REU 2019 Day 3

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TA office hours 1:30pm today

Greg & Andy

R-systems

Let $G=(V,E)$ be a directed edge-weighted graph with no loops or multiple edges.

Consider a variable x_v for each vertex $v \in V$.

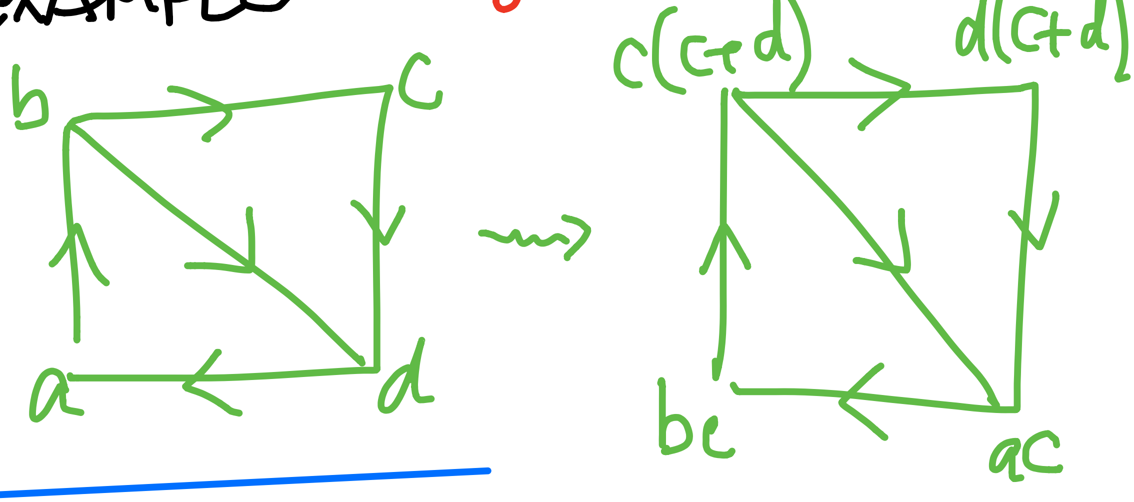
Given $(x_v)_{v\in V}$, we want to find a new set of variables $(x'_v)_{v\in V}$ that satisfy

$$x_v x_v' \overset{(*)}{=} \left(\sum_{(u,v)\in E} \frac{wt(u,v)}{x_u'}\right)^{-1}\left(\sum_{(v,w)\in E} wt(v,w)\, x_w\right)$$

More symmetrically,

$$\sum_{(u,v)\in E} wt(u,v)\frac{x_v'}{x_u'} = \sum_{(v,w)\in E} wt(v,w)\frac{x_w}{x_v}$$

EXAMPLE (edges with no weight are 1)



Check bottom left vertex:

$$\mathrm{LHS}(*) = abc$$

$$\mathrm{RHS}(*) = b\left(\frac{1}{ac}\right)^{-1} = abc$$

Check top left vertex:

$$\mathrm{RHS}(*) = bc(c+d)$$

$$\mathrm{LHS}(*) = (c+d)\left(\frac{1}{bc}\right)^{-1}$$

Why do this?

This generalizes a process called "birational rowmotion" defined by Einstein & Propp,
(David, not Albert)

- an operation on (variables assigned to elements of) posets
- combines ideas of birational toggling (a well-studied operation) and rowmotion (toggling elements in and out of poset ideals)

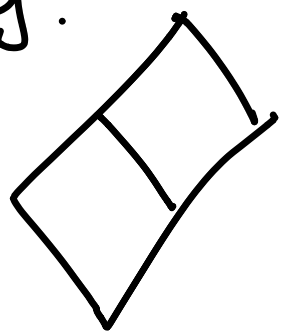
Consider a poset (= partially ordered set) as a directed graph with all edges oriented upward.

Add an an additional vertex s.

Add edges

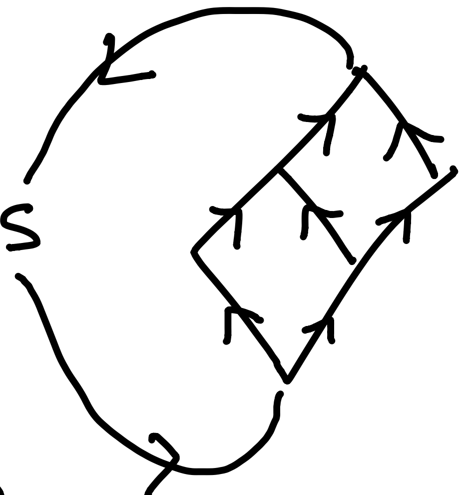
- from s to any source
- from any sink to s

e.g.



poset

⇝

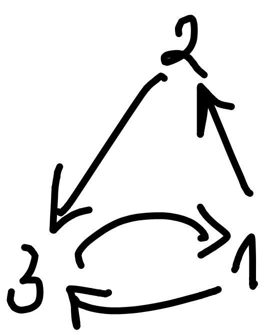


Applying the transformation $(x_v) \longrightarrow (x'_v)$ where we fix $x_s = x'_s = 1$ is the same as doing birational rowmotion on the poset.

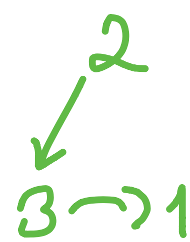
DEF'N: A graph $G=(V,E)$ is strongly connected if $\forall u,v \in V$ $\exists$ a directed path u to v in G.

DEF'N: For a strongly connected G, an arborescence rooted at v is a spanning tree directed toward v.

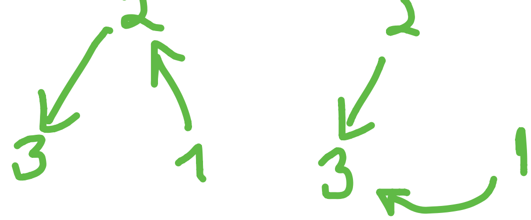
EXAMPLE:



has one rooted at 1:



but two rooted at 3:



DEFN: The **weight** of an arborescence T is
$$\mathrm{wt}(T,X) := \prod_{(u,v)\in T} \mathrm{wt}(u,v)\frac{x_v}{x_u}$$

$$T = \begin{tikzcd} & 2 \arrow[dl] \\ 3 \arrow[r] & 1 \end{tikzcd} \qquad \mathrm{wt}(T,X) = \left(\frac{x_1}{x_3}\right)\left(\frac{x_3}{x_2}\right)$$

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 3 & & 1 \arrow[ul] \end{tikzcd} \qquad\qquad \begin{tikzcd} 2 \arrow[d] & \\ 3 & 1 \arrow[l] \end{tikzcd}$$

$$\left(\frac{x_3}{x_2}\right)\left(\frac{x_2}{x_1}\right) \qquad\qquad \left(\frac{x_3}{x_1}\right)\left(\frac{x_3}{x_2}\right)$$

THM (Galashin–Pylyavskyy) 2017

Let $G=(V,E)$ be strongly connected. Given $(x_v)_{v\in V}$ there exists a unique solution for $(x'_v)_{v\in V}$ up to rescaling:

$$x'_v = \frac{x_v}{\displaystyle\sum_{\substack{\text{arborescences } T\\ \text{rooted at } v}} wt(T,x)}$$

EXAMPLE: G from before has

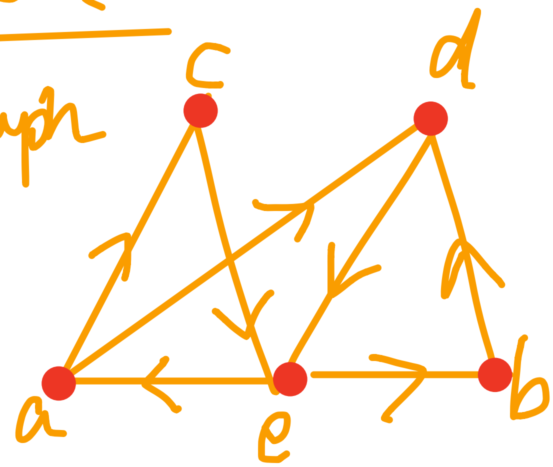
$$x'_1 = \frac{x_1}{\left(\frac{x_1}{x_2}\right)} = x_2$$

$$x'_2 = \frac{x_2}{\left(\frac{x_2}{x_3}\right)} = x_3$$

$$x'_3 = \frac{x_3}{\frac{x_3}{x_1} + \frac{x_3^2}{x_1 x_2}} = \frac{x_1 x_2}{x_2 + x_3}$$

REU Exercise 6

Consider the graph



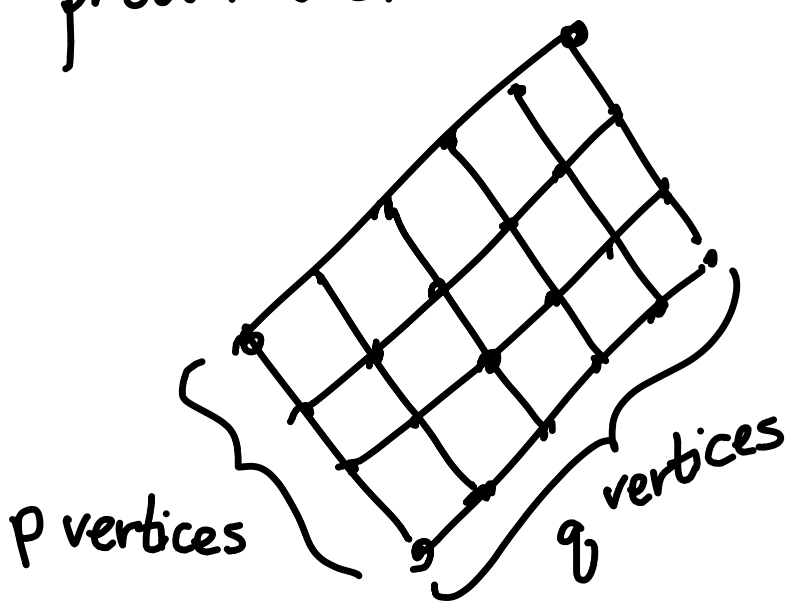
use the arboresence formula to

find (x_v') corresponding

to $(x_v) = (a,b,c,d,e)$ above.

DEF'N: For a strongly connected G define ϕ to be the map $(x_v) \to (x_v')$. Then the R-system associated with G is the discrete dynamical system obtained by iterating ϕ

DEF'N: A rectangle poset is the product of two chain posets



THM (Grinberg-Roby 2016)

Birational rowmotion is periodic on rectangular posets, with period $p+q$.

This can be proven using a "τ sequence"

τ-sequence idea: write our X variables as Laurent monomials (= monomials with positive and negative exponents allowed) in some other variables $\{\tau_i\}_{i \in \mathbb{Z}_{>0}}$ where the τ_i's form a recursive sequence and each τ_i is a Laurent polynomial in the original values.

EXAMPLE:

$$(x_1, x_2, x_3) \longmapsto \left(x_2, x_3, \frac{x_1 x_2}{x_2 + x_3}\right)$$

$$\begin{array}{ccc} \| & \| & \| \\ u_1 & u_2 & u_3 \end{array} \qquad \begin{array}{ccc} \| & \| & \| \\ u_2 & u_3 & u_4 \end{array}$$

$$\longmapsto (u_3, u_4, u_5) \longmapsto \cdots$$

Let $\tau_0 = x_1 x_2 x_3$

$\tau_1 = x_2 x_3$

$\tau_2 = x_3$

$\tau_3 = 1$

and $\underbrace{\tau_n \tau_{n+4} = \tau_{n+1}\tau_{n+3} + \tau_{n+2}^2}$

the Somos-4 sequence,

known to have Laurent phenomenon

(can write all τ_i's as Laurent polynomials in $\tau_0, \tau_1, \tau_2, \tau_3$)

CLAIM: $u_n = \dfrac{\tau_n}{\tau_{n+1}}$

$$u_{n+3} = \frac{\tau_{n+3}}{\tau_{n+4}} = \frac{\tau_{n+3}\,\tau_n}{\tau_{n+1}\tau_{n+3} + \tau_{n+2}^2}$$

versus

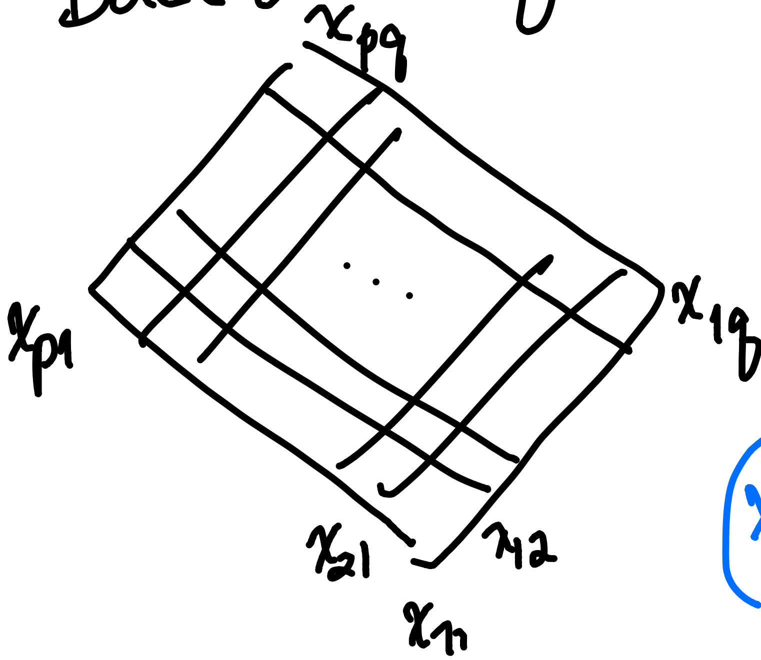
$$u_{n+3} = \frac{u_n u_{n+1}}{u_{n+1} + u_{n+2}} = \frac{\left(\frac{\tau_n}{\tau_{n+1}}\right)\left(\frac{\tau_{n+1}}{\tau_{n+2}}\right)}{\left(\frac{\tau_{n+1}}{\tau_{n+2}}\right) + \left(\frac{\tau_{n+2}}{\tau_{n+3}}\right)}$$

from the map ϕ

$= \ldots =$ expression above ✓

induction

Back to rectangles



$$(x_{ij}(t)) \xrightarrow{\phi} (x_{ij}(t+1))$$

There is a τ-sequence

$$\alpha_{ij}(t)\,\alpha_{ij}(t+1) = \left\{ \text{see Sunita's scanned notes!} \right\}$$

and $$x_{ij}(t) = \frac{\alpha_{i+1,j+1}(t)}{\alpha_{ij}(t+1)}$$

REU Exercise 7:

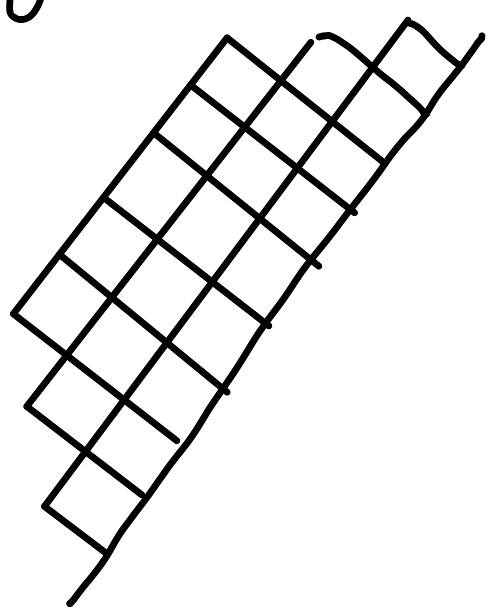
(a) List the relations that must be true for this system to be consistent.

(b) Check the relations for

$$2 \leq i \leq p$$
$$2 \leq j \leq q$$

We can prove periodicity using this T-sequence
(part of this will be done in this afternoon's TA session).

DEF'N: A trapezoid poset is a rectangle poset with the sides cut off by diagonals



REU Problem 3a

Prove birational rowmotion is periodic on all trapezoid posets (ask Sylvester what the conjectural period is).

DEF'N: A **k-fold cover** of a graph $G=(V_G,E_G)$ is a graph $H=(V_H,E_H)$ with a map $\pi: H \to G$ such that

(1) π maps vertices to vertices, edges to edges

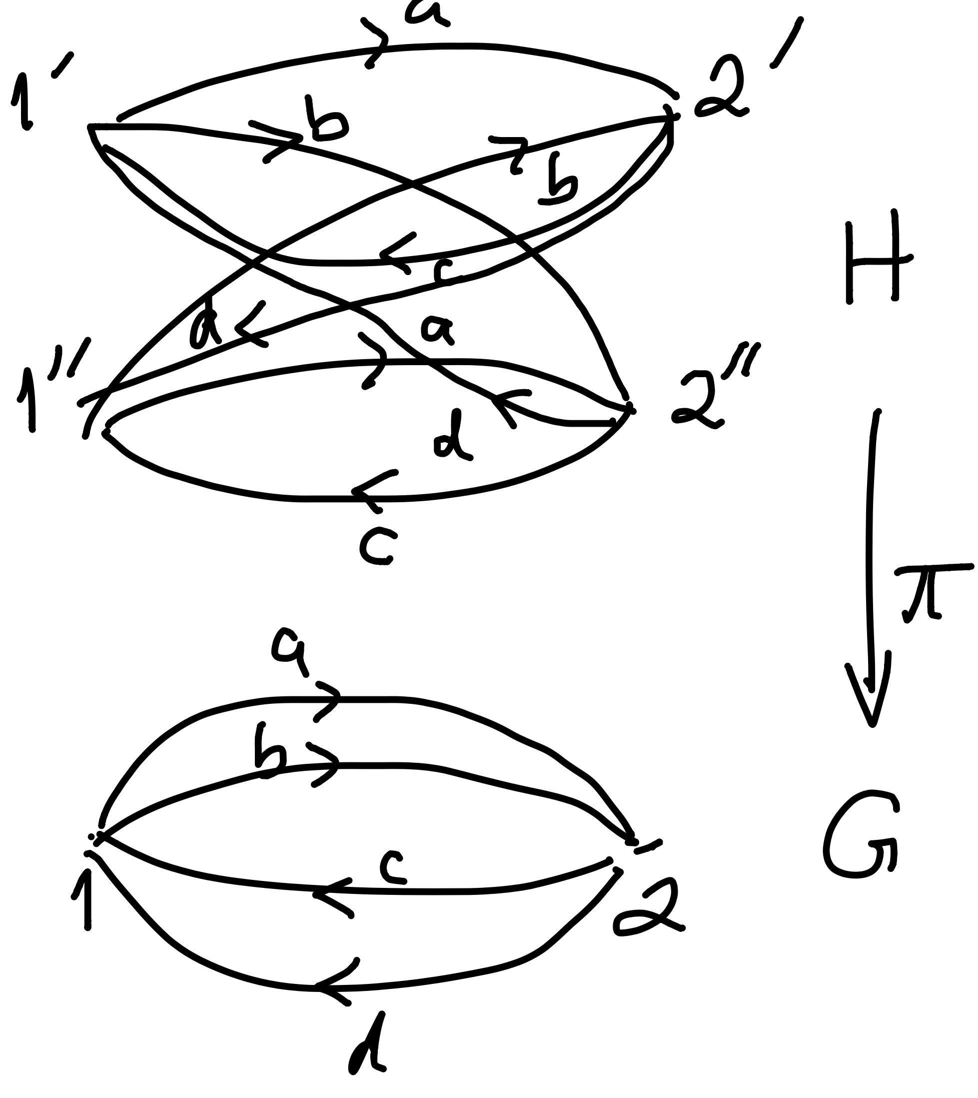
(2) $|\pi^{-1}(v)|=|\pi^{-1}(e)|=k \quad \forall v\in V_G, e\in E_G$

(3) For $e=(u,v)\in E_H$,

$$\pi(e)=(\pi(u),\pi(v))\in E_G$$

(4) For $e\in E_H$, $wt(e)=wt(\pi(e))$.

EXAMPLE:



THM: If H is a k-fold cover of G and v' covers v, then

$$\frac{\displaystyle\sum_{\substack{\text{arborescences}\\ T \text{ rooted at } v \text{ in } G}} wt(T)}{\displaystyle\sum_{\substack{\text{arborescences}\\ T \text{ rooted at } v' \text{ in } H}} wt(T)}$$

does not depend upon the choice of v', nor on v.

In above example, picking $v=1, v'=1'$

get $\dfrac{c+d}{cbc+adc+bcd+dad} = \dfrac{1}{bc+ad}$

but picking $v=2, v'=2'$

get $\dfrac{a+b}{ada+cba+dab+bcb} = \dfrac{1}{bc+ad}$

same!

REU Problem 3b

Write this ratio as a determinant, and investigate the positivity of the coefficients.

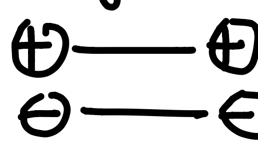
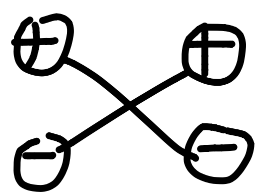
Why might one think it's related to a determinant?

DEF'N: The Laplacian matrix of a graph $G=(V,E)$ is $L=(L_{ij})_{1\le i,j\le n}$, where $V = \{v_1,\ldots,v_n\}$,

where
$$L_{ij} = \begin{cases} \sum\limits_{e \text{ with source } v_i} wt(e) & \text{if } i=j \\ -\sum\limits_{e=(v_i,v_j)} wt(e) & \text{if } i\neq j \end{cases}$$

EXAMPLE: G from before

$$L = \begin{bmatrix} a+b & -a-b \\ -c-d & c+d \end{bmatrix}$$

Consider the 2-fold cover case, and label one lift of each vertex as ⊕, the other as ⊖. Call an edge of G **positive** if its lifts are  , **negative** if its lifts are  .

In the Laplacian of G, switch the sign of the weights of the positive edges off the diagonal.

EXAMPLE

$$\begin{bmatrix} a+b & a-b \\ c-d & c+d \end{bmatrix}$$

"CONJECTURE"

Our ratio is $\left[\frac{1}{2}(\text{det of this matrix})\right]^{-1}$

EXAMPLE

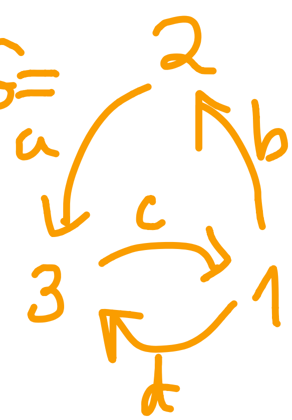
$$\det\begin{bmatrix} a+b & a-b \\ c-d & c+d\end{bmatrix} = (a+b)(c+d) - (a-b)(c-d)$$

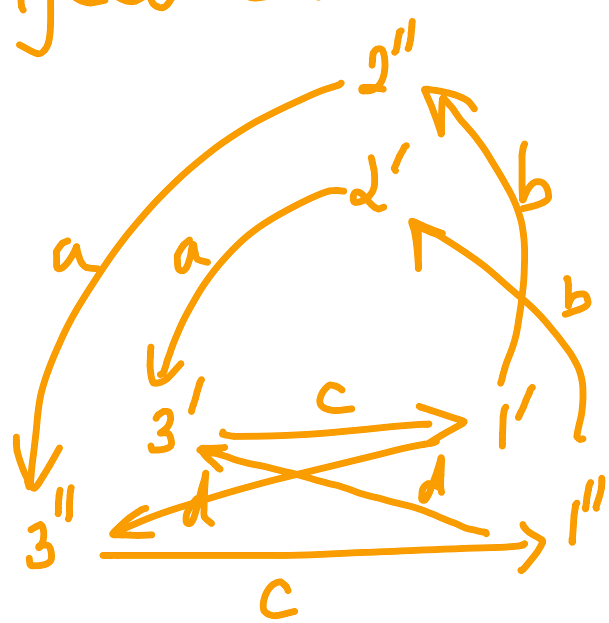
$$= \cancel{ac}+ad+bc+\cancel{bd}-\cancel{ac}+ad+bc-\cancel{bd}$$

$$=2(ad+bc)$$

REN Exercise 8

Check the conjecture for

G= 

H= 

(maybe it fails?)