Real-rootedness of Polynomials from Planar Graphs

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Outline

Log-concave sequences

Definition

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Another interesting example is the sequence of the (absolute values of the) coefficients of the chromatic polynomial of a finite graph (Huh 2012).

Pólya frequency sequences

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Example

Each row of Pascal's triangle forms a PFS: the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ gives the polynomial $(1+t)^n$, which has only real roots.

Theorem (Aissen–Schoenberg–Whitney)

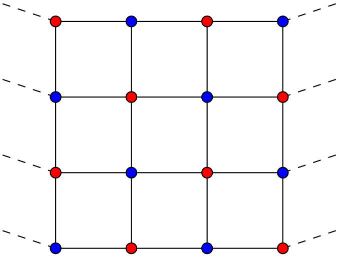
The sequence $(a_i)_{i=0}^n$ is a Pólya frequency sequence if and only if the associated Aissen–Schoenberg–Whitney matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \cdots \\ 0 & 0 & a_0 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative.

Graphs on a cylinder

Throughout, our graphs will be planar, bipartite, and embedded on a cylinder.



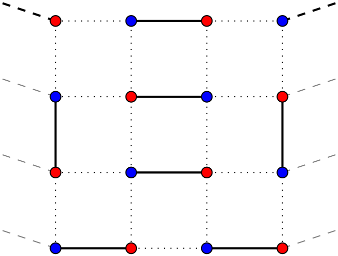
Interested in “dimer covers” on these graphs.

Dimer covers

Definition

A *dimer cover* (or *perfect matching*) of a graph G is a subgraph which contains every vertex of G, and in which every vertex has degree 1.

Graphs on a cylinder



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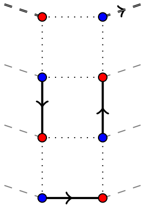
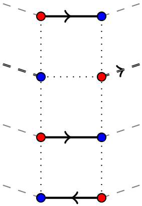


π_1



$\pi_2^\vee$

Example

Note that for all dimer covers π_1, π_2 of G, the union $\pi_1 \cup \pi_2^\vee$ is a union of vertex-disjoint directed cycles.

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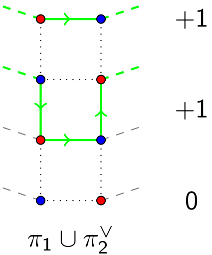
Definition

The *(relative) height* $\mathrm{ht}(\pi_1, \pi_2)$ of two dimer covers π_1, π_2 of G equals the number of positively oriented cycles of $\pi_1 \cup \pi_2^\vee$ minus the number of negatively oriented cycles of $\pi_1 \cup \pi_2^\vee$.

Relative height 2 (previous slide):



Absolute height function

Lemma

For any three dimer covers π_1, π_2, π_3 of G, we have
$\text{ht}(\pi_1, \pi_3) = \text{ht}(\pi_1, \pi_2) + \text{ht}(\pi_2, \pi_3)$.

Absolute height function

Lemma

For any three dimer covers π_1, π_2, π_3 *of* G*, we have* $\mathrm{ht}(\pi_1, \pi_3) = \mathrm{ht}(\pi_1, \pi_2) + \mathrm{ht}(\pi_2, \pi_3)$.

- Thus, there exists a dimer cover π_0 of G such that $\mathrm{ht}(\pi, \pi_0) \geq 0$ for all dimer covers π.

Definition

The *absolute height* of a dimer cover π of G is given by $\mathrm{ht}(\pi) := \mathrm{ht}(\pi, \pi_0)$.

Absolute height of π is independent of the choice of π_0.

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Definition

The *absolute height* of a dimer cover π of G is given by $\text{ht}(\pi) := \text{ht}(\pi, \pi_0)$.

Absolute height of π is independent of the choice of π_0. Also, it follows that

$$\text{ht}(\pi_1, \pi_2) = \text{ht}(\pi_1) - \text{ht}(\pi_2)$$

Height sequence

- We define the *height sequence* $a_0, a_1, \ldots$ of G by letting a_i be the number of dimer covers of G with absolute height i.

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$$a_i := \sum_{\substack{\text{dimer covers } \pi \\ \mathrm{ht}(\pi)=i}} \mathrm{wt}(\pi).$$

Outline

2×2 minors–weighted

Proposition

The 2×2 minors of the ASW matrix of (a_i) are nonnegative. In particular, (a_i) is log-concave.

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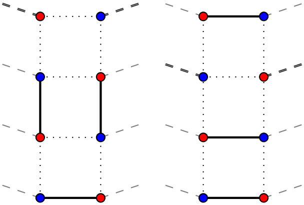
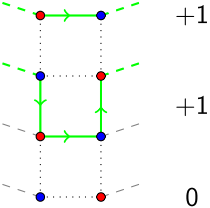
Main idea: let T_i be the set of dimer covers of height i. Then there is a weight-preserving injection

$$T_{i+1} \times T_{i-1} \to T_i \times T_i$$

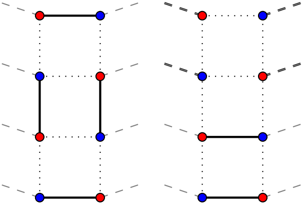
Example

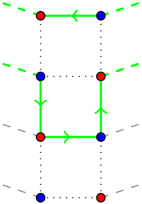




Now look at "running sum" from the top down.

Example (cont).





Certain 3×3 minors–weighted

Earlier, we stated that we knew the 2×2 minors of the ASW matrix of (a_i) are nonnegative. We also know that two certain 3×3 minors are also nonnegative:

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Proposition

We have

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{pmatrix} \geq 0, \qquad \det \begin{pmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{pmatrix} \geq 0.$$

PFS–unweighted grid graphs

Proposition

G is an unweighted grid graph $\implies$ (a_i) is a PFS.

This is a real-rootedness proof as opposed to one about total nonnegativity.

Acknowledgments

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