Day 4 : Dihedral Sieving Phenomena

6/6/19

TAs: Sarah and Andy

Session today at 1:30pm in Vincent 209

Based on work by Rao & Suk 2017 REU

1. Cyclic Sieving & q-analogues
   (a) subsets, q-binomials
   (b) polygon dissections, q-Catalans
2. Dihedral Sieving
   (a) subsets
   (b) triangulations, (q,t)-Catalans
3. REU Problem 4

1. Cyclic Sieving

Have in mind a finite set X e.g. $X = \binom{[n]}{k}$ = k-subsets of $[n] := \{1, \ldots, n\}$, and some polynomial $X(q) \in \mathbb{Z}[q]$ that is a q-analogue of $\#X$ meaning: $[X(q)]_{q=1} = \#X$

e.g. $\#X = \#\binom{[n]}{k} = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$X(q) := \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$ where $[n]!_q = [n]_q \cdot [n-1]_q \cdots [1]_q$

and $[n]_q = 1 + q + \cdots + q^{n-1}$

Note $[n]_{q=1}! = n!$, and so $\begin{bmatrix} n \\ k \end{bmatrix}_{q=1} = \binom{n}{k}$

e.g. $$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \frac{[6]_q [5]_q [4]_q}{[3]_q [2]_q [1]_q} = \frac{(1+q+\cdots+q^5)(1+q+\cdots+q^4)(1+q+q^2+q^3)}{(1+q+q^2)(1+q)(1)}$$

$$= (1+q^3)(1+q^2)[5]_q$$

$$= 1+q+2q^2+3q^3+3q^4+3q^5 \overset{+3q^6}{\wedge} +2q^7+q^8+q^9$$

$\in \mathbb{Z}[q]$

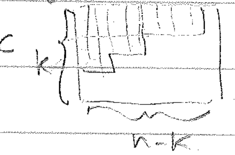
$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = \cdots = 1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7+q^8$$

REU Exercise 9

(a) show $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ "q-Pascal"

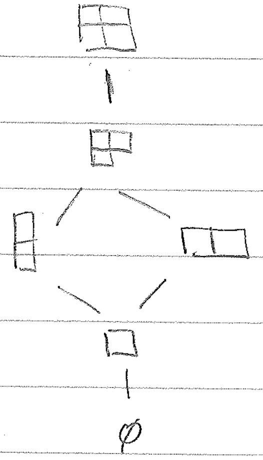
(b) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum\limits_{\substack{\text{Ferrers/Young} \\ \text{diagrams}}} q^{|\lambda|}$.

$\lambda \subset$



e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$: $k=2$, $n-k=2$.

Possible λ:



q^4

$+ q^3$

$+ 2q^2$

$+ q$

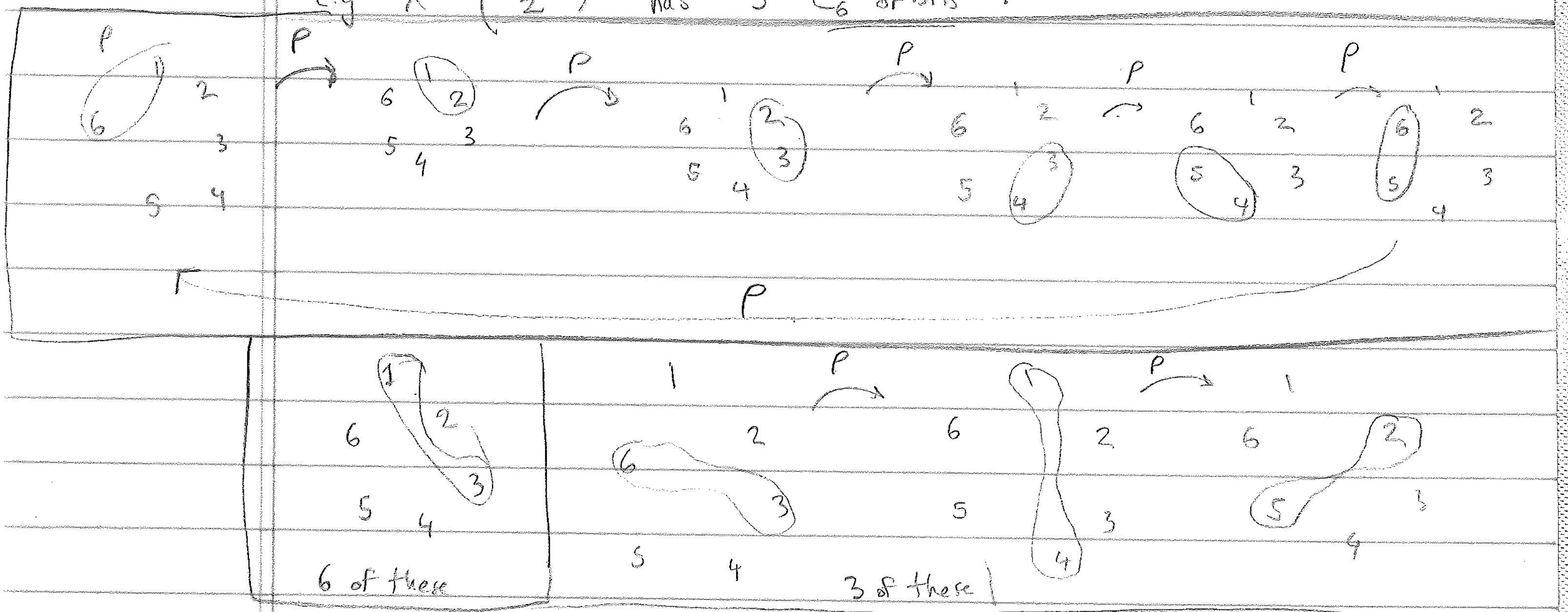
1

(c) Show $\begin{bmatrix} n \\ k \end{bmatrix}_{q=p^d}$ where p is prime (so ∃ a finite field $\mathbb{F}_q = \mathbb{F}_{p^d}$ of size $q = p^d$)
is $= \# \left\{ \begin{matrix} k\text{-dim'l } \mathbb{F}_q\text{-linear subspaces} \\ \text{of } (\mathbb{F}_q)^n \end{matrix} \right\}$.

Now assume X has the action of a cyclic group $C := C_m \cong \mathbb{Z}/m\mathbb{Z}$
(written $C \circlearrowright X$) $= \langle \rho \rangle = \{1, \rho, \rho^2, \ldots \rho^{m-1}\}$
"acts on"

permuting X.
e.g. $C_n \circlearrowright X = \binom{[n]}{k}$ via $\rho \cdot \{i_1, \ldots i_k\} = \{i_1+1, \ldots i_k+1\}$
↖ read the values mod n.

e.g $X = \binom{[6]}{2}$ has 3 C_6-orbits :



Def/. (Stanton-White-R, 2014). Say $(X \circlearrowleft C_m, X(q))$ exhibits a cyclic sieving phenomenon (CSP) if for every $\rho^k \in C_m \overset{\shortparallel}{} \langle \rho \rangle$.
one has $\# X^{\rho^k} := \#\{x \in X : \rho^k(x) = x\}$

$$= [X(q)]_{q = (\zeta_m)^k}, \quad \zeta = e^{2\pi i/m}.$$

(a) Thm (RSW . 2004) $\left(X = \binom{[n]}{k} \circlearrowleft C_n, X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q\right)$ exhibits a CSP.

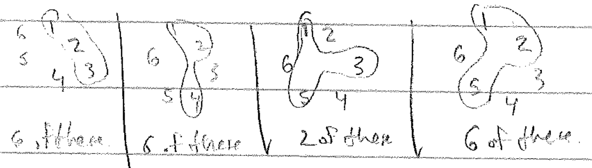
$(1+q+q^4)[5]_q = \frac{[6]_q[4]_q}{[2]_q} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}_q$: $q = 1 \rightsquigarrow \binom{6}{2} = 15 = \# X = \# X^{\rho^0}$ (no surprise).

$q = \zeta_6^3 = -1 \rightsquigarrow 3 \overset{\checkmark}{=} \# X^{\rho^3}$

$q = \zeta_6^2 \rightsquigarrow 0$

$q = \zeta_6 \rightsquigarrow 0.$

$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q$: $q = 1 \; \binom{6}{3}$. orbits:



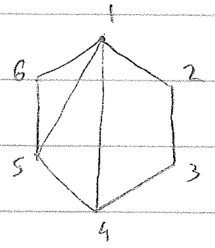
$q = -1 \rightsquigarrow 0$. (nothing fixed by 180° rotation).

$q = \zeta_6^2 = e^{2\pi i/3} \rightsquigarrow 2.$

$q = \zeta_6 \rightsquigarrow 0.$

$$\text{(b). Thm}\begin{pmatrix}\text{Kirkman 1857}\\ \text{Cayley 1890}\end{pmatrix} \ \#\left\{\begin{array}{c} X := \\ \text{dissections of} \\ n+2\text{-sided polygon} \\ \text{using } k\text{-noncrossing} \\ \text{diagonals}\end{array}\right\} = \frac{1}{n}\binom{n}{k+1}\binom{n+k+1}{k}$$

eg. $n=4$ $k=2$

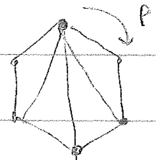


When $k=n-1 \rightsquigarrow \#\left\{\begin{array}{c}\text{triangulations of}\\ (n+2)\text{-gon}\end{array}\right\} = \frac{1}{n}\binom{n}{n}\binom{2n+1}{n-1}$

$= \frac{1}{n+1}\binom{2n}{n}$

= Catalan #s.

eg: $n=4$ $k=3$:



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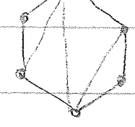

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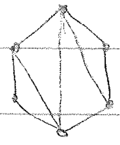

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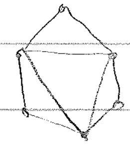

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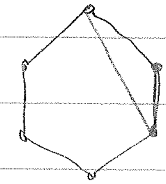
$6+3+3+2 = 14 = \frac{1}{4+1}\binom{8}{4} = \frac{1}{5}\binom{8}{4}$.

$n=4$ $k=1$.



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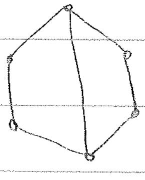

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$6+3 = 9 = \frac{1}{4}\binom{4}{2}\binom{6}{1}$

Thm (RSW 2004) $\left(\underset{\text{nec}}{X} = \circlearrowleft C_n,\ X(q) := \begin{bmatrix}n\\k+1\end{bmatrix}_q \begin{bmatrix}n+k+1\\k\end{bmatrix}_q \frac{1}{[n]_q}\right)$

exhibits a CSP.

$n=4$ $k=1$.: $X(q) = \frac{1}{[4]_q}\begin{bmatrix}4\\2\end{bmatrix}_q\begin{bmatrix}6\\1\end{bmatrix}_q = (1+q^2+q^4)(1+q+q^2)$

$q=1 \rightsquigarrow 9$

$q=-1 \rightsquigarrow 3$

$q=\zeta_6^2 \rightsquigarrow 0$

$q=\zeta_6^3 \rightsquigarrow 0$

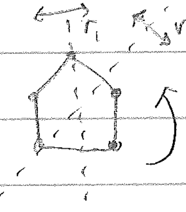
2. Dihedral Sieving

Lots of sets X with C_m-action actually have an action of $I_2(m)$

$\underline{I_2(m) := \text{(linear) symmetries of a regular } m\text{-gon}}$

dihedral group of ~~size~~ $2m$

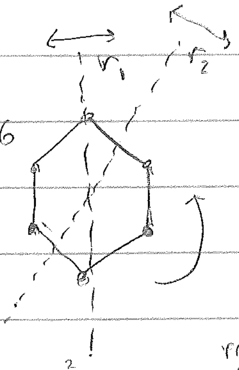
m=5



$\rho = $ rotation through $2\pi/m$.

$= r_1 r_2$.

m=6



ρ = rotation through $2\pi/m$

$$I_2(m) = \{\underbrace{1, \rho, \ldots \rho^{m-1}}_{m \text{ rotations}}, \underbrace{\overset{r}{\overset{\shortparallel}{r_1}}, \overset{r\rho}{\overset{\shortparallel}{r_1\rho}}, \overset{r\rho^2}{\overset{\shortparallel}{r_1\rho^2}} \ldots \overset{r\rho^{m-1}}{\overset{\shortparallel}{r_1\rho^{m-1}}}}_{m \text{ reflections}}\}$$

(call $r_1 = r$).

REU Exercise 10

(a) Show $I_2(m)$ has this (abstract) presentation

$$I_2(m) \cong \langle \underbrace{r, \rho}_{\text{generators}} \mid \underbrace{1 = r^2 = \rho^m, \; \underset{r\rho r}{r\rho r^{-1}} = \rho^{-1}}_{\text{relations}} \rangle$$

(b). Show that the defining representation $I_2(m) \xrightarrow{\varphi_{def}} GL_2(\mathbb{R})$ $(\subset GL_2(\mathbb{C}))$

(= group homomorphism $I_2(m) \to GL(V)$)

$$r \longmapsto \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & -1 & 0 \\ e_2 & 0 & 1 \end{array}$$

$$\rho \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = e^{2\pi/m}$$

is equivalent via some change of basis matrix $P \in GL_2(\mathbb{C})$ to

this representation:

$$I_2(m) \xrightarrow{\varphi^{(1)}} GL_2(\mathbb{C})$$

$$r \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho \longmapsto \begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix}$$

meaning

$P^{-1}\varphi_{dep}(w)P = \varphi^{(1)}(w)$

for all $w \in W$

homomorphisms into $GL_1(\mathbb{C}) = \mathbb{C}^\times$

(c). Show that all of the inequivalent $\overset{\text{(linear)}}{\text{characters}}$ are these:

only when m is even: χ_1, χ_2

	r_1	r_2	$\rho = r_1 r_2$
$\mathbb{1}$	$+1$	$+1$	$+1$
χ_1	-1	$+1$	-1
χ_2	$+1$	-1	-1
det.	-1	-1	$+1$.

(d) Show that $I_2(m) \xrightarrow{\varphi^{(k)}} GL_2(\mathbb{C})$ for $k = 1, \ldots, m-1$.

$$r \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho \longmapsto \begin{bmatrix} \zeta_m^k & \\ & \zeta_m^{-k} \end{bmatrix}$$

define representations of $I_2(m)$, and they are inequivalent for $k = 1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor$

Q: Is there a dihedral sieving phenomenon (DSP)?

A: Yes, at least for m odd!

def/ (Rao-Suk 2017) Given a finite set $X \circlearrowleft I_2(m)$ with $\underline{\underline{m \text{ odd}}}$ and a polynomial $X(q,t) \in \mathbb{Z}[q,t]$ that is symmetric in q,t (i.e. $X(q,t) = X(t,q)$).

Say $(X \circlearrowleft I_2(m), X(q,t))$ exhibits a DSP if

$$\forall w \in I_2(m) \quad \# X^w = [X(q,t)]_{\substack{q=\lambda_1 \\ t=\lambda_2}}$$

where $\{\lambda_1, \lambda_2\}$ are eigenvalues of w in φ_{def} or $\varphi^{(1)}$

i.e. $\det(x I_2(m) - \varphi_{def}(w)) = x^2 - (\lambda_1 + \lambda_2)x + \lambda_1 \lambda_2$

in other words, $\# X^w = \begin{cases} X(q,t)|_{q=1,\ t=-1} & \text{if } w = r \\ X(q,t)|_{q=\zeta_m^k\ t=\zeta_m^{-k}} & \text{if } w = \rho^k \end{cases}$

(a) Subsets. Thm (Rao - Suk 2019) For n odd,

$$\left(X = \binom{[n]}{k} \circlearrowleft I_2(m),\ X(q,t) = \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,t}\right) \text{ exhibits a DSP.}$$

where $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\circ} = \frac{\{n\}!}{\{n-k\}!\{k\}!}$ $\quad \{n\}! = \{n\}\{n-1\}\cdots\{1\}$.

$$\{n\} = q^{n-1} + q^{n-2}t + \cdots + qt^{n-1} + t^{n}$$

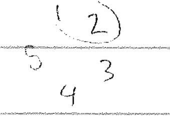
e.g. $n=5$ $k=2$. $X(q,t) = \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} = \frac{\{5\}_q\{4\}_q}{\{2\}_q} = q^6 + q^5t + 2q^4t^2 + 2q^3t^3 + 2q^2t^4 + qt^5 + t^6.$

$q,t = 1,1 \rightsquigarrow \binom{5}{2} = 10$

$q,t = \zeta_5, \zeta_3^{-1} \rightsquigarrow 0$

$q,t = 1,-1 \rightsquigarrow 2$

orbits



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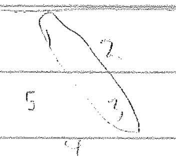

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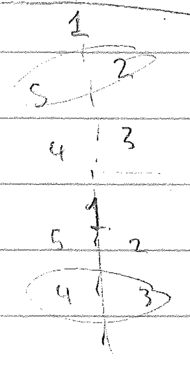


(b) Triangulations Thm (Rao - Suk 2019) For n odd

$$\left(X = \left\{\begin{matrix}\text{triangulations of}\\ (n+2)\text{-gon}\end{matrix}\right\} \circlearrowleft I_2(n+2),\ X(q,t) = (qt)^{\binom{n}{2}} \mathrm{Cat}_n(q,t)\right)$$

exhits a DSP where $\mathrm{Cat}_n(q,t)$ = Garsia - Haiman's (q-t)-Catalan polynomial

:= bigraded Hilbert series

for the sgn-isotypic component

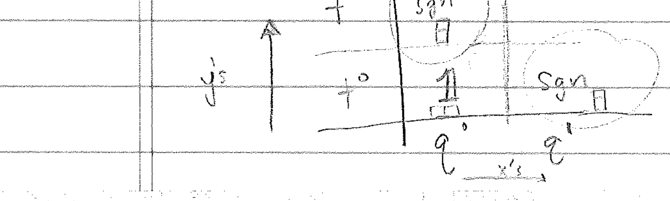
of the ring of diagonal harmonics

$$\underset{\text{diagonally}}{\overset{S_n}{\text{acts}}} \circlearrowright = \mathbb{C}[\overset{\underline{x}}{x_1 \ldots x_n}, \overset{\underline{y}}{y_1 \ldots y_n}] \Big/ \underbrace{\mathbb{C}[\underline{x},\underline{y}]^{S_n}_{+}}_{\substack{\text{ideal gen by}\\ \text{Sym poly of positive}\\ \text{degree!}}}$$

e.g. $n=2$. $\mathbb{C}[x_1,x_2,y_1,y_2] \Big/ \big((x_1+x_2), (y_1+y_2), x_1^2+x_2^2, y_1^2+y_2^2, (x_1y_1+x_2y_2)\big)$

$S_2 \circlearrowleft$

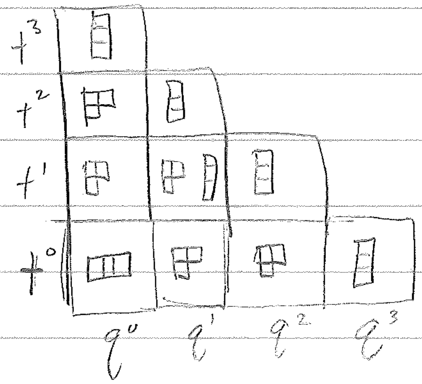
$= \mathbb{C}$-span of $\{1, \underset{-x_2}{\overset{\shortparallel}{x_1}}, \underset{-y_2}{\overset{\shortparallel}{y_1}}\}$.



$\rightsquigarrow \mathrm{Cat}_2(q,t) = q + t$

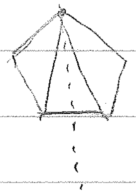
$\Rightarrow X(q,t) = (qt)(q+t) = q^2t + t^2q$

n=3.



$X(q,t) = (qt)^{\binom{3}{2}} (q^3+q^2t+qt^2+t^3+qt)$

$q=1, t=-1 \rightsquigarrow 1.$ corr. to

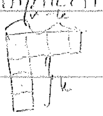


3. REU Problem 4

(a) Prove <u>Conj</u> (Rao–Suk, 2019) For n odd, one also has a DSP:

$\left(X = \left\{ \begin{array}{l} \text{dissections of } (n+2)\text{-gon} \\ \text{with } k \text{ diagonals} \end{array} \right\} \circlearrowright I_2(n+2),\ X(q,t) = (qt)^{\underset{\circ}{P}} \hat{\tilde{S}}_{n,n-1+k}(q,t) \right)$

$\downarrow$ "little" q-t Schroder polynomial giving the bigraded multiplicities for



in $\mathbb{C}[x,y] / \mathbb{C}[x,y]^{S_n}$.

i.e. $\hat{\tilde{S}}_{3,1}(q,t) = q^2+qt+t^2+q+t$. for ⊞

(b). Generalize to $X(q,t) = (q,t)^{\underset{\circ}{P}} \left((q,t)\text{-Fuss-Catalan poly} \right)$

and $X = \left\{ \begin{array}{c} \text{quadrangulations, pentagonal rations.} \\ \text{of polygons} \end{array} \right\}$

CSP version of this is in Gu-Fu. $\longrightarrow$ .. and to (q,t)-Fuss-Schröder polynomials and Fuss dissections.

(c). Maybe even a rational (q-t)-Catalan version?

$$\boxed{\text{(d) Q: What is a DSP for } \underline{n \text{ even}}\ \underset{\circ}{P}}$$