REU 2019 Day 6

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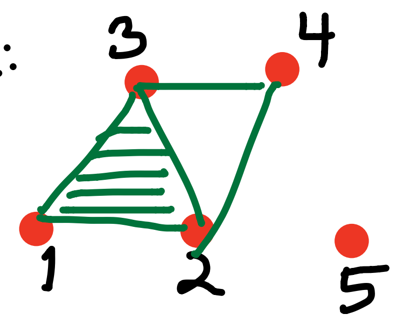
Virtual Stanley-Reisner Theory

Example: A simplicial complex Δ on $\{1,2,3,4,5\}$:

$$\Delta = \{\text{all subsets of } \{1,2,3\}, \{2,4\}, \{3,4\}, \{5\}\}$$

$$= \{\text{subsets above plus } \{1\},\{2\},\{3\},\{4\}, \{1,3\},\{2,3\},\{1,2\}, \emptyset\}$$

Picture:



DEF'N: An abstract simplicial complex Δ on a vertex set $\{1, 2, \ldots, n\}$ is a collection of subsets, called faces or simplices, that is closed under taking subsets.

A simplex/face $\sigma \in \Delta$ of cardinality $|\sigma| = i+1$ has dimension i and is called an i-face of Δ.

CONVENTION: $\dim(\{\emptyset\}) = -1$

The dimension of Δ, $\dim(\Delta)$ is the maximum $\max_{\sigma \in \Delta} \dim(\sigma)$, except $\dim(\{\}) = -\infty$

the void complex (!)

- Unless $\Delta = \{\}$, $\emptyset \in \Delta$

- Δ is completely determined by its **facets**, i.e. its maximal faces with respect to inclusion

(e.g. facets are $\{1,2,3\}, \{2,4\}, \{3,4\}, \{5\}$ in above example)

Stanley-Reisner rings

$\mathbb{k}$ any field (e.g. $\mathbb{k} = \mathbb{C}$)

$S := \mathbb{k}[x_1, x_2, \ldots, x_n]$

$\sigma \subseteq \{1,2,\ldots,n\} \rightsquigarrow x^\sigma := \prod_{i \in \sigma} x_i$

a **squarefree** monomial

DEF'N: The Stanley-Reisner ideal of a simplicial complex Δ is the squarefree monomial ideal

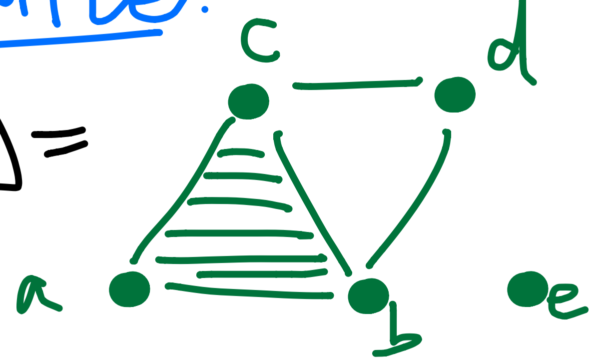
$$I_\Delta := \langle x^\tau \mid \tau \notin \Delta \rangle \quad (**)$$

Then $\Bbbk[\Delta] := S/I_\Delta$ is the Stanley-Reisner ring of Δ

Here I_Δ is generated by the x^τ where τ is a minimal non-face of Δ

EXAMPLE:

$\Delta =$



$$I_\Delta \subset k[a,b,c,d,e] = S$$

$$I_\Delta = \langle ad, bcd, ac, be, ce, de \rangle$$

$$= \langle d,e \rangle \cap \langle a,b,e \rangle \cap \langle a,c,e \rangle \cap \langle a,b,c,d \rangle$$

facets $\{a,b,c\}$ $\quad$ $\{c,d\}$ $\quad$ $\{b,d\}$ $\quad$ $\{e\}$

PROPOSITION (*):

$$I_\Delta = \bigcap_{\text{faces } \sigma \in \Delta} \langle x_i \mid i \notin \sigma \rangle$$

actually facets are enough

Minimal free resolution:

$$F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_n} F_n \leftarrow 0 : F$$

with

$$F_i = \bigoplus_{\tau \in \mathbb{Z}^n} \underbrace{S(-\tau)}_{S \text{ with } 1 \text{ in degree } \tau}^{\beta_{i,\tau}}$$

Here $\operatorname{degree}(x_i) = \bar{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i^{th} \text{ position} \\ \\ \\ \\ \end{array} \in \mathbb{Z}^n$

e.g. $\operatorname{degree}(x_1^2 x_2 x_4) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^4$

... and $H_i F := \ker \varphi_i / \operatorname{im} \varphi_{i+1}$

should satisfy $H_i F = 0 \quad \forall i \geq 1$

$H_0 F = S/I_\Delta = \Bbbk[\Delta]$

Switched to Macaulay 2 demo computing the minimal free resolution

$$0 \leftarrow S \xleftarrow{\varphi_1} S^6 \xleftarrow{\varphi_2} S^9 \xleftarrow{\varphi_3} S^5 \xleftarrow{\varphi_4} S^1 \leftarrow 0$$

$\varphi_1 = [ad, ae, be, ce, de, bcd]$

φ_2: a 6×9 matrix

φ_3: a 9×5 matrix

$\varphi_4 = \begin{bmatrix} -d \\ c \\ -b \\ a \\ 0 \end{bmatrix}$

The entries in the φ_i's are not unique, but the degrees of its columns are, determined by some topology of Δ...

Let $F_i(\Delta) = \{i\text{-faces of } \Delta\}$

apologies for conflict of terminology with F_i in resolution!

e.g. $F_{-1}(\Delta) = \{\emptyset\}$

$F_0(\Delta) = \{a, b, c, d, e\}$

$F_1(\Delta) = \{ab, ac, bc, bd, cd\}$

$F_2(\Delta) = \{abc\}$

stopped writing set braces, commas

DEF'N: The augmented or reduced chain complex of Δ:

$$\tilde{C}_\cdot(\Delta; \mathbb{k}): \quad 0 \leftarrow \mathbb{k}^{F_{-1}(\Delta)} \xleftarrow{\partial_0} \mathbb{k}^{F_0(\Delta)} \xleftarrow{\partial_1} \mathbb{k}^{F_1(\Delta)} \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_{n-1}} \mathbb{k}^{F_{n-1}(\Delta)} \leftarrow 0$$

with $\partial_i(e_\sigma) := \sum_{j \in \sigma} \underset{(-1)^r \text{ if } j \text{ is the } r^{\text{th}} \text{ element of } \sigma}{\underset{\shortparallel}{\operatorname{sign}(j,\sigma)}} e_{\sigma \setminus j}$

$$0 \leftarrow \overset{|F_{-1}(\Delta)|}{\Bbbk^1} \leftarrow \overset{|F_0(\Delta)|}{\Bbbk^5} \leftarrow \overset{|F_1(\Delta)|}{\Bbbk^5} \leftarrow \overset{|F_2(\Delta)|}{\Bbbk^1} \leftarrow 0$$

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline \emptyset & -1 & -1 & -1 & -1 & -1 \end{array}$$

$$\begin{array}{c|ccccc} & bc & ac & ab & db & dc \\ \hline a & 0 & 1 & 1 & 0 & 0 \\ b & 1 & 0 & -1 & 1 & 0 \\ c & -1 & -1 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & -1 & -1 \\ e & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|c} & abc \\ \hline bc & -1 \\ ac & 1 \\ ab & -1 \\ db & 0 \\ dc & 0 \end{array}$$

$$\tilde{H}_i(\Delta;\Bbbk) := \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}} \cong \begin{cases} \Bbbk^1 & \text{if } i = 0,1 \\ 0 & \text{if } i \neq 0,1 \end{cases}$$

Roughly speaking

$$\dim_{\Bbbk} \tilde{H}_i(\Delta, \Bbbk) = \#\ \text{(independent) } i\text{-dimensional holes in } \Delta$$

THM: [Hochster 1977]

Δ a simplicial complex on $\{1, 2, \dots, n\}$.

View $S = \Bbbk[x_1, \dots, x_n]$ with $\mathbb{Z}^n$-grading

Then $\Bbbk[\Delta]$ has min. free resolution

$$F : F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n-1} \leftarrow 0$$

with $F_i = \bigoplus_{\tau \in \{0,1\}^n} S(-\tau)^{\beta_{i,\tau}}$ where

$$\beta_{i,\tau} = \dim_{\Bbbk} \tilde{H}_{|\tau|-i-1}(\Delta|_\tau, \Bbbk)$$

Here $\Delta|_\tau := \Delta$ restricted to vertices in τ

Switched to Macaulay 2 demo...

e.g. $\tau = (1,1,1,1,1)$

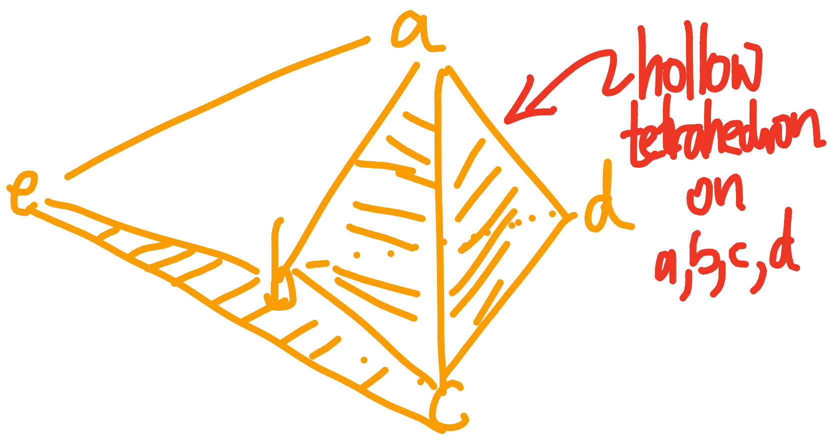
$\Delta|_\tau = \Delta$ has $\tilde{H}_i(\Delta|_\tau) = \begin{cases} k^1 & \text{if } i = 0,1 \\ 0 & \text{else} \end{cases}$

so multidegree τ occurs once in resolution at $F_{|\tau|-0-1} = F_{5-0-1} = F_4$ and also at $F_{|\tau|-1-1} = F_{5-1-1} = F_3$

REU Exercise 13

$\Delta' := \{abc, abd, acd, bcd, bce, ae$
$+$ all subsets thereof $\}$

$=$



(a) Compute $I_{\Delta'}$ in 2 ways:
via definition and Proposition (*)

(b) Compute MFR of $S/I_{\Delta'}$
with $\mathbb{Z}^5$-grading, note $\beta_{i,\tau}$'s

(c) Compute $\tilde{H}_i(\Delta'|_\tau, \mathbb{k})$ for all $\tau \subseteq \{a,b,c,d,e\}$ and compare with Hochster's formula.

Note here:

$$\tilde{H}_i(\Delta', \mathbb{k}) = \begin{cases} 0 & \text{if } i=0 \\ \mathbb{k}' & \text{if } i=1 \\ \mathbb{k}' & \text{if } i=2 \\ 0 & \text{if } i\geq 3 \end{cases}$$

Back to Virtual resolutions...

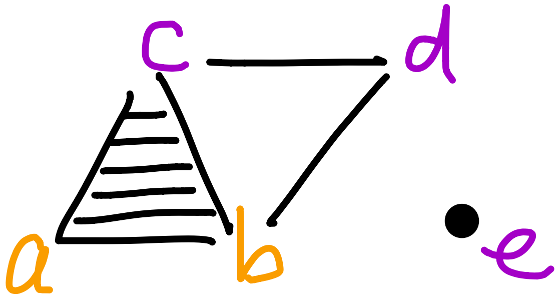
$$X = \mathbb{P}^{\bar{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \dots \times \mathbb{P}^{n_r}$$

$$S = \Bbbk\left[\left\{ x_{ij} \;\middle|\; \begin{matrix} 1 \leq i \leq r \\ 0 \leq j \leq n_i \end{matrix} \right\}\right]$$

with degree $(x_{ij}) := \bar{e}_i \in \mathbb{Z}^r$

EXAMPLE: $X = \mathbb{P}^1 \times \mathbb{P}^2$

$$S = \Bbbk[\underbrace{\overset{a}{x_{10}}, \overset{b}{x_{11}}}_{\deg\binom{1}{0}}, \underbrace{\overset{c}{x_{20}}, \overset{d}{x_{21}}, \overset{e}{x_{22}}}_{\deg\binom{0}{1}}]$$

$\Delta =$ 

$B = \langle a,b\rangle \cap \langle c,d,e\rangle$

Recall the Irrelevant ideal

$$B = \bigcap_{i=1}^{r} \langle x_i \mid 0 \le j \le n_i \rangle \subset S$$

$$\Updownarrow$$

$$V(B) = \emptyset \text{ in } \mathbb{P}^{\bar{n}}$$

Recall $X = \left(\mathbb{k}^{|n|+r} \setminus V(B)\right) \Big/ (\mathbb{k}^{\times})^{r}$

Want to again use B to make the resolutions smaller

$\forall$ ideals $I \subseteq S$, $\forall \bar{a} \in \mathbb{Z}^r_{\geq 0}$

$$\mathcal{V}(I) = \mathcal{V}(I) \cup \underbrace{\mathcal{V}(B^{\bar{a}})}_{= \cancel{\varnothing}}$$

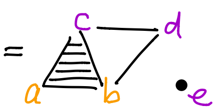
$$\overset{\text{Exercise 1(c)}}{=} \mathcal{V}(I \cap B^{\bar{a}})$$

where $B^{\bar{a}} := \bigcap_{i=1}^{r} \langle x_{ij} \mid j = 1, \ldots, n_i \rangle^{a_i}$

e.g.

$$I_\Delta = \langle d,e\rangle \cap \langle a,b,e\rangle \cap \langle a,c,e\rangle \cap \langle a,b,c,d\rangle$$

$$\langle a,b,e\rangle \supseteq \langle a,b\rangle \qquad \langle a,b,c,d\rangle \supseteq \langle a,b\rangle$$

$\Delta =$ 

$B = \langle a,b\rangle \cap \langle c,d,e\rangle$

$$V(\langle a,b,c\rangle) \subseteq V(\langle a,b\rangle) \subset V(B)$$

since $\langle a,b\rangle \supset B$

Hence $V(I_\Delta) = V(\langle d,e\rangle \cap \langle a,c,e\rangle)$

This is in the Macaulay 2 code as

saturate(I_Δ, B)

REU Exercise 14

Δ a simplicial complex

$X = \mathbb{P}^m \times \mathbb{P}^n$ (or $\mathbb{P}^{\bar{n}}$ if you like)

Describe how to modify the definition (**) of I_Δ and Proposition (*) to obtain J_Δ, K_Δ respectively, with

$$V(I_\Delta) = V(K_\Delta) = V(J_\Delta).$$

The goal is to use the fewest monomials and intersectands, respectively.

REU Problem 6 : $X = \mathbb{P}^n$

How does Δ encode properties of virtual resolutions of S/I_Δ?

- the shortest length of a vres of S/I_Δ?
- if/when is it the shortest length $= |\bar{n}| - \dim V(I_\Delta)$ (virtual Cohen-Macaulay)
- if/when $\exists$ a vres which is a Koszul complex (virtual complete intersection)
- can we compute $\beta_{i,\tau}$'s in various vres's of S/I_Δ, e.g. res $S/I_\Delta \cap B^{\bar{a}}$
  res $S/\text{saturate}(I_\Delta, B)$
  virtualOfPair
  res$(S/\langle \text{subset of gens of } I_\Delta \rangle)$

$\Bigg\}$ (***)

Example: $X = \mathbb{P}^1 \times \mathbb{P}^1$

$([x_0 : x_1], [y_0 : y_1])$

$$I = \langle x_0, y_0 \rangle \cap \langle x_1, y_1 \rangle$$

$\mapsto V(I) = 2$ points

resolution of I:

$I \cap \langle x_0, x_1 \rangle$

$$S/I \twoheadleftarrow S \leftarrow \begin{matrix} S\begin{pmatrix}-2\\0\end{pmatrix}^1 \\ \oplus \\ S\begin{pmatrix}-1\\-1\end{pmatrix}^2 \\ \oplus \\ S\begin{pmatrix}0\\-2\end{pmatrix}^1 \end{matrix} \leftarrow \begin{matrix} S\begin{pmatrix}-2\\-1\end{pmatrix}^2 \\ \oplus \\ S\begin{pmatrix}-1\\-2\end{pmatrix}^2 \end{matrix} \leftarrow S\begin{pmatrix}-2\\-2\end{pmatrix}^1 \leftarrow 0$$

$I \cap \langle y_0, y_1 \rangle$

$$I = \langle x_0 x_1, x_0 y_1, x_1 y_0, y_0 y_1 \rangle$$

$I \cap B$: $S^1 \leftarrow S\binom{-1}{-1}^2 \leftarrow S\binom{-2}{-2}^1$

Koszul complex

REU Exercise 15

$X = \mathbb{P}^1 \times \mathbb{P}^1$

$S = [a,b,c,d]$

$\cup|$

$B = \langle a,b \rangle \cap \langle c,d \rangle$

(a) Write out all Δ's with $V(I_\Delta) \neq \emptyset$ (non-irrelevant Δ's)

(b) What conditions on Δ make $V(I_\Delta) \neq \emptyset$?

(c) Find shortest/nice vres's for S/I_Δ.

(use ideas from (***))

- when is it vCM? (virtual Cohen-Macaulay)
- when is it a vCI? (virtual complete intersection)