Ribbon Lattices and Ribbon Functions

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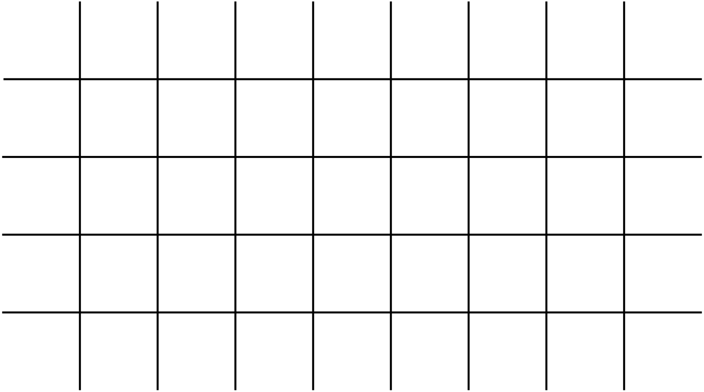
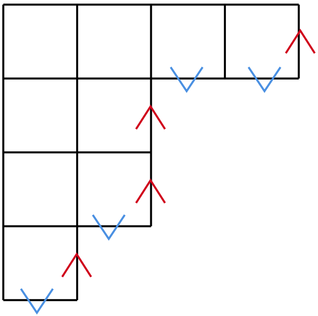
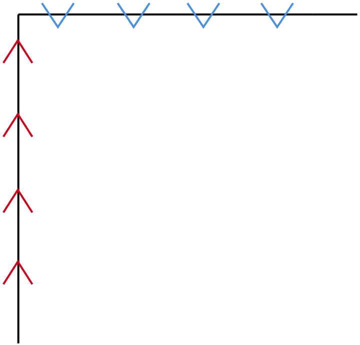
Boundary Conditions: For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ create a grid with r rows and $\lambda_1 + r$ columns. For example, if $\lambda = (4, 2, 2, 1)$ then $r = 4$ and $\lambda_1 + r = 8$

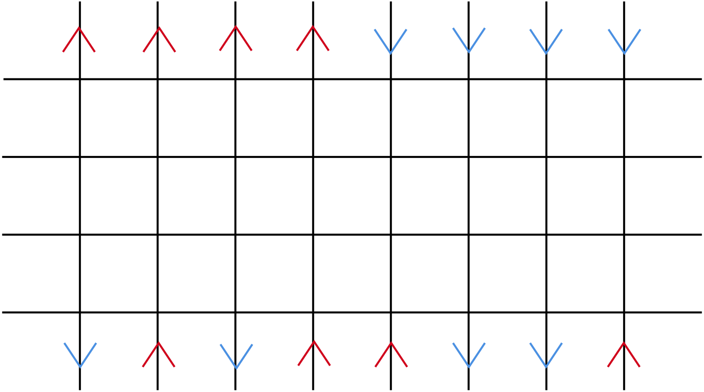


Take the path sequence of λ and the empty partition:

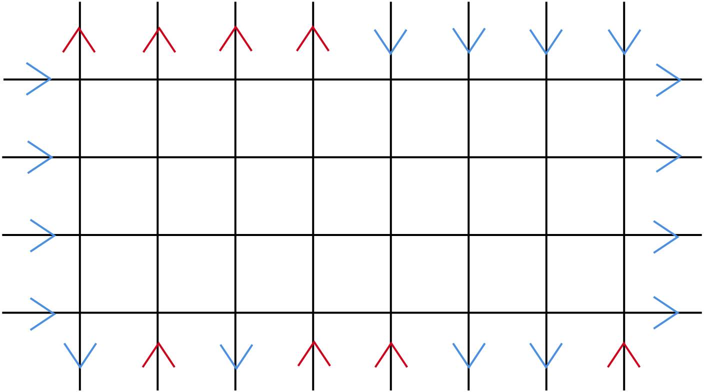




Place the path sequence of I at the bottom of the grid and the path sequence of $\emptyset$ at the top of the grid



Finally place only right arrows along the horizontal boundary:



Theorem

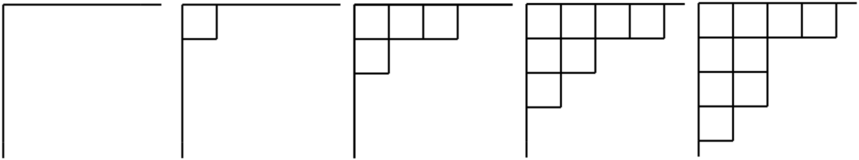
Denote the partition function with these boundary conditions $\mathcal{Z}_\lambda$. Then $\mathcal{Z}_\lambda$ is equal to the Schur function s_λ for any partition λ.

- Idea of Proof: Construct a weight preserving bijection between semistandard Young tableaux of shape λ and fillings of the lattice with the boudary conditions just described.

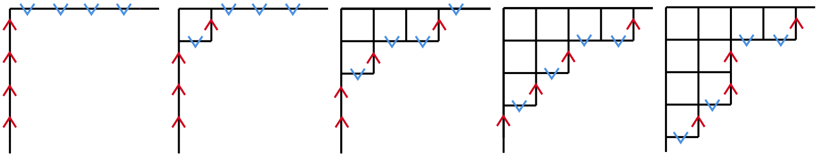
Example:

1	2	2	3
2	3		
3	4		
4			

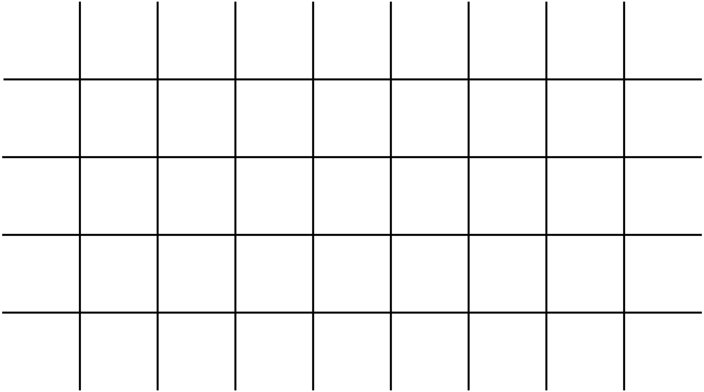
Write the tableaux as a sequence of partitions:



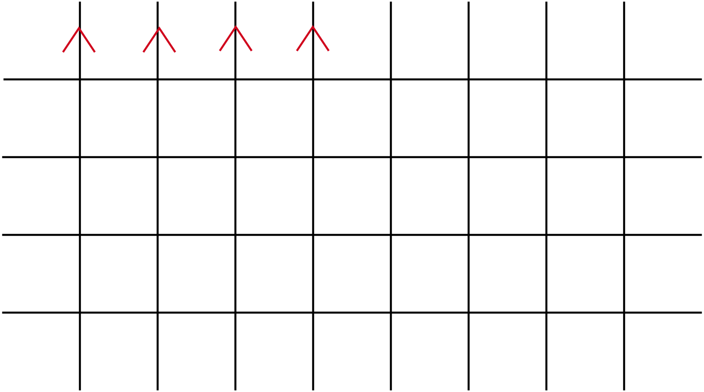
Take the path sequence of each partition:



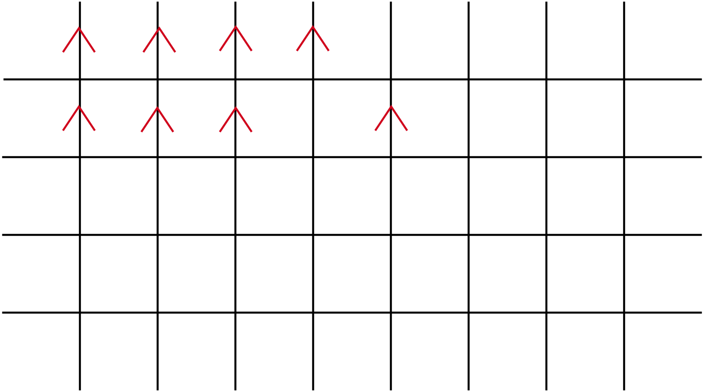
Start with 4 by 8 lattice as before:



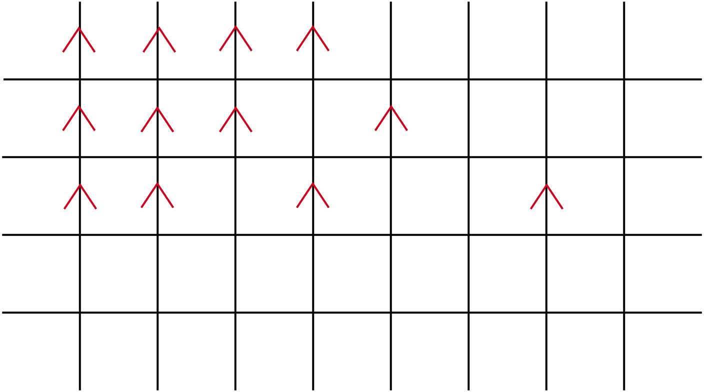
Add path sequence of empty partition to the top:

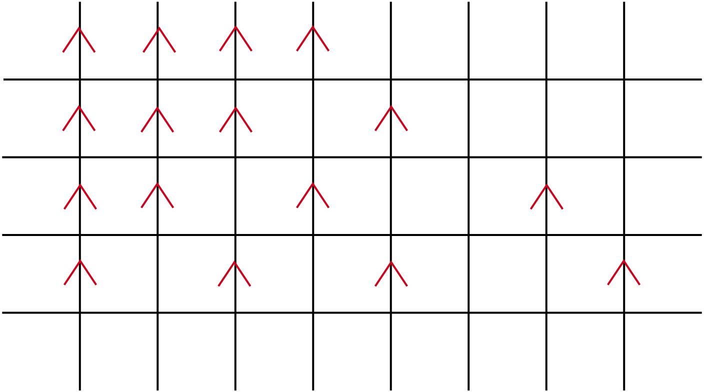


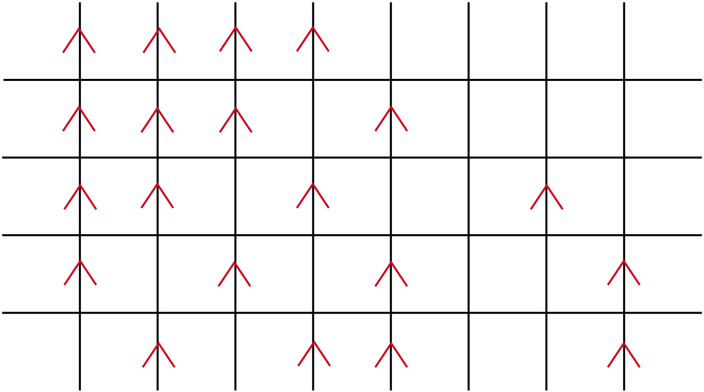
Add path sequence of second partition to the next row:



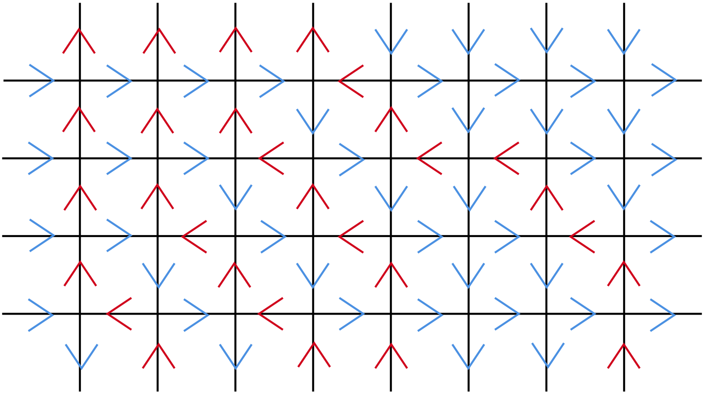
Continue:







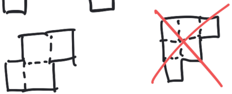
Then there is only one possible admissible state with this choice of up arrows.



Ribbon Tableaux

- **n-ribbon** := Some skew-shape containing n "unit boxes" without  2×2 square.

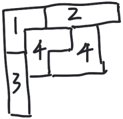
  e.g. 3-ribbon 

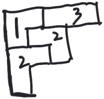
  4-ribbon 

- **Semistandard n-ribbon Tableaux**

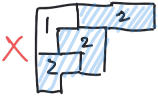
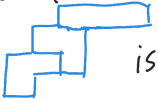
  Tile a Young diagram with n-ribbons then fill each ribbon with numbers like a SSYT

  ☆ The part with same number forms a **horizontal strip** (define later)

Eg.  $\in RT_3^4(\lambda)$ $\lambda = 5, 4, 4, 1, 1$

~~Eg.~~  ,  is not a Young diagram.

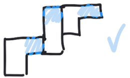
Infact, NO RT for this diagram.

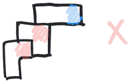
  is not a horizontal strip.

A horizontal strip is a Collection of ribbons which forms a skew shape, Such that

- The Upper Right box of each ribbon has to touch the air, i.e. Nothing above it.

E.g.





Spin

- The spin of a ribbon is height -1.

  e.g. spin() $= 2$

- The spin of a ribbon Tableau is sum of spin.

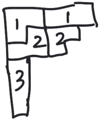
  e.g spin() $= 1 + 1 + 2 = 4$

Ribbon Function (Lascoux, Leclerc & Thibon)

- Let λ/μ be a skew partition tilable by n-ribbons.

$$G^{n}_{\lambda/\mu}(\underline{x}, q) = \sum_{T} q^{\mathrm{spin}(T)} \underline{x}^{\mathrm{wt}(T)}$$

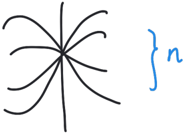
e.g.



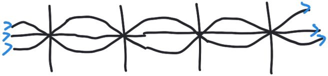
$\Rightarrow q^{6} x_1^2 x_2^2 x_3$

(Thm) Ribbon F'ns are Symmetric

Ribbon Lattice Models. for n-ribbon f'n.

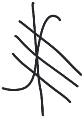


, with (n+1) in-arrow (out-arrow)

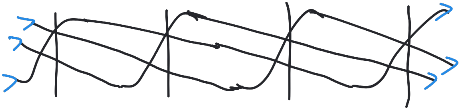


....

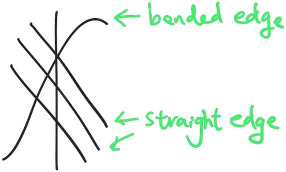
Think of the vertex as:



... then



The weight of Ribbon vertex



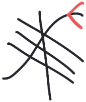
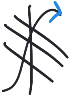
$$\text{weight}(v) = \delta \cdot q^{\delta(v)} x_i^{\varepsilon(v)}$$

(if in the ith row)

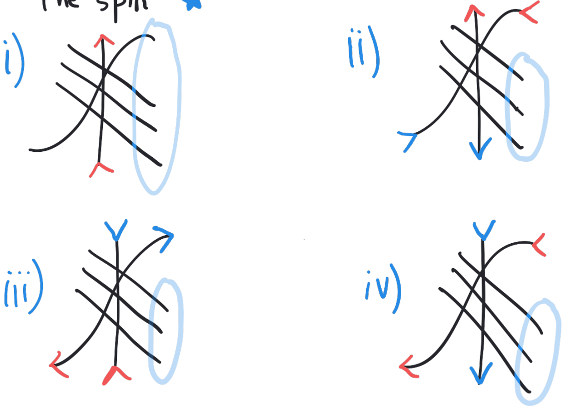
① Don't allow changing arrow on straight edge :

e.g $\delta = 1$ if  $\delta = 0$ if 

② $\varepsilon(v) = 1$ if a left arrow entering through bended edge

$\varepsilon($  $) = 1$ $\varepsilon($  $) = 0$

③ The spin ★

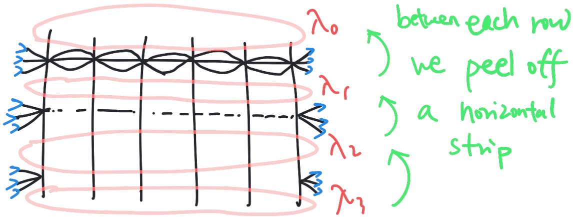


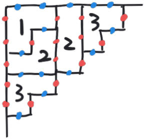
$\sigma(v) = \#$ of $<$ in ◯

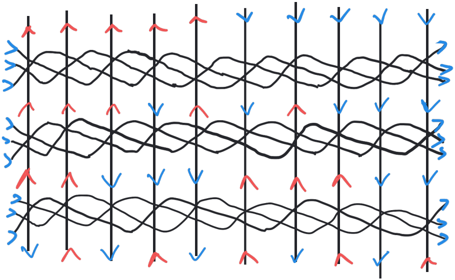
From Ribbon Tableaux to Lattice model.

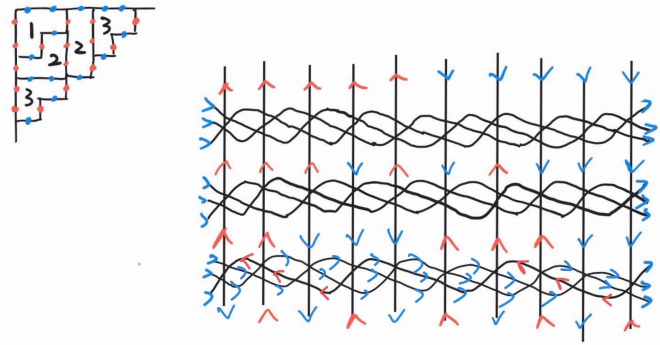
- Think of Ribbon Tableaux as sequence of partitions.

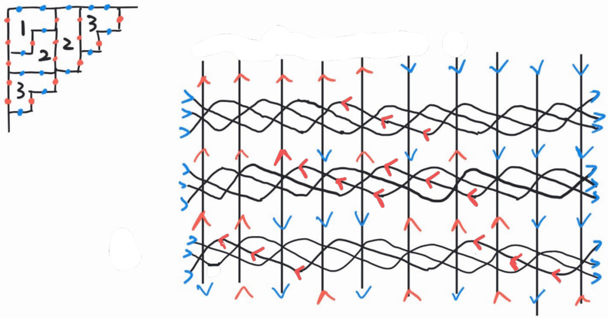
- Same boundary condition

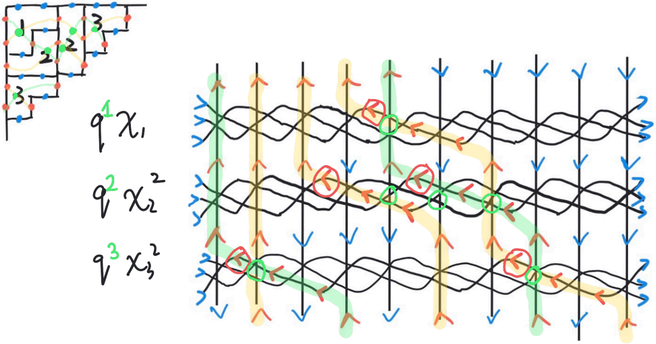



1
2
2
3
3

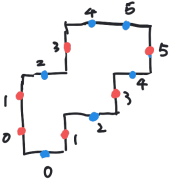


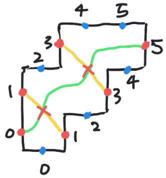

1
2
2
3
3


1
2
2
3
3


$q^1 x_1$
$q^2 x_2^2$
$q^3 x_3^2$

• peeling off one n-ribbon (if time)

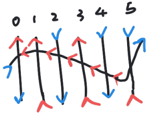




i) numbering the two edge sequence (blue red dots) from 0 to n

ii) The n-th • is moved to the 0-th •, everything else stays.

iii) in the Lattice



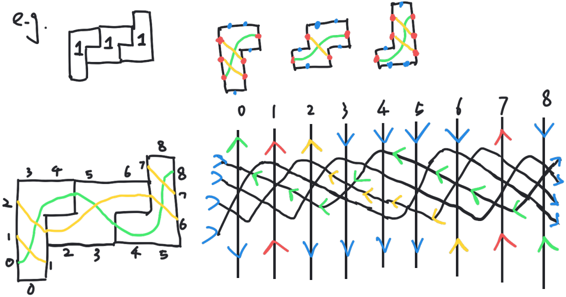
iv) # intersection = spin

peeling off one horizontal ribbon strip. (if time)

- b/c the top-right box of each ribbon has nothing above it.

  we can glue small ribbons up to make the entire strip.

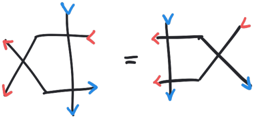
e.g.



Yang-Baxter Equation (a.k.a star-triangle equality)

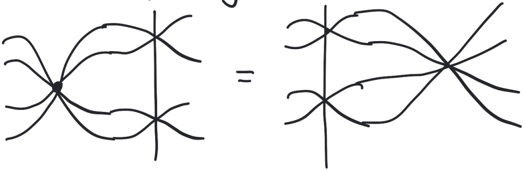
want a new set of vertex  .... with certain weight.

Such that



$$\sum\left(wt(LHS)\right) = \sum\left(wt(RHS)\right)$$ for all boundary

Star-triangle for larger ribbon looks like:



- We <u>conjecture</u> that our lattice model is solvable i.e. there exist YBEs.
- The YBE for 1, 2, 3-ribbon lattice is computed via SAGE.

Application of the Lattice model.

· We can derive various identities of Ribbon F'n using our lattice.

e.g. dual Cauchy identity

· Pieri rule ·

$q=1$ ribbon f'n is product of Schur f'ns.

Thank You!