REU Problem 3: Questions on Generating Functions via Cluster Algebras

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REU in Algebra and Combinatorics

Introduction to Cluster Algebras

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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Binomial Exchange Relations via Quivers (Directed Graphs)

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For example, if $Q =$ $\begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, then

$$x_1 x_1' = 1 + x_2^2 \qquad\qquad x_2 x_2' = x_1^2 x_3 + x_4$$

$$x_3 x_3' = x_4 + x_2 \qquad\qquad x_4 x_4' = x_2 + x_3.$$

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If Q has n vertices, we obtain n new seeds (startng from the initial seed) by mutating in n directions: e.g.

$$\{x_1, x_2, x_3, x_4\}$$

$$\{x_1', x_2, x_3, x_4\} \quad \{x_1, x_2', x_3, x_4\} \quad \{x_1, x_2, x_3', x_4\} \quad \{x_1, x_2, x_3, x_4'\}$$

Exchange Patterns for New Seeds via Quiver Mutation

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Note: Mutation is an **involution**, meaning that $\mu_j^2 Q = Q$ for any vertex j.

Exchange Matrices Representing Quivers (Directed Graphs)

Given a quiver Q (i.e. a directed graph) with n vertices, we build an n-by-n skew-symmetric matrix $B_Q = [b_{ij}]_{i=1,\ j=1}^{n}$ whose entries are

$$b_{ij} = (\#\text{arrows from } i \text{ to } j) - (\#\text{arrows from } j \text{ to } i).$$

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If $Q = 1 \rightarrow 2$, then $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, if $Q = 1 \Rightarrow 2$, then $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$,

and if $Q = $ $\begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, then $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$.

Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices B_Q. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the **mutation of** $B_Q = [b_{ij}]$ **at k**, by

$$b'_{ij} = \begin{cases} -b_{ij} \text{ if } i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+ \text{ otherwise} \end{cases}$$

using $[\alpha]_+ = \max(\alpha, 0)$.

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Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

$$\mu_1(B) = \mu_2(B) = -B \ \text{ and } \ x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

$$\mu_1(B) = \mu_2(B) = -B \quad \text{and} \quad x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

Example 1 $(b = c = 1)$:

$$x_3 = \frac{x_2 + 1}{x_1}. \quad x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2}.$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1 x_2} + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2 (x_2 + 1)} =$$

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

Quivers and Exchange Matrices with Principal Coefficients

Given a quiver Q on n vertices, and its associated n-by-n matrix B_Q, we build the corresponding $2n$-by-n exchange matrix with principal coefficients via $\widetilde{B_Q} = \begin{bmatrix} B_Q \\ I_n \end{bmatrix}$, where I_n denotes the n-by-n identity matrix.

Equivalently, $\widetilde{B_Q}$ corresponds to the exchange matrix of the framed quiver $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ with a single arrow from $i' \to i$ for each $1 \leq i \leq n$.

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Example:

If $Q = $ $\begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, then $\widetilde{Q} = $ $\begin{tikzcd} 1' \arrow[d] & 2' \arrow[d] & 3' \arrow[d] & 4' \arrow[d] \\ 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$

$$B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \widetilde{B_Q} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Examples of mutation with principal coefficients

As framed quivers (for the case of a type A_2 quiver):

$$\begin{tikzcd} 1' \arrow[d] & 2' \arrow[d] & \xrightarrow{\mu_1} \\ 1 \arrow[r] & 2 & \end{tikzcd}$$

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As framed quivers (for the case of a type A_2 quiver):

$$
\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] \\
1 \arrow[r] & 2
\end{tikzcd}
\rightarrow^{\mu_1}
\begin{tikzcd}
1' \arrow[dr] & 2' \arrow[d] \\
1 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\rightarrow^{\mu_2}
$$

As framed quivers (for the case of a type A_2 quiver):

$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] &
\end{tikzcd}$$

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1' \arrow[d] & 2' \arrow[d] & \rightarrow^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \rightarrow^{\mu_2} & 1' & 2' \arrow[dl] & \rightarrow^{\mu_1} & 1' & 2' & \rightarrow^{\mu_2} \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] &
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1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} & 1' & 2' & \to^{\mu_2} & 1' \arrow[dr] & 2' & \to^{\mu_1} \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[ul] \arrow[u] & & 1 \arrow[ur] & 2 \arrow[ul] \arrow[l] & & 1 \arrow[ur] \arrow[r] & 2 &
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\begin{tikzcd}[column sep=small]
1' \arrow[d] & 2' \arrow[d] \\
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\end{tikzcd}
\rightarrow^{\mu_1}
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1 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\rightarrow^{\mu_2}
\begin{tikzcd}[column sep=small]
1' & 2' \arrow[dl] \\
1 \arrow[r] & 2 \arrow[ul, crossing over] \arrow[u]
\end{tikzcd}
\rightarrow^{\mu_1}
\begin{tikzcd}[column sep=small]
1' & 2' \\
1 \arrow[ur] & 2 \arrow[l] \arrow[ul, crossing over]
\end{tikzcd}
\rightarrow^{\mu_2}
\begin{tikzcd}[column sep=small]
1' \arrow[dr] & 2' \\
1 \arrow[r] \arrow[ur, crossing over] & 2
\end{tikzcd}
\rightarrow^{\mu_1}
\begin{tikzcd}[column sep=small]
1' \arrow[dr] & 2' \arrow[dl, crossing over] \\
1 & 2 \arrow[l]
\end{tikzcd}
$$

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\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} & 1' & 2' & \to^{\mu_2} & 1' \arrow[dr] & 2' & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[ul] \arrow[u] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & & 1 \arrow[r] \arrow[ur] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}
$$

As $2n$-by-n exchange matrices:

$$
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1}
$$

Examples of mutation with principal coefficients

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$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \rightarrow^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \rightarrow^{\mu_2} & 1' & 2' \arrow[dl] & \rightarrow^{\mu_1} & 1' & 2' & \rightarrow^{\mu_2} & 1' \arrow[dr] & 2' & \rightarrow^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & & 1 \arrow[r] \arrow[ur] & 2 & & 1 & 2 \arrow[l]
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1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & & 1 \arrow[r] \arrow[ur] & 2 & & 1 & 2 \arrow[l]
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Examples of mutation with principal coefficients

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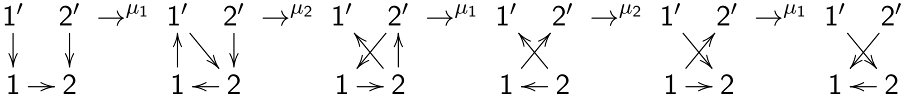

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$$\rightarrow^{\mu_2}$$

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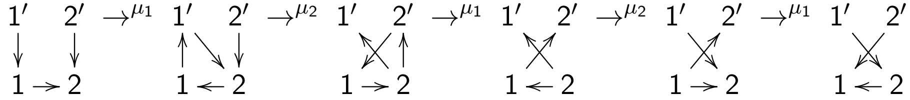

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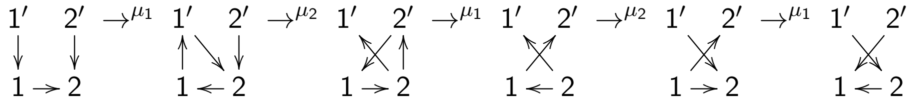

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Examples of mutation with principal coefficients

Starting with the framed quiver for the case of the Kronecker quiver

$$\begin{array}{ccc} 1' & & 2' \\ \downarrow & & \downarrow \\ 1 & \rightrightarrows & 2 \end{array}$$

As $2n$-by-n exchange matrices:

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1}$$

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$\rightarrow^{\mu_2}$

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$$\to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \to^{\mu_1}$$

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$$\to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \to^{\mu_2}$$

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Examples of mutation with principal coefficients

Starting with the framed quiver for the case of the Kronecker quiver

$$\begin{array}{ccc} 1' & & 2' \\ \downarrow & & \downarrow \\ 1 & \Rightarrow & 2 \end{array}$$

As $2n$-by-n exchange matrices:

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 3 & -2 \\ 2 & -1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -3 & 4 \\ -2 & 3 \end{bmatrix}$$

$$\to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 7 & -6 \\ 6 & -5 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ -7 & 8 \\ -6 & 7 \end{bmatrix} \to \cdots$$

Cluster Variables with Principal Coefficients

Framed quivers for a type A_2 quiver:

$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \xrightarrow{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \xrightarrow{\mu_2} & 1' & 2' \arrow[dl] & \xrightarrow{\mu_1} & 1' & 2' & \xrightarrow{\mu_2} & 1' \arrow[dr] & 2' & \xrightarrow{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & & 1 \arrow[r] \arrow[ur] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}$$

$$\{x_1, x_2\} \to \{x_3, x_2\} \to \{x_3, x_4\} \to \{x_5, x_4\} \to \{x_5, x_1\} \to \{x_2, x_1\}$$

$$x_3 = \frac{y_1 + x_2}{x_1},$$

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$$x_3 = \frac{y_1 + x_2}{x_1}, \quad x_4 = \frac{y_1 y_2 + x_3}{x_2} =$$

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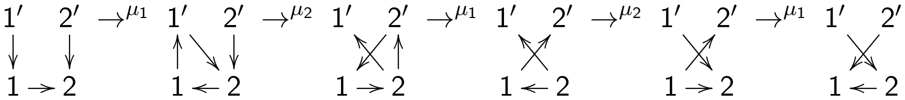

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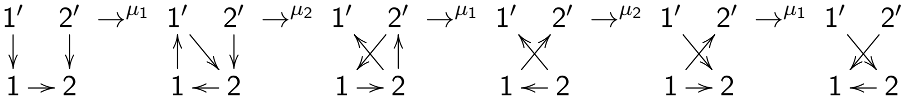

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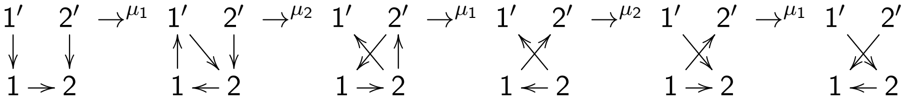

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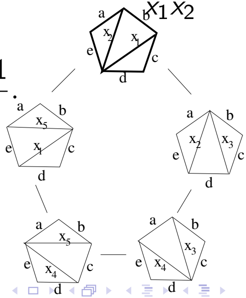

Cluster Variables for the Kronecker quiver, i.e. $\mathcal{A}(2,2)$

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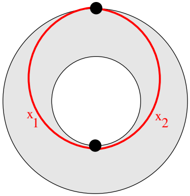

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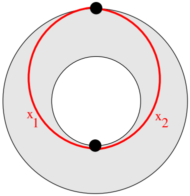

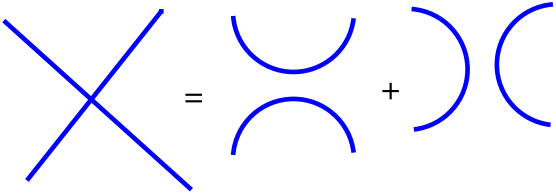

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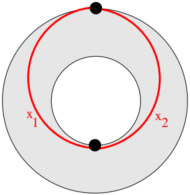

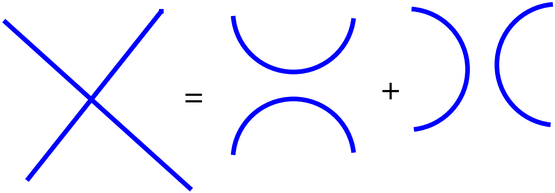

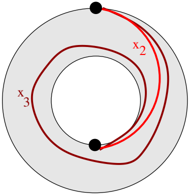

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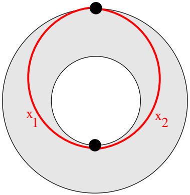

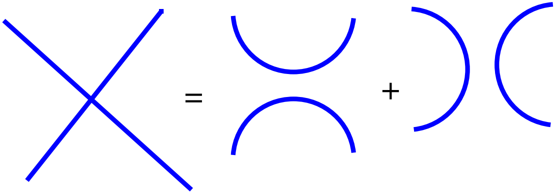

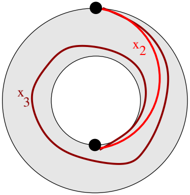

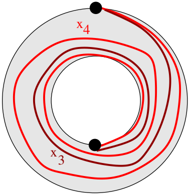

Cluster variables x_n's correspond to arcs that wind around the annulus.

Example of Type A_3 with Principal Coefficients

Example 3: Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ and the initial exchange pattern

$$x_1x_1' = y_1 + x_2, \quad x_2x_2' = x_1x_3y_2 + 1, \quad x_3x_3' = y_3 + x_2, \text{ i.e. } \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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REU Exercise # 3.1: Each seed of $\mathcal{A}$ corresponds to a triangulation of a hexagon such that chords correspond to cluster variables. Furthermore, $\mathcal{A}$ is a cluster algebra of finite type, with cluster variables given as:

$$\left\{ x_1, x_2, x_3, \frac{y_1 + x_2}{x_1}, \frac{x_1 x_3 y_2 + 1}{x_2}, \frac{y_3 + x_2}{x_3}, \frac{x_1 x_3 y_1 y_2 + y_1 + x_2}{x_1 x_2}, \right.$$
$$\left. \frac{x_1 x_3 y_2 y_3 + y_3 + x_2}{x_2 x_3}, \frac{x_1 x_3 y_1 y_2 y_3 + y_1 y_3 + x_2 y_3 + x_2 y_1 + x_2^2}{x_1 x_2 x_3} \right\}.$$

Relationship with Total Positivity

Given a 2-by-2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2$, what is a sufficient condition to check whether it is totally positive, meaning that all minors are positive?
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Together, these 5 algebraic elements generate a cluster algebra structure of type A_1 (i.e. a binomial exchange between a and d with b, c, Δ frozen).

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Warning: Even if $a > 0, c > 0, d > 0, ad - bc > 0$, it is still possible $b \leq 0$. (Ditto if we leave out c or $\Delta = ad - bc$.)

Relationship with Total Positivity

Given a 3-by-3 matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in GL_3$, how do you check whether it is totally positive, meaning that all minors are positive?

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Answer: It is sufficient to check that $c > 0$, $g > 0$, $bf - ce > 0$, $dh - eg > 0$ and four other conditions

(for a total of 8 verifications rather than all 19 minors).

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Answer: It is sufficient to check that $c > 0$, $g > 0$, $bf - ce > 0$, $dh - eg > 0$ and four other conditions

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There are exactly 50 such overlapping sets of four conditions. These 50 algebraic elements generate a cluster algebra structure of type D_4 (with binomial exchange relations among the elements).

More Matrix Minors: Coordinate Ring of Grassmannian

Let $Gr_{2,n+3} = \{V | V \subset \mathbb{C}^{n+3},\ \dim V = 2\}$ planes in $(n+3)$-space

Elements of $Gr_{2,n+3}$ represented by 2-by-$(n+3)$ matrices of full rank.

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Plücker coordinates $p_{ij}(M) =$ det of 2-by-2 submatrices in columns i and j.

The coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ is generated by all the p_{ij}'s for $1 \leq i < j \leq n+3$ subject to the Plücker relations given by the 4-tuples

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \text{ for } i < j < k < \ell.$$

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Claim. $\mathbb{C}[Gr_{2,n+3}]$ has the structure of a type A_n cluster algebra. Clusters are each maximal algebraically independent sets of p_{ij}'s.

Each have size $(2n+3)$ where $(n+3)$ of the variables are frozen and n of them are exchangeable.

More Matrix Minors: Coordinate Ring of Grassmannian

Cluster algebra structure of $Gr_{2,n+3}$ as a triangulated $(n+3)$-gon.

Frozen Variables / Coefficients $\longleftrightarrow$ sides of the $(n+3)$-gon

Cluster Variables $\longleftrightarrow \{p_{ij} : |i-j| \neq 1 \mod (n+3)\} \longleftrightarrow$ diagonals

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Seeds $\longleftrightarrow$ triangulations of the $(n+3)$-gon

Clusters $\longleftrightarrow$ Set of p_{ij}'s corresponding to a triangulation

Can exchange between various clusters by flipping between triangulations.

From Cluster Variables to F-polynomials

If we start with a framed quiver $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ and the intial cluster $\{x_1, \ldots, x_N\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we iterate cluster mutation with the extra restriction of disallowing mutation at vertices i'.

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Consequently, the binomial exchange relation for cluster mutation

$$x'_k = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{k=1}^{n} x_i^{[-b_{ik}]_+}}{x_k} = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k}$$

will involve $y_1, y_2, \ldots, y_n$ in the numerator, but never in the denominator.

From Cluster Variables to F-polynomials

If we start with a framed quiver $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ and the intial cluster $\{x_1, \ldots, x_N\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we iterate cluster mutation with the extra restriction of disallowing mutation at vertices i'.

Consequently, the binomial exchange relation for cluster mutation

$$x_k' = \frac{\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{k=1}^n x_i^{[-b_{ik}]_+}}{x_k} = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k}$$

will involve $y_1, y_2, \ldots, y_n$ in the numerator, but never in the denominator.

By letting $x_1 = x_2 = \cdots = x_n = 1$, and iterating cluster mutation, we replace cluster variables (which are Laurent polynomials in x_i's and y_i's) with polynomials in $y_1, y_2, \ldots, y_n$, which are called **F-polynomials**.

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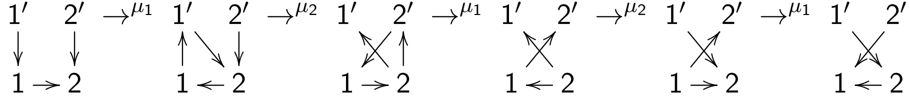
Example 1: Cluster Algebra of Type A_2 with principal coefficients



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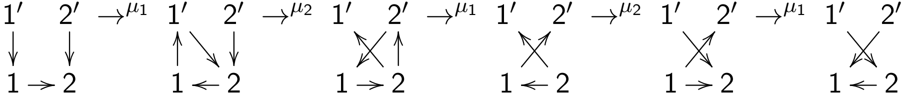

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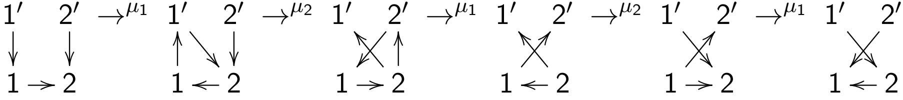

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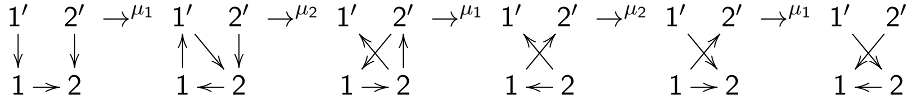

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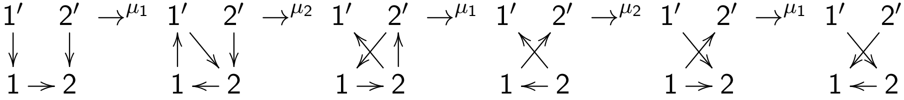

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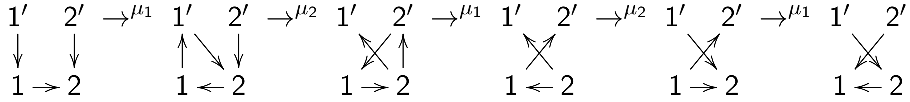

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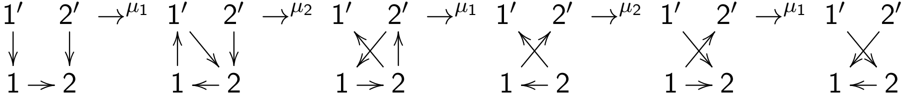

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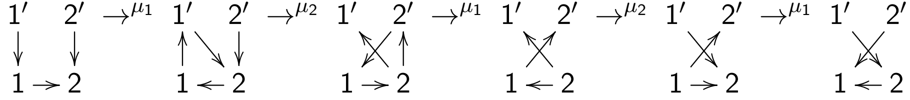

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c-vectors

Given a framed quiver $\widetilde{Q}$ and its images under a sequence of mutations, we define the c-vectors associated to the seed t by

$$\mathbf{c_{j,t}} = [c_{1j}, c_{2j}, \ldots, c_{nj}]^T$$

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In particular, the initial c-vectors, for seed t_0, equal unit vectors

$$\mathbf{c_{1,t_0}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{c_{2,t_0}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{c_{n,t_0}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

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and then recursively c-vectors mutate alongside quivers and exchange matrices. Letting $\mathbf{c_{j,\ \mu_k t}} = [c'_{1j}, c'_{2j}, \ldots, c'_{nj}]^T$ for each $1 \leq j \leq n$, we have

$$c'_{ij} = \begin{cases} -c_{ij} = -c_{ik} \text{ if } j = k \\ c_{ij} + [c_{ik}]_+[b_{kj}]_+ - [-c_{ik}]_+[-b_{kj}]_+ \text{ otherwise} \end{cases}.$$

Example 1 Revisited: c-vectors for $1 \to 2$

$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} & 1' & 2' & \to^{\mu_2} & 1' \arrow[dr] & 2' & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[ul] \arrow[u] & & 1 \arrow[ur] & 2 \arrow[ul] \arrow[l] & & 1 \arrow[ur] \arrow[r] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}$$

$$t_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} t_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} t_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

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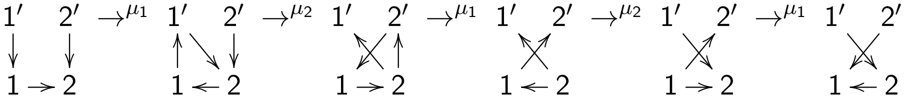

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\to^{\mu_2}
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\to^{\mu_1}
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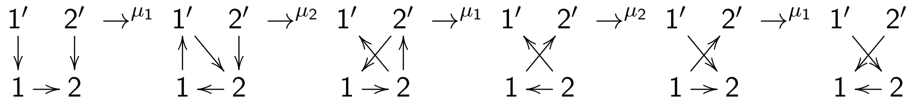

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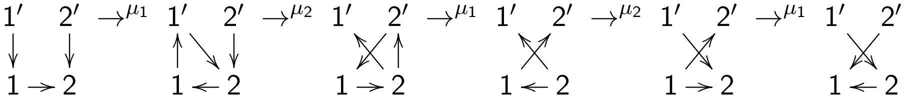

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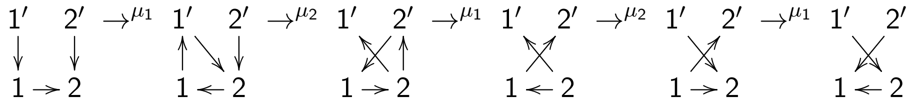
Example 1 Revisited: c-vectors for $1 \to 2$



$$t_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} t_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} t_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\to^{\mu_1} t_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \to^{\mu_2} t_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \to^{\mu_1} t_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{c}_{\mathbf{1},\mathbf{t_0}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_0}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_1}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_1}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_2}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{c}_{\mathbf{1},\mathbf{t_3}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_3}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_4}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_4}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_5}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_5}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example 2 Revisited: c-vectors for $1 \Rightarrow 2$

$$t_0 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow^{\mu_1} t_1 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow^{\mu_2} t_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 3 & -2 \\ 2 & -1 \end{bmatrix}$$

$$\rightarrow^{\mu_1} t_3 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -3 & 4 \\ -2 & 3 \end{bmatrix} \rightarrow^{\mu_2} t_4 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \rightarrow^{\mu_1} t_5 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \rightarrow \dots$$

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c-vector Sign Coherence

For $1 \to 2$ and $\mu_1\mu_2\mu_1\mu_2\mu_1$,

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Theorem (Derksen-Weyman-Zelevinsky 2010) Each c-vector consists exclusively of nonnegative entries or exclusively of nonpositive entries.

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Sign Coherence implies we can assign a sign $\epsilon_{j,t_r} \in \{\pm 1\}$ to each $\mathbf{c_{j,t_r}}$.

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Note: Conjectured by Fomin-Zelevinsky in *Cluster Algebras IV*, 2006, and proven in the skew-symmetrizable case by Gross-Hacking-Keel-Kontsevich.

F-polynomials from C-Vectors

Theorem (Based on Gupta '18): Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \ldots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

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Then the F-polynomial resulting from the final mutation, i.e. $F_{i_\ell;t_\ell}$, is expressible as a product of recursively defined formulas, dependent only on c-vectors (and g-vectors), followed by a monomial specilization:

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Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$.

$$\text{Then } F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\Big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}.$$

Also see [Nagao10], [Keller12], and [Reading18].

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Here, $\mathbf{c_p}$ (resp. $|\mathbf{c_p}|$ or $\mathbf{g_p}$) denotes the pth c-vector (resp. the normalized c-vector $\epsilon_p\mathbf{c_p}$ or the g-vector) along the mutation sequence $\overline{\mu}$,

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F-polynomials from C-Vectors

Theorem (Based on Gupta '18): Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \dots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

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Also see [Nagao10], [Keller12], and [Reading18].

Note: Before the monomial specialization, the L_j's and F_{i_ℓ,t_ℓ}'s may be rational functions in the z_i's.

Note 2: g-vectors to be discussed later.

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

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Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$.

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$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1,$$

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q |\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q |\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q |\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q |\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2 (1 + z_1)^{-1} =$$

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2(1 + z_1)^{-1} = \frac{1 + z_1 + z_2}{1 + z_1}$$

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q |\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q |\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q |\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q |\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2 (1 + z_1)^{-1} = \frac{1 + z_1 + z_2}{1 + z_1}$$

$$L_3 = 1 + z_3 L_1^{-1} L_2^{-1} =$$

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2(1 + z_1)^{-1} = \frac{1 + z_1 + z_2}{1 + z_1}$$

$$L_3 = 1 + z_3 L_1^{-1} L_2^{-1} = 1 + \frac{z_3}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} =$$

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q |\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q |\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q |\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q |\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2 (1 + z_1)^{-1} = \frac{1 + z_1 + z_2}{1 + z_1}$$

$$L_3 = 1 + z_3 L_1^{-1} L_2^{-1} = 1 + \frac{z_3}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} = \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2}$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 =$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q |\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q |\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q |\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q |\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} =$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q |\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q |\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q |\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q |\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$L_5 = 1 + z_5 L_1^{-1} L_2^{-1} L_3^0 L_4^1 =$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$L_5 = 1 + z_5 L_1^{-1} L_2^{-1} L_3^0 L_4^1 = 1 + \frac{z_5}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$=$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$L_5 = 1 + z_5 L_1^{-1} L_2^{-1} L_3^0 L_4^1 = 1 + \frac{z_5}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$= \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1 z_5 + z_4 z_5 (1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1 + z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1 + z_1,$$

Type A_2 Quiver Example (continued)

$$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1. \qquad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1 + z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2 = 1+z_1+z_2,$$

$$F_3 = L_2L_3 =$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2 = 1+z_1+z_2,$$

$$F_3 = L_2L_3 = \frac{1+z_1+z_2+z_3}{1+z_1},$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1 + z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2) + z_5 + z_1z_5 + z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1+z_1+z_2+z_3}{1+z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 =$$

Type A_2 Quiver Example (continued)

$$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1. \qquad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$$

$$L_1 = 1+z_1,\quad L_2 = \frac{1+z_1+z_2}{1+z_1},\quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2},\quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2 = 1+z_1+z_2,$$

$$F_3 = L_2L_3 = \frac{1+z_1+z_2+z_3}{1+z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2 = 1+z_1+z_2,$$

$$F_3 = L_2L_3 = \frac{1+z_1+z_2+z_3}{1+z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1+z_2+z_3)}$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

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$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\dots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

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$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Based on $\epsilon_3 = -1$, $\epsilon_4 = +1$, $\epsilon_5 = +1$, and B_Q as above, we get

$$F_3F_1 = F_2 + z_3, \quad F_4F_2 = z_4F_3 + 1, \quad F_5F_3 = z_5F_4 + 1,$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)},$$

$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Based on $\epsilon_3 = -1$, $\epsilon_4 = +1$, $\epsilon_5 = +1$, and B_Q as above, we get

$$F_3F_1 = F_2 + z_3, \quad F_4F_2 = z_4F_3 + 1, \quad F_5F_3 = z_5F_4 + 1,$$

and these recurrences are valid for these expressions as rational functions.

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)},$$

$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Letting $z_1 = y_1$, $z_2 = y_1y_2$, $z_3 = y_2$, $z_4 = y_1$, $z_5 = y_2$, we get polynomials

$$F_1 = y_1 + 1,\ F_2 = y_1y_2 + y_1 + 1,\ F_3 = y_2 + 1,\ F_4 = 1,\ F_5 = 1.$$

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)},$$

$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Motivation for REU Problem 3: What is a combinatorial or geometric interpretation of the rational functions L_1, L_2, L_3, L_4, L_5 or F_1, F_2, F_3, F_4, F_5 (in terms of z_i's), the latter of which specialize to F-polynomials?

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

$$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1},$$

$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)},$$

$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Motivation for REU Problem 3: What is a combinatorial or geometric interpretation of the rational functions L_1, L_2, L_3, L_4, L_5 or F_1, F_2, F_3, F_4, F_5 (in terms of z_i's), the latter of which specialize to F-polynomials?

Five Minute Coffee Break

Theorem (Based on Gupta '18) : Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \ldots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$

$$\text{and } F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}.$$

Note: g-vectors to be discussed later.

F-polynomials from C-Vectors (2nd Version)

Theorem (Based on Gupta '18) : Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \ldots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$

$$\text{and } F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}.$$

Note: g-vectors to be discussed later.

REU Exercise # 3.2: Use the Generalized Binomial Theorem and the above product expansion for $F_{i_\ell;t_\ell}$ to derive the following power series expansion (which appears in a slightly different form in Gupta '18):

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Kronecker Quiver Example (via Power Series Expansion)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j} \cdot \left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j |\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$.

Kronecker Quiver Example (via Power Series Expansion)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\ldots$, $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,
and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\ldots$, $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$.

Kronecker Quiver Example (via Power Series Expansion)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\ldots$, $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,
and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\ldots$, $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$. Hence

$$\mathbf{c_j}\cdot\mathbf{g}_\ell = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -\ell \\ \ell+1 \end{smallmatrix}\right] = \ell - j + 1,\ \ \mathbf{c_j}\cdot B_Q|\mathbf{c_k}| = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -2k+2 \\ -2k \end{smallmatrix}\right] = 2(j-k).$$

Kronecker Quiver Example (via Power Series Expansion)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, ..., $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,
and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, ..., $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$. Hence

$$\mathbf{c_j}\cdot\mathbf{g}_\ell = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -\ell \\ \ell+1 \end{smallmatrix}\right] = \ell - j + 1,\ \ \mathbf{c_j}\cdot B_Q|\mathbf{c_k}| = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -2k+2 \\ -2k \end{smallmatrix}\right] = 2(j-k).$$

Consequently, we simplify the formula in the Kronecker case to

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \ y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=}$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=} 1 + y_1$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=} 1 + y_1$$

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} \stackrel{?}{=}$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=} 1 + y_1$$

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} \stackrel{?}{=} 1 + 2y_1 + y_1^2 + y_1^2 y_2.$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=} 1 + y_1$$

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} \stackrel{?}{=} 1 + 2y_1 + y_1^2 + y_1^2 y_2.$$

$$F_{1;t_3} = \sum_{m_1,m_2,m_3\in\mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} \stackrel{?}{=}$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} \stackrel{?}{=} 1 + y_1$$

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} \stackrel{?}{=} 1 + 2y_1 + y_1^2 + y_1^2 y_2.$$

$$F_{1;t_3} = \sum_{m_1,m_2,m_3\in\mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} \stackrel{?}{=}$$

$$1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2.$$

This power series expansion of $F_{i_\ell;t_\ell}$ leaves the polynomiality (finiteness of the sum) and positivity of the coefficients as surprising consequences.

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \, y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = \underline{1 + 2y_1 + y_1^2} + \underline{y_1^2 y_2}.$$

The two underlined contributions correspond to $m_2 = 0$ and $m_2 = 1$, respectively. Analogously, there are no contributions for $m_2 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = \underline{1 + 2y_1 + y_1^2} + \underline{y_1^2 y_2}.$$

The two underlined contributions correspond to $m_2 = 0$ and $m_2 = 1$, respectively. Analogously, there are no contributions for $m_2 \geq 2$.

The first three terms correspond to $m_1 = 0, m_1 = 1, m_1 = 2$, respectively, and there are no contributions for $m_1 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1+3y_1+3y_1^2+y_1^3+2y_1^2y_2+2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3\in\mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1+3y_1+3y_1^2+y_1^3+2y_1^2y_2+2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1+3y_1+3y_1^2+y_1^3} + \underline{2y_1^2y_2+2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1} \binom{2-2m_3}{m_2} \binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2y_2 + 2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1 + 3y_1 + 3y_1^2 + y_1^3} + \underline{2y_1^2y_2 + 2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

However, in addition we get an **infinite** number of contributions

$$\sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+4} y_2^2 + \sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+3} y_2^2; \quad \text{recall} \quad \binom{-1}{m_1} = (-1)^{m_1}$$

arising when $m_2 = 2, m_3 = 0$ or $m_2 = 0, m_3 = 1$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2y_2 + 2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1 + 3y_1 + 3y_1^2 + y_1^3} + \underline{2y_1^2y_2 + 2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

However, in addition we get an **infinite** number of contributions

$$\sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+4} y_2^2 + \sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+3} y_2^2; \quad \text{recall} \quad \binom{-1}{m_1} = (-1)^{m_1}$$

arising when $m_2 = 2, m_3 = 0$ or $m_2 = 0, m_3 = 1$. This telescoping infinite sum vanishes except for the term of $y_1^3y_2^2$ for $m_1 = 0, m_2 = 0, m_3 = 1$.

Kronecker Quiver Example (continued)

The formulae continue as

$$F_{2;t_4} = \sum_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_{\geq 0}} \binom{4-2m_2-4m_3-6m_4}{m_1} \binom{3-2m_3-4m_4}{m_2}$$

$$\times \quad \binom{2-2m_4}{m_3} \binom{1}{m_4} y_1^{m_1+2m_2+3m_3+4m_4} y_2^{m_2+2m_3+3m_4}$$

$$F_{1;t_5} = \sum_{m_1,m_2,m_3,m_4,m_5 \in \mathbb{Z}_{\geq 0}} \binom{5-2m_2-4m_3-6m_4-8m_5}{m_1} \binom{4-2m_3-4m_4-6m_5}{m_2} \times$$

$$\binom{3-2m_4-4m_5}{m_3} \binom{2-2m_5}{m_4} \binom{1}{m_5} y_1^{m_1+2m_2+3m_3+4m_4+5m_5} y_2^{m_2+2m_3+3m_4+4m_5}$$

Kronecker Quiver Example (continued)

The formulae continue as

$$F_{2;t_4} = \sum_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_{\geq 0}} \binom{4-2m_2-4m_3-6m_4}{m_1}\binom{3-2m_3-4m_4}{m_2}$$

$$\times \quad \binom{2-2m_4}{m_3}\binom{1}{m_4} y_1^{m_1+2m_2+3m_3+4m_4} y_2^{m_2+2m_3+3m_4}$$

$$F_{1;t_5} = \sum_{m_1,m_2,m_3,m_4,m_5 \in \mathbb{Z}_{\geq 0}} \binom{5-2m_2-4m_3-6m_4-8m_5}{m_1}\binom{4-2m_3-4m_4-6m_5}{m_2} \times$$

$$\binom{3-2m_4-4m_5}{m_3}\binom{2-2m_5}{m_4}\binom{1}{m_5} y_1^{m_1+2m_2+3m_3+4m_4+5m_5} y_2^{m_2+2m_3+3m_4+4m_5}$$

$F_{1;t_5}$ includes terms such as $6y_1^5y_2^3 \ - \ 2y_1^5y_2^3 = 4y_1^5y_2^3$ in its expansion, corresponding to $(m_1, m_2, m_3, m_4, m_5) = (0, 1, 1, 0, 0)$ and $(1, 0, 0, 1, 0)$, respectively. In particular, the contributions from negative binomial coefficients yield a positive term, yet arises from a non-trivial difference.

More on the Kronecker Quiver Example

For general $\ell \geq 1$, recall the power series expansion formula we derived for $1 \rightrightarrows 2$ is

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \, y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

More on the Kronecker Quiver Example

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We now switch gears and discuss a formula for q-binomial coefficients

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

For general $\ell \geq 1$, recall the power series expansion formula we derived for $1 \Rrightarrow 2$ is

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

We now switch gears and discuss a formula for q-binomial coefficients

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)m_{k-j} + m_{k-i} \\ m_{k-i} \end{bmatrix}_q$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of k which contains m_i occurrences of i,

Onto q-Binomial Coefficients

For general $\ell \geq 1$, recall the power series expansion formula we derived for $1 \Rrightarrow 2$ is

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

We now switch gears and discuss a formula for q-binomial coefficients

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda\vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)m_{k-j} + m_{k-i} \\ m_{k-i} \end{bmatrix}_q$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of k which contains m_i occurrences of i, and $n(\lambda) = \sum_{i=1}^{k}(i-1)\lambda_i = \sum_{i=1}^{k}\binom{\lambda_i'}{2}$.

Onto q-Binomial Coefficients

For general $\ell \geq 1$, recall the power series expansion formula we derived for $1 \Rrightarrow 2$ is

$$F_{i_\ell; t_\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell} (j-i) m_j}{m_i} y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell} (i-1) m_i}.$$

We now switch gears and discuss a formula for q-binomial coefficients

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j) m_{k-j} + m_{k-i} \\ m_{k-i} \end{bmatrix}_q$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of k which contains m_i occurrences of i, and $n(\lambda) = \sum_{i=1}^{k} (i-1)\lambda_i = \sum_{i=1}^{k} \binom{\lambda_i'}{2}$.

Note that $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_K')$ is the conjugate partition and if we write $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, then $m_i = \lambda_i' - \lambda_{i+1}'$ as well.

Onto q-Binomial Coefficients

For general $\ell \geq 1$, recall the power series expansion formula we derived for $1 \Rrightarrow 2$ is

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

We now switch gears and discuss a formula for q-binomial coefficients

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda\vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)m_{k-j} + m_{k-i} \\ m_{k-i} \end{bmatrix}_q$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of k which contains m_i occurrences of i, and $n(\lambda) = \sum_{i=1}^{k}(i-1)\lambda_i = \sum_{i=1}^{k}\binom{\lambda_i'}{2}$.

Note that $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_K')$ is the conjugate partition and if we write $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, then $m_i = \lambda_i' - \lambda_{i+1}'$ as well.

This is known as the KOH (Kathleen O' Hara) Formula.

Onto q-Binomial Coefficients

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Possible Paper Presentations: Kathleen O' Hara, "Unimodality of Gaussian Coefficients: A Constructive Proof" in JCTA (1990)

Zeilberger, "A One-line High School Algebra Proof of the Unimodality of the Gaussian Polynomials ....", q-Series and Partitions, IMA Volumes in Mathematics and its Applications, Springer-Verlag, New York (1989).

I.G. Macdonald, "An Elementary Proof of a q-Binomial Identity", q-Series and Partitions, IMA Volumes in Mathematics and its Applications, Springer-Verlag, New York (1989).

Comparing Kronecker Quiver Example and KOH

Recall that if we let $y_1 = y_2 = 1$ for the Kronecker Quvier $1 \rightrightarrows 2$, then the F-polynomials $F_{i_\ell;t_\ell}$ specialize to every-other Fibonacci numbers $1, 1, 2, 5, 13, 34, 89, \ldots,$ (or specialize cluster variables as $x_1 = x_2 = 1$)

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Furthermore, Fibonacci numbers can be decomposed into sums of binomial coefficients: if $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$, then

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See Hoggatt-Lind ,"Fibonacci and Binomial Properties of Weighted Compositions" from Journal of Combinatorial Theory (1968), or

Gessel-Li, "Compositions and Fibonacci Identities" from Journal of Integer Sequences (2013):

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$$\text{Note that we also have} \quad F_n = \sum_{k=1} \binom{n-k}{k-1}.$$

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b) Prove that for $n \geq 3$, $F_n(q) = F_{n-1}(q) + q^{n-2} F_{n-2}(q)$.

c) Give and prove a combinatorial interpretation for $F_n(q)$ in terms of counting integer partitions.

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What if we instead define $\widetilde{F}_n(q) = \sum_{k=1} q^{(k-1)} \binom{n-k}{k-1}$?

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REU Exercise 3.4: a) What are $\widetilde{F}_n(q)$ for $3 \leq n \leq 7$?

b) Describe a combinatorial interpretation for the $\widetilde{F}_n(q)$'s.

c) Describe a $\mathbb{Z}[q]$-specialization of the F-polynomials for the Kronecker quiver such that for each $\ell \geq 3$, we have $F_{i_\ell;t_\ell}$ specializes to $\widetilde{F}_\ell(q)$.

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Carlitz: $F_n(q) = \sum_{k=1} \begin{bmatrix} n-k \\ k-1 \end{bmatrix}_q$, Variant: $\widetilde{F}_n(q) = \sum_{k=1} q^{(k-1)}\binom{n-k}{k-1}$.

REU Problem # 3.1: Develop a (q,t)-analogue of KOH formula for binomial coefficients and identify the associated algebraic transformation such that the analogous sum of (q,t)-binomial coefficients match the formulas for $F_{i_\ell;t_\ell}(y_1, y_2)$ for the Kronecker quiver.

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda\vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + m_{k-i} + \sum_{j=0}^{i-1} 2(i-j)m_{k-j} \\ m_{k-i} \end{bmatrix}_q$$

REU Problem # 3.2: KOH vs MACKOH

There is also hope that a better understanding of how the above power series formula for F-polynomials for Kronecker quivers and the KOH formula for q-Binomial Coefficients and/or q-Fibonacci numbers would help solve an open problem of Dennis Stanton!

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However, if we instead evaluate $\begin{bmatrix} N \\ s \end{bmatrix}_q$, for negative N, as a generalized binomial coefficient, i.e. $\begin{bmatrix} N \\ s \end{bmatrix}_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}$, then this identity is known as MACKOH (due to Ian Macdonald's work).

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Open Problem 5.8 of Dennis Stanton: Find an involution that proves the MACKOH identity implies the KOH. (See http://www-users.math.umn.edu/˜ stant001/PAPERS/Prob2019.pdf.)

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Also see the $M = N$ conjecture from mathematical physics as in P. Di Francesco and R. Kedem, "Proof of the Combinatorial Kirillov-Reshetikhin Conjecture", `arXiv:0710.4415.pdf`

Formula for general Rank Two, i.e. r-Kronecker Case

For the case of $B_Q = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$,

$$F_{i_\ell,t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{s_{\ell-i} - r\sum_{j=i+1}^{\ell} s_{j-i-1}m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} s_{i-1}m_i} y_2^{\sum_{i=1}^{\ell} s_{i-2}m_i}$$

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REU Problem # 3.3: Combinatorics for the r-Kronecker

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Possible Paper Presentations:

Kyungyong Lee "On Cluster Variables of Rank Two Acyclic Cluster Algebras", Annals of Combinatorics (2012)
Lee-Schiffler "A combinatorial formula for rank 2 cluster variables", Journal of Algebraic Combinatorics (2013)
Lee-Li-Zelevinsky "Greedy elements in rank 2 cluster algebras", Selecta Mathematica (2014)

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Note: K. Lee's formulas therein utilize binomial coefficients that are set to zero when the top of the binomial coefficient is negative.

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Note: K. Lee's formulas therein utilize binomial coefficients that are set to zero when the top of the binomial coefficient is negative. Hence we see KOH-like behavior where our above power series formulas were assuming generalized binomial coefficients and exhibited MACKOH-like behavior.

Further afield, but two other related open-ended REU problems on this topic

REU Problem # 3.4: F-polynomial formulas in the limit

Consider the original power series expansion for general quivers and mutation sequences:

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j} \cdot \left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |\mathbf{c_k}|\right)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j |\mathbf{c_j}|}.$$

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In the TA session this afternoon g-vectors will be discussed, and how there are "holes" in the cluster fan in the case of infinite type cluster algebras.

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In the TA session this afternoon g-vectors will be discussed, and how there are "holes" in the cluster fan in the case of infinite type cluster algebras. For example, for the Kronecker example, the g-vectors of the form $\begin{bmatrix} n \\ -n \end{bmatrix}$ for $n \geq 1$ will never occur as $\mathbf{g}_\ell$ associated to the result of finite length mutation sequence.

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However, for $1 \Rightarrow 2$ if we let $\overline{\mu}$ be the infinite sequence $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots$ and $\mathbf{g}_\ell = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we get an infinite power series as a result, which can also be expressed as a ratio of two series taken to a limit.

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In fact, such expressions are examples of infinite path-ordered products in scattering diagrams.

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Possible Paper Presentation: Sections 3.2 and 3.3 of Nathan Reading, "A combinatorial appraoch to scattering diagrams", `arXiv:1806.05094`.

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In the TA session this afternoon g-vectors will be discussed, and how there are "holes" in the cluster fan in the case of infinite type cluster algebras.

REU Problem # 3.4: Develop power series formulas (or expressed as ratios) for missing g-vectors beyond the case of the Kronecker quiver.

REU Problem # 3.4: F-polynomial formulas in the limit

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$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \; \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

In the TA session this afternoon g-vectors will be discussed, and how there are "holes" in the cluster fan in the case of infinite type cluster algebras.

REU Problem # 3.4: Develop power series formulas (or expressed as ratios) for missing g-vectors beyond the case of the Kronecker quiver.

Alternatively, see Sections 6 and 7 of M. Gupta, "A formula for F-Polynomials in terms of C-Vectors and Stabilization of F-Polynomials" for a different approach to obtaining such limits.

Can we better understand the combintorics behind such formulas?

More Open-ended Question: Are there different specializations of the z_i's in the formuals for L_k's or F_{i_ℓ, t_ℓ}'s, which were naturally rational functions in terms of the z_i's which lead to different families of polynomials that are also of interest?

REU Problem # 3.5: Other Specializations

More Open-ended Question: Are there different specializations of the z_i's in the formuals for L_k's or F_{i_ℓ, t_ℓ}'s, which were naturally rational functions in terms of the z_i's which lead to different families of polynomials that are also of interest?

Or are there other ways to understand these rational functions as generating functions or partition functions (i.e. think statistical mechanics or weighted paths in networks) that would be meaningful in the theory of cluster algebras?

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Or are there other ways to understand these rational functions as generating functions or partition functions (i.e. think statistical mechanics or weighted paths in networks) that would be meaningful in the theory of cluster algebras?

As motivation for this last question, cutting edge research of Hamed-He-Lam "Cluster configurations spaces of finite type" in `arXiv:2005.11419` discussed a family of rational functions known as f_γ's and a different family of variables (u-variables) that are relevant to both mathematics and physics alike.

Further References

- Meghal Gupta, *A formula for F-polynomials in terms of C-Vectors and Stabilization of F-polynomials*, REU 2018, `arXiv:1812.01910`
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