Algebraic Monoids and Their Hecke Algebras

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Outline

Introduction

- In this presentation, we explore algebraic monoids, their Hecke algebras, and their representations.
- We seek to produce analogous results from finite algebraic group representation theory in the setting of algebraic monoids.
- We focus on the representation theory of the rook monoid R_n and the symplectic rook monoid RSp_{2n}, and their Hecke algebras, $\mathcal{H}(R_n)$ and $\mathcal{H}(RSp_{2n})$, respectively.

Background on Monoids

Definition

A monoid is a semigroup (assoc. mult.) with identity.

Contained in every monoid, M, is a group of units (i.e., invertible elements) $G(M)$. By studying M, we gain valuable insight into the action of $G(M)$, informing its representation theory.

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Definition

M is an algebraic monoid if it is a Zariski-closed subset of $\mathrm{Mat}_n(F)$ for some $n \in \mathbb{Z}$ and F a field. Furthermore, M is reductive if G(M) is a reductive group and M is an irreducible algebraic variety.

Properties of reductive monoids

If M is reductive, $G(M)$ has a Borel subgroup B, e.g. the invertible upper triangular matrices in the case of $\mathrm{Mat}_n(F)$.

Furthermore, M has a Renner decomposition as the disjoint union of double cosets of B:

$$M = \bigsqcup_{r \in R} B\underline{r}B \tag{1}$$

where R, the Renner monoid of M, encodes vital structural information about M.

The group of units of R is the Weyl group of $G(M)$. Furthermore, R has the decomposition

$$R = G(R)E(\overline{T}) \tag{2}$$

where $E(\overline{T})$ is a set of idempotents.

Rook Monoid

The “Rook Monoid” is the Renner monoid of the algebraic monoid $\mathrm{Mat}_n(F)$.

- R_n is realized as the set of all $n \times n$ matrices with entries 0 and 1 such that each row and column has at most one nonzero entry.
- We call this the Rook monoid because if we view the ones as rooks, then this monoid is the set of all $n \times n$ chessboard with at most n non-attacking rooks.
- Its unit group $G(R_n)$ is isomorphic to the symmetric group, S_n.

Rook Monoid Examples

Example

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} \in R_3$$

Rook Monoid Examples

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$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \in R_3$$

Example (er... Non-example)

$$\begin{pmatrix}1&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} \notin R_3$$

Symplectic Rook Monoid

Similarly, the symplectic Rook monoid is the Renner monoid for the more complicated algebraic monoid whose unit group is the symplectic group $\mathrm{Sp}_{2n}(F)$. Further, The B_n Weyl group embeds as $G(RSp_{2n})$.

Nice presentation:

Theorem

$$RSp_{2n} \cong \{A \in R_{2n} \mid AJA^T = 0 \ or \ J\}, \quad J = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ & \dots & \dots & \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

Symplectic Rook Monoid Examples

Example

$$\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\in RSp_4$$

Symplectic Rook Monoid Examples

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Representations of Monoids

Let M, N be monoids. A map $\varphi : M \to N$ is a homomorphism of monoids if the following hold:

- For all $m_i \in M$, $\pi(m_1 m_2) = \pi(m_1)\pi(m_2)$.
- For e_M, e_N the identity elements of M and N respectively, $\pi(e_M) = e_N$.

Let V be a vector space over k. A morphism $\pi : M \to End_k(V)$ is called a representation of M. We denote representations as the pair (π, V).

A representation is irreducible if it has no proper subrepresentations.

If V is finite dimensional, we define the character $\chi : M \to k$ of π as the function defined by $\chi(m) = tr(\pi(m))$ for all $m \in M$.

Induced Representations

Let N be a submonoid of M and (π, V) a representation of N. We have that (π, V) induces a representation $(\mathrm{Ind}_N^M \pi, \mathrm{Ind}_N^M V)$ of M. Define

- $\mathrm{Ind}_N^M V = \{f : M \to V \mid f(nm) = \pi(n)f(m)\} \quad \forall n \in N, m \in M$
- $(\mathrm{Ind}_N^M \pi)(m)f(x) = f(xm) \quad \forall x,\ m \in M.$

We proved that the following result holds in the case of monoids:

Frobenius Reciprocity for finite monoids

If N is a submonoid of M, (π, V) a representation of N, and (σ, W) a representation of M, then

$$\mathrm{Hom}_M(\mathrm{Ind}_N^M V, W) \cong \mathrm{Hom}_N(V, W) \tag{3}$$

as vector spaces over F

Rook Monoid Representations [Solomon, 2002]

- The irreducible representations of R_n are indexed by partitions of at most n.
- Further, these representations are derived from representations of S_k for $k \in \{0, \ldots, n\}$.
- Let λ be a partition of k, and let V^λ be the corresponding irreducible representation of S_k.
  - There exists an irreducible representation W^λ of R_n.
  - $dim(W^\lambda) = \binom{n}{k} dim(V^\lambda)$

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  - There exists an irreducible representation W^λ of R_n.
  - $dim(W^\lambda) = \binom{n}{k} dim(V^\lambda)$
- We note that "conjugacy classes" of the monoid are also indexed by partitions of at most n.
- It turns out the character table of any Renner monoid is block upper triangular, when the representations are the columns and conjugacy classes are the rows.

Character Table of R_n

Let Ch_k be the character table of S_k. Then define Y_n to be the following block diagonal matrix:

$$Y_n = \begin{pmatrix} Ch_n & & & & \\ & Ch_{n-1} & & & \\ & & \dots & & \\ & & & Ch_1 & \\ & & & & Ch_0 \end{pmatrix}$$

Let M_n be the character table of R_n. Solomon found explicit descriptions of the matrices A and B such that

$$M_n = AY_n = Y_nB \tag{4}$$

The A matrix comes from combinatorics of cycle structures.

The B matrix comes from the Pieri rules for induced representations.

Pieri Rules and Induced Representations

Our motivation in this section comes from restricting our monoid representations to their corresponding group of units. Using [Solomon, 2002] and [Li et al., 2008], we obtain the following result:

Theorem

Let W_n be a Weyl group of type A_n, B_n, C_n, or D_n, with corresponding Renner monoids RW_n. Let χ be a character of S_r, and χ^* the associated character of W_n. Then

$$\chi^*|_{W_n} = \mathrm{Ind}_{S_k \times W_{n-k}}^{W_n}(\chi \otimes \eta_{n-k})$$

In particular, when the Weyl group is A_n, the above restriction produces the well-known Pieri rules. From this result, we can now describe the B matrix as Solomon does.

B matrix for R_n

Let λ and μ index partitions of at most n. Recall that the rows and columns were also indexed by partitions. Thus, we can describe the B matrix entries by the partitions. Solomon finds the B matrix to be:

$$B_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda - \mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases}$$

This comes exactly from the Pieri rules for type A found in [Geck et al., 2000].

Example from R_3 Character Table

$$M_3 = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 3 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y_3 B_3 = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Symplectic Rook Monoid Representations

- Similar story in the Symplectic Rook monoid case.
- The irreducible representations of RSp_{2n} are indexed by pairs of partitions, (λ, μ), such that $|\lambda| + |\mu| = n$, as well as partitions, ν, of $\{0, \ldots, n\}$.
- The representations are derived from representations of B_n and S_k.

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- The representations are derived from representations of B_n and S_k.
- Let (λ, μ) be as above, and let $V^{(\lambda,\mu)}$ be the corresponding irreducible representation of B_n.
    - There exists an irreducible representation $W^{(\lambda,\mu)}$ of RSp_{2n}.
- Let ν be as above, and let V^{ν} be the corresponding irreducible representation of S_k.
    - There exists an irreducible representation W^{ν} of RSp_{2n}.
    - $dim(W^{\nu}) = 2^k \binom{n}{k} dim(V^{\nu})$
- We note that "conjugacy classes" of the monoid are also indexed by partitions of at most n and pairs of partitions whose sum is n.

Character Table of RSp_{2n}

Let X_n be the character table of B_n, and let Ch_k be the character table of S_k. Then define Y_n to be the following block diagonal matrix:

$$Y_n = \begin{pmatrix} X_n & & & & & \\ & Ch_n & & & & \\ & & Ch_{n-1} & & & \\ & & & \dots & & \\ & & & & Ch_1 & \\ & & & & & Ch_0 \end{pmatrix}$$

Let $CRSp_{2n}$ be the character table of RSp_{2n}. In the spirit of Solomon, we derive explicit descriptions of the matrices A and B such that

$$CRSp_{2n} = AY_n = Y_nB \tag{5}$$

The A matrix comes from combinatorics of cycle structures.

The B matrix comes from the Pieri rules for induced representations.

B matrix for RSp_{2n}

We determine the character table to be the following:

$$CRSp_{2n} = \begin{bmatrix} X_n & * \\ 0 & M_n \end{bmatrix} \tag{6}$$

We are able to determine the B matrix in a similar way to the rook matrix. In particular:

$$B = \begin{bmatrix} Id & P \\ 0 & B^* \end{bmatrix} \tag{7}$$

where B^* is the B matrix for R_n, and P comes from Pieri rules in the type B case.

Pieri Coefficients for type B

Theorem

Let $\nu \vdash k$ index a representation of S_k. Then,

$$\mathrm{Ind}_{S_k \times B_{n-k}}^{B_n}(\chi_\nu \boxtimes \eta_{n-k}) = \sum_{\substack{\gamma,\mu \\ \gamma+\mu \vdash n}} \left(\sum_{\substack{\lambda \\ \gamma-\lambda \ is \\ n-k \ horiz. \ strip}} c_{\lambda,\mu}^{\nu} \right) \chi_{\gamma,\mu} \qquad (8)$$

The coefficients obtained from the above formula are the numbers in the P matrix on the previous slide.

What the Hecke?

- It turns out, we can form Hecke algebras from R_n and RSp_{2n}.
- $\mathcal{H}(R_n)$
  - Representations of $\mathcal{H}(R_n)$ are described by [Halverson, 2004].
  - The character table is described in [Dieng et al., 2003].
  - We show that the character table can be decomposed into

$$\mathcal{M}_n = Y_n B \tag{9}$$

    where Y_n is a block diagonal matrix with Hecke algebra character table blocks, and B is the same B matrix we computed for R_n.

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- $\mathcal{H}(RSp_{2n})$
  - Representations have not been described before.
  - We give a first description of the character table.
  - We show that the character table can be decomposed into

$$\mathcal{M}_{2n} = Y_n B \tag{10}$$

    where Y_n is a block diagonal matrix with Hecke algebra character table blocks, and B is the same B matrix we computed for RSp_{2n}.

The Iwahori-Hecke algebra of a reductive monoid

Let M be a reductive monoid over a finite field F. Recall that M unit group $G(M)$, Borel subgroup B, and Renner monoid R.

Definition

The **Hecke algebra** $\mathcal{H}(M, B)$ over $\mathbb{C}$ is the algebra

$$\mathcal{H}(M, B) = \{f : M \to \mathbb{C} \mid f(b_1 x b_2) = f(x)\ \forall b_1, b_2 \in B,\ x \in M\} \quad (11)$$

under addition and convolution of functions, with convolution given by

$$(f * g)(x) = \sum_{yz=x} f(y)g(z). \quad (12)$$

Properties of Hecke algebras

- The Hecke algebra of a monoid has a basis over $\mathbb{C}$ given by, for all $r \in R$, $1_{B\underline{r}B}$ defined to be the characteristic function of the double coset of $\underline{r}$.
- Let M be a reductive monoid with Renner monoid R. Then $\mathcal{H}(M, B) \cong \mathbb{C}[R]$ as $\mathbb{C}$-algebras.
- Let (π, V) be a representation of M. Then V has a $\mathcal{H}(M, B)$-module structure under the following action: for $f \in \mathcal{H}(M, B)$,

$$\pi(f)v = \sum_{x \in M} f(x)\pi(x)v \tag{13}$$

- Let $V^B = \{v \in V \mid \pi(b)v = v \, \forall b \in B\}$ be the space of vectors fixed pointwise by a Borel subgroup. The Hecke algebra of an algebraic monoid M encodes information about representations of M with V^B nonzero.

The Borel-Matsumoto Theorem

The Borel-Matsumoto theorem for finite monoids

- *Let (π, V) be an irreducible representation of M with $V^B \neq \{0\}$. Then V^B is irreducible as an $\mathcal{H}(M, B)$-module.*
- *If (π, V) and (σ, W) are two irreducible representations of M with V^B and W^B nonzero and isomorphic as $\mathcal{H}(M, B)$-modules, then $(\pi, V) \cong (\sigma, W)$.*

The Borel-Matsumoto theorem allows us to reduce questions about representations of our algebraic monoid M with V^B nonzero to questions about the representations of $\mathcal{H}(M, B)$.

Since $\mathcal{H}(M, B) \cong \mathbb{C}[R]$ for R, the Renner monoid of M, its representation theory is markedly simpler than that of M itself.

In theory, we could use $\mathcal{H}(M, B)$ to classify irreducible representations of M with V^B nonzero.

Further Questions

- How do representations of R_{2n} restrict to RSp_{2n}?
- What does this process look like for type D Renner monoids?
- Can we construct the irreducible representations of a reductive monoid M with V^B nonzero guaranteed by the Borel-Matsumoto theorem?
- Is there a Deligne-Lusztig theory for finite monoids of Lie type?
- Is there a Borel-Matsumoto theorem for p-adic reductive monoids?
- Does the comparatively simple geometry of algebraic monoids help us with their representation theory?
- What other aspects of the theory of group Hecke algebras hold in the case of monoid Hecke algebras?

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Questions

Any questions?