 -$

$1, \cdots, t_{u-1} - 1, t_u)$ is called the *string type* of ς. Moreover, we say that the negative string $[j]$, consisting of only one link of length 1, has the negative composition (1) as its *string type*.

If $\sigma \in V$ corresponds to the symplectic partition (3), then (3) is referred to as the *cycle-string type* of σ. For completeness we state the traditional result, taken from [8], about conjugacy classes of the unit group W of $\mathbf{RSp}_n$.

Theorem 44. *There is a bijection between the conjugacy classes in W and the signed partitions of l.*

We refer the reader to [6] for the formulas for calculating the number of conjugacy classes and the order of each class.

What are the group conjugacy classes of the even special orthogonal Renner monoid $\mathbf{RSO}_n$. Let W be the unit group of the symplectic Renner monoid $\mathbf{RSp}_n$ and A_n the unit group of the even rook monoid $\mathbf{RSO}_n$ with $n = 2l$. Thus A_n is the subgroup of W consisting of all even permutations in W. Let $V = \mathbf{RSp}_n \setminus W$ and $V' = \mathbf{RSO}_n \setminus A_n$. Then V' is the submonoid of V consisting of even special orthogonal injective partial transformations in V.

We consider the restriction to V' of the conjugation action of W on V. A simple calculation yields that if $\sigma \in V'$ and $\rho \in W$, then $\rho\sigma\rho^{-1} \in V'$. So the restriction to V' of the conjugation action of W on V induces an action of W on V', the conjugation action of W on V'. For now, let C be a W conjugacy class in V. It follows from [13, Lemma 6.3] that two elements of C are A_n conjugate if and only if there is $\sigma \in C$ that commutes with an odd permutation in W. We also know that if there are two elements in C not A_n conjugate, then C is a disjoint union of two A_n conjugacy classes with equal cardinality.

We define a class function c on V'. Let $\sigma \in V'$ with domain $I(\sigma) = \{i_1, \cdots, i_r\}$ and range $J(\sigma) = \{j_1, \cdots, j_r\}$. Define $c(\sigma)$ to be the cardinality of the set $\{|i_1|, \cdots, |i_r|, |j_1|, \cdots, |j_r|\}$. For example, if $n = 8$ and σ maps 3 to 5 and 7 to 2 and leaves the rest unchanged, then $c(\sigma) = 3$ since $I(\sigma) = \{3, 7\}$ and $J(\sigma) = \{5, 2\}$ and $\{|3|, |7|, |5|, |2|\} = \{2, 3, 4\}$. Clearly, if $\sigma, \tau \in V'$ are W conjugate, then $c(\sigma) = c(\tau)$.

Theorem 45 ([13, Theorem 6.8]). *Let C be a W conjugacy class in V'. If $c(C) < l$, then C is an A_n conjugacy class. If $c(C) = l$, then C is a disjoint union of two A_n conjugacy classes with equal number of elements.*

How can one determine if an element of V is in V' using its cycle-link type?

Theorem 46 ([13, Theorem 6.9]). *If $\sigma \in V$ has cycle-link type*

$$\left(\lambda_1, \ldots, \lambda_s \,\middle|\, \mu_{11}, \ldots, \mu_{1p_1}; \ldots; \mu_{t1}, \ldots, \mu_{tp_t} \,\middle|\, \nu_{11}, \ldots, \nu_{1q_1}; \ldots; \nu_{u1}, \ldots, \nu_{uq_u}; 1^{(v)}\right)$$

then $\sigma \in V'$ if and only if $u + v > 0$, or u=v=0 and $p_1 + \cdots + p_t$ is even.

4.9 Munn Conjugacy

The set of $i \in I(\sigma)$ such that $\sigma^k(i)$ is defined for all $k \geq 1$ is called the *stable domain* of $\sigma \in R$, and is denoted by $I^\circ(\sigma)$. That is,

$$I^\circ(\sigma) = \bigcap_{k\geq 1}^{\infty} I(\sigma^k).$$

The restriction of σ to $I^\circ(\sigma)$ induces a permutation σ° of $I^\circ(\sigma)$. This permutation is an element of R. If $\sigma^\circ \in WeW$ for some $e \in \Lambda$, then e is referred to as the subrank of σ.

Definition 10. Two elements $\sigma, \tau \in R$ are called Munn *conjugate*, denoted by $\sigma \approx \tau$, if there exists $w \in W$ such that $w^{-1}\sigma^\circ w = \tau^\circ$.

The Munn conjugacy class of σ is denoted by $[\sigma]$. All elements of $[\sigma]$ have the same subrank, and $[\sigma]$ meets one and only one parabolic subgroup of the form $\{W^*(f) \mid f \in \Lambda\}$. More specifically, $[\sigma]$ meets $W^*(e)$ where e is the subrank of σ.

Theorem 47 ([41, Theorems 4.16 and 4.17]). *There is a bijection between the set of Munn conjugacy classes of a Renner monoid R and the set of all group conjugacy classes of $W^*(e)$ for all $e \in \Lambda$.*

As a consequence, a Renner monoid has as many Munn conjugacy classes as inequivalent irreducible representations over an algebraically closed field of characteristic zero.

Theorem 48 ([6, Theorem 7.2]). *Let W be the unit group of* $\mathbf{RSp}_n$ *and* $V = \mathbf{RSp}_n \setminus W$. *Then two elements in V are Munn conjugate if and only if they have the same cycle part in their cycle-string types. Furthermore, the number of Munn classes is $\sum_{r=0}^{m} p(r)$.*

We describe the relationship between Munn conjugacy and other conjugacies in semigroup theory. Notice that there are different conjugacy relations in semigroups. We are interested in semigroup conjugacy, action conjugacy, character conjugacy, and McAlister conjugacy.

Let S be a semigroup. Then elements $\sigma, \tau \in S$ are called *primary S-conjugate* if there are $x, y \in S$ for which $\sigma = xy$ and $\tau = yx$. This latter relation is reflexive and symmetric, but not transitive. Let $\equiv$ be its transitive closure, called *semigroup conjugacy*. In general, group conjugacy is finer than semigroup conjugacy. But, in a group they are the same, equal to the usual group conjugacy.

Kudryavtseva and Mazorchuk [30] study action conjugacy and character conjugacy. To define action conjugacy, consider the partial action of S^1 on S

$$\sigma \cdot x = \begin{cases} \sigma x \sigma^{-1}, & \text{if } \sigma^{-1}\sigma \geq e_x; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

It follows from [30, Lemma 1] that if $\sigma, \tau \in S^1$ and $x \in S$ then $\tau\sigma \cdot x$ is defined if and only if $\sigma \cdot x$ and $\tau \cdot (\sigma \cdot x)$ are both defined, in which case $\tau\sigma \cdot x = \tau \cdot (\sigma \cdot x)$. We call $x, y \in S$ *primary action conjugate* if there is $\sigma \in S^1$ for which $y = \sigma \cdot x$ or $x = \sigma \cdot y$. This relation is reflexive and symmetric, but not necessarily transitive. Its transitive closure is called *action conjugacy*.

Two elements x, y in a semigroup S are referred to as *character conjugate* if for every finite-dimensional complex representation ϕ of S we have $\chi_\phi(x) = \chi_\phi(y)$, where χ_ϕ is the character of ϕ.

McAlister [43] introduces a conjugacy. Let a be an element in a finite semigroup S and $\bar{a} = ae$, where e is the unique idempotent in the subgroup $\langle a \rangle$ generated by a. Then a, b are conjugate if $\bar{b} = x'\bar{a}x$ and $\bar{a} = x\bar{b}x'$ for some regular element x with inverse x'.

Theorem 49 ([41, Corollary 4.5]). *The action conjugacy, character conjugacy, McAlister conjugacy, Munn conjugacy, and semigroup conjugacy are all the same in a Renner monoid.*

4.10 Representations

What can we say about the representations of the Renner monoid R? We will state the main result of [40] first, and then provide some related information on the representation theory of finite monoids. For any $e \in \Lambda$, let B_e be the group algebra of $W^*(e)$ over F, a field of characteristic 0.

Theorem 50 ([40, Theorem 3.1]). *The inequivalent irreducible representations of R over F are completely determined by those of B_e, where $e \in \Lambda$.*

We briefly elaborate on how to achieve the above result. Let

$$FR = \left\{ \sum_{\sigma \in R} \alpha_\sigma \sigma \;\middle|\; \alpha_\sigma \in F \right\}$$

be the monoid algebra of R over F. The key is to show that FR is isomorphic to the direct sum $\bigoplus_{e \in \Lambda} A_e$, where $A_e = M_{d_e}(B_e)$ in which $d_e = |W|/|W(e)|$. Therefore, FR is a semisimple algebra. To this end, an explicit description of the Möbius function of R is found and a precise formula for Solomon central idempotents is obtained. We refer the reader to [40] for the details.

The work of [40] is a generalization of Munn [47] and Solomon [86] from representations of rook monoids to all Renner monoids. Munn initiates the study of irreducible representations of rook monoid $\mathbf{R}_n$ in terms of irreducible representations of certain symmetric groups contained in the monoid. Solomon [86] investigates these representations using central idempotents of $F\mathbf{R}_n$, and then studies many other aspects related to these representations as well. Steinberg [88,89] discusses representations of finite inverse semigroups S and shows that there is an algebra isomorphism between the monoid algebra of S and the groupiod algebra of S.

Putcha [63, 66–68] has developed a systematic representation theory of finite monoids, including representations of any finite monoid, irreducible characters of full transformation semigroups, highest weight categories and blocks of the complex algebra of the full transformation semigroups. In particular, he provides an explicit isomorphism between the monoid algebra of the Renner monoid and the monoid Hecke algebra introduced by Solomon [84]. Putcha and Okninski describe complex representations of matrix semigroups in [49]. Putcha and Renner study irreducible modular representations of M in [73, 81]. Munn [45, 46] investigates semigroup algebras and matrix representations of semigroups.

4.11 Generating Functions

We investigate the generating functions associated with the orders of classical Renner monoids. Let $r_n = |\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$. It follows from [2] that the generating function $r(x) = \sum_{n=0}^{\infty} \frac{r_n}{n!} x^n$ is convergent to the solution of the differential equation

$$\frac{r'(x)}{r(x)} = \frac{2-x}{(1-x)^2}.$$

This result is generalized in [37] to study the generating functions of the orders of the symplectic and orthogonal Renner monoids.

Let $s_n = \sum_{i=0}^{n} a^i \binom{n}{i}^2 i!$, where a is a nonzero real number. The following recursive formula for s_n, taken from [37], is a variant of [2]. It allows us to calculate the generating function of s_n. Clearly, $s_0 = 1$ and $s_1 = a + 1$.

$$s_n = [a(2n-1)+1]s_{n-1} - a^2(n-1)^2 s_{n-2}, \quad \text{for } n \geq 2. \tag{4}$$

Theorem 51 ([37, Theorem 3.1]). *Let $s(x) = \sum_{n=0}^{\infty} \frac{s_n}{a^n n!} x^n$. If $a \geq 1$, then $s(x)$ converges for $|x| < 1$ to the function $\frac{1}{1-x} e^{x/a(1-x)}$. Also, $s(x)$ satisfies the differential equation*

$$\frac{s'(x)}{s(x)} = \frac{a+1-ax}{a(1-x)^2}.$$

The generating function of s_n is closely related to the Laguerre polynomials. Let $l_n(t)$ be the nth Laguerre polynomial. Then

$$\frac{s_n}{a^n} - \frac{ns_{n-1}}{a^{n-1}} = l_n\Big(\frac{1}{a}\Big), \text{ for } a \geq 1.$$

Corollary 4. *Let r_n be the order of the symplectic Renner monoid, and let $r(x) = \sum_{n=0}^{\infty}\big(\frac{r_n}{4^n n!}\big)x^n$, the generating function of r_n. Then $r(x)$ converges for $|x| < 1$ to the function $\frac{1}{1-x}e^{x/4(1-x)} + \frac{2}{2-x}$.*

Corollary 5. *Let d_n be the order of the even special orthogonal Renner monoid, and let $d(x) = \sum_{n=0}^{\infty}\big(\frac{d_n}{4^n n!}\big)x^n$, the generating function of d_n. Then $d(x)$ converges to the function $\frac{1}{1-x}\big[e^{x/4(1-x)} - \frac{x}{2(2-x)}\big]$ for $|x| < 1$.*

Remark. The generating function of the order of the odd special orthogonal Renner monoid is the same as that of the symplectic Renner monoid.

4.12 Generalized Renner Monoids

It is convenient, for the moment, to let R be temporally a *factorizable monoid* with unit group W acting on the set $E(R)$ of idempotents by conjugation. Denote by Λ a transversal of $E(R)$ for this action. For each $e \in E(R)$ let

$$W(e) = \{w \in W \mid we = ew\}$$
$$W_*(e) = \{w \in W \mid we = ew = e\}.$$

Godelle introduces the concept of *generalized Renner monoids*, a class of factorizable monoids.

Definition 11. A generalised Coxeter-Renner system is a triple (R, W, S) such that

(1) R is a factorizable monoid and (W, S) is Coxeter system.
(2) Λ a sub-semilattice of $E(R)$.
(3) For each pair $e_1 \leq e_2$ in $E(R)$ there exists $w \in W$ and $f_1 \leq f_2$ in Λ such that $wf_iw^{-1} = e_i$ for $i = 1, 2$.
(4) For every $e \in \Lambda$ the subgroups $W(e)$ and $W_*(e)$ are standard Coxeter subgroups of W.
(5) The map $\Lambda \to 2^S : e \mapsto \lambda^*(e) = \{s \in S \mid se = es \neq e\}$ satisfies: if $e \leq f$ then $\lambda^*(e) \leq \lambda^*(f)$.

The monoid R in a Coxeter-Renner system is referred to as a generalized Renner monoid. Godelle [16] introduces a different length function on R, and he used this function to investigate the generic Hecke algebra $H(R)$ over $Z[q]$, which are deformations of the monoid Z-algebra of R. If M is a finite reductive monoid with a Borel subgroup B and Renner monoid R, he then finds the associated Iwahori-Hecke algebra $H(M, B)$ by specialising q in $H(R)$ and tensoring by C over Z. The Renner monoid of a reductive monoid and the Renner monoid of a finite monoid of Lie type are examples of generalized Renner monoids. Mokler [44] studies a different type of discrete monoids constructed from Kac-Moody Lie groups and algebras, called Weyl monoids. The Weyl monoids are generalized Renner monoids [16].

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