Toric ideals & regularity

(Toric Reg)

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$\boxed{\text{Jay Yang}}$ (Wed.)

REU Problem: I_A = toric ideal

When does $\operatorname{reg}(I_A) = \deg(S/I_A) - \operatorname{codim}(I_A) + 1$?

Outline:

I. Some Commutative Algebra

II. Bounding Regularity

III. Toric ideals

IV. Regularity of toric ideals

Some Commutative Algebra

Ex: $\langle x-y, y-z\rangle = \langle x-z, y-z\rangle$

$x-z = (x-y) + (y-z)$

$S = \mathbb{C}[x_1, x_2, \dots, x_n] \ni f_1, f_2, \dots, f_r$ homogeneous

(homog) (grd) ideal $\langle \underbrace{f_1, \dots, f_r}_{\text{generators}} \rangle := \{ g_1 f_1 + \dots + g_r f_r \mid g_i \in S \} = I$

Ex: $S = \mathbb{C}[u, v, w, x, y, z]$

(a) $I' = \langle x^2, xy^2, y^3 \rangle = \left\{ ax^2 + bxy^2 + cy^3 \,\middle|\, \begin{matrix} a,b,c \\ \in S \end{matrix} \right\}$

(b) $J = \langle x^2 - xz,\ xy - xz - yz + z^2,\ y^2 - xz - z^2 \rangle$

(c) $K = \langle y^2 - zu,\ xu - w^2,\ v^3 - u^2 w \rangle$

$\mathrm{maxdeg}(\overset{\text{ideal}}{I})$:= maximal degree of a minimal generator of I

(a) 3

(b) 2

(c) 3

Question: $\max\deg(I) \le$?

HARD?

Instead:

Question: degrees of all syzygies of $I \le$?

$\| $

$\mathrm{reg}(I)$

$\left\{\begin{array}{l}\text{graded minimal} \\ \text{free resolution} \\ \text{(gmfr) of } S/I\end{array}\right.$

$$F: \quad F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_n} F_n \longleftarrow 0$$

← syzygies

$$H_i F := \frac{\ker \varphi_i}{\operatorname{im} \varphi_{i+1}} = \begin{cases} S/I & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}$$

ⓐ $I' = \langle x^2, xy^2, y^3 \rangle$

$$S \xleftarrow{\varphi_1 = \begin{bmatrix} x^2 & xy^2 & y^3 \end{bmatrix}} \begin{array}{c} S(-2) \\ \oplus \\ S(-3)^2 \end{array} \xleftarrow{\begin{bmatrix} y^2 & 0 \\ -x & y \\ 0 & -x \end{bmatrix}} S(-4)^2 \longleftarrow 0$$

$$y^2 \cdot x^2 - x \cdot xy^2 + 0 \cdot y^3 = 0$$

ψ

$$\begin{pmatrix} x^2 & xy^2 & y^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{a x^2 + b x y^2 + c y^3}$$

$\Rightarrow \operatorname{im} \varphi_1 = I'$

graded minimal free resolution (gmfr) of S/I & $\varphi_i(F_i) \subseteq \langle x \rangle \cdot F_{i-1}$

$$F: \; F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_n} F_n \longleftarrow 0$$

← syzygies

$$H_i F := \frac{\ker \varphi_i}{\operatorname{im} \varphi_{i+1}} = \begin{cases} S/I & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

(a) $I' = \langle x^2, xy^2, y^3 \rangle$

$$S^1 \xleftarrow{\varphi_1 = \begin{bmatrix} x^2 & xy^2 & y^3 \end{bmatrix}} \begin{matrix} S(-2)^1 \\ \oplus \\ S(-3)^2 \end{matrix} \xleftarrow{\begin{bmatrix} y^2 & 0 \\ -x & y \\ 0 & -x \end{bmatrix}} S(-4)^2 \longleftarrow 0$$

Betti table

$\beta(S/I')$:

	0	1	2
0	1	-	-
1	-	1	-
2	-	2	2

$\operatorname{maxdeg}(I') = \operatorname{reg}(I') = 3$

$\operatorname{pdim}(S/I') = 2$

regularity

$\operatorname{reg}(I) :=$ # nonzero rows of $\beta(S/I)$

projective dimension

$\operatorname{pdim}(S/I) :=$ # nonzero maps in gmfr of S/I

(b) $J = \langle \overset{f_1}{y^2 + xz - z^2}, \overset{f_2}{xy - xz - yz + z^2}, \overset{f_3}{x^2 - xz} \rangle$

FIXED

$$S^1 \xleftarrow{\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} 0 & x-z \\ x & -y-z \\ -y+x & -z \end{bmatrix}} S(-3)^2 \leftarrow 0$$

$\beta(S/J)$:

	0	1	2
0	1	–	–
1	–	3	2

$\mathrm{reg}(J) = 2$

$\mathrm{pdim}(J) = 2$

(c) $K = \langle \overset{g_1}{y^2 - zu}, \overset{g_2}{xu - w^2}, \overset{g_3}{v^3 - u^2 w} \rangle$

$$S^1 \xleftarrow{\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}} \begin{matrix} S(-2)^2 \\ \oplus \\ S(-3)^1 \end{matrix} \xleftarrow{\begin{bmatrix} -xu+w^2 & -v^3+u^2w & 0 \\ y^2-zu & 0 & -v^3+u^2w \\ 0 & y^2-zu & xu-w^2 \end{bmatrix}} \begin{matrix} S(-4)^1 \\ \oplus \\ S(-5)^2 \end{matrix} \xleftarrow{\begin{bmatrix} v^3-u^2w \\ -xu+w^2 \\ y^2-zu \end{bmatrix}} S(-7)^1 \leftarrow 0$$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\beta(S/K)$:

	0	1	2	3
0	1	–	–	–
1	–	2	–	–
2	–	1	1	–
3	–	–	2	–
4	–	–	–	1

5 rows

$\mathrm{reg}(K) = 5$

$\mathrm{pdim}(S/K) = 3$

$\pi : \mathbb{C}^n \setminus \{\vec{0}\} \twoheadrightarrow \mathbb{P}^{n-1} := \left(\mathbb{C}^n \setminus \{\vec{0}\}\right) / \mathbb{C}^\times$

$\vec{x} \longmapsto \bar{x} = [x_1 : \dots : x_n] := \{\text{equiv. class of } \vec{y} = \underline{\lambda \vec{x}},\ \lambda \in \mathbb{C}^\times\}$

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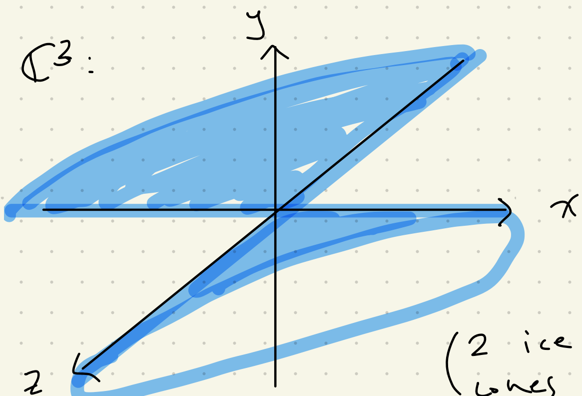
grd. ideal $I \subseteq S = \mathbb{C}[x_1, \dots, x_n]$

variety of I $\quad V(I) := \{\underline{\bar{x}} \in \mathbb{P}^{n-1} \mid f(x) = 0 \quad \forall f \in I \text{ w/ } \pi(x) = \bar{x}\}$

Ex: $\quad I = \langle xz - y^2 \rangle$

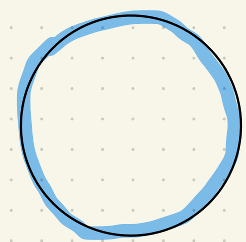
In $\mathbb{C}^3$:



(2 ice cream cones w/ points @ $\vec{0}$)

In $\mathbb{P}^2$:

$\longrightarrow$



Algebraic Geometry:

algebra of S/I $\rightsquigarrow$ geometry of $\mathrm{Var}(I)$

e.g.

I prime $\iff$ S/I domain $\leftrightsquigarrow$ $\mathrm{Var}(I)$ irred.

I is prime if $\forall a, b \in S$, $ab \in I \implies a \in I$ or $b \in I$

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dimension $\dim(S/I) :=$ $1 +$ geom. dim of $\mathrm{Var}(I) \subseteq \mathbb{P}^{n-1}$

$=$ geom. dim of $\pi^{-1}(\mathrm{Var}(I))$

codimension $\mathrm{codim}(I) := n - \dim(S/I)$

Turns out: $\operatorname{pdim}(S/I) \geq \operatorname{codim}(I)$

I is Cohen-Macaulay (CM) if $\operatorname{pdim}(S/I) = \operatorname{codim}(I)$

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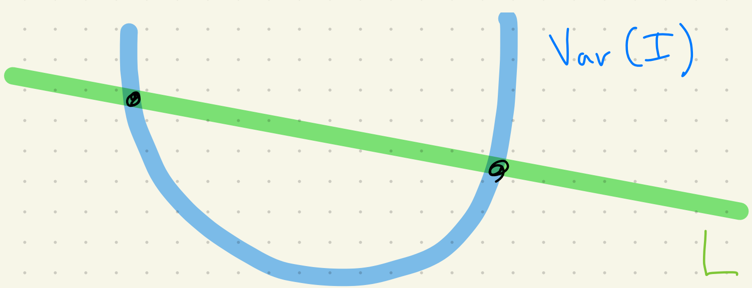
degree $\deg(S/I) :=$ # of points in $\operatorname{Var}(I) \cap L$

w/ L = general hyperplane of $\dim = \operatorname{codim}(I)$

e.g.



$\Rightarrow \deg(S/I) = 2$

Macaulay 2:

www.macaulay2.com ← download / install

→ www.web.macaulay2.com ← web interface
(Use this today!)

REU Exercise 5.1:

Given: (i) $\langle x, y\rangle \subseteq \mathbb{C}[x,y]$

(ii) $\langle x^2, xy, y^2, xw^2 + yz^2\rangle \subseteq \mathbb{C}[w,x,y,z]$

(iii) $\langle wy, wz, xy, xz\rangle \subseteq \mathbb{C}[w,x,y,z]$

(iv) $\langle w^2 + x^2, y^2 + z^2, wz - xy, wy - xz\rangle \subseteq \mathbb{C}[w,x,y,z]$

(a) Use Macaulay2 to compute gmfr, betti tables, pdim, reg, deg, & codim in all examples.

(b) * One more slide first *

Bounding regularity

Thm: $\operatorname{reg}(I) \leq \left(2 \cdot \operatorname{maxdeg}(I)\right)^{2^{n-2}}$ & is nearly optimal

(1993) (1998)

(1984)

Eisenbud-Goto Conjecture:

no linear polys in I

I grd. **prime** ideal in S s.t. $I \subseteq \langle \underline{x} \rangle^2$

$$\Longrightarrow \quad \operatorname{reg}(I) \leq \underbrace{\deg(S/I) - \operatorname{codim}(I) + 1}_{\geq 0} \quad \leftarrow$$

Thm: EG Conj holds when I is grd. **CM** ideal w/ $I \subseteq \langle \underline{x} \rangle^2$.

(2018)

Thm: EG Conj. is FALSE!

REU Exercise 5.1:

Given: (i) $\langle x,y\rangle \subseteq \mathbb{C}[x,y]$

(ii) $\langle x^2, xy, y^2, xw^2+yz^2\rangle \subseteq \mathbb{C}[w,x,y,z]$

(iii) $\langle wy, wz, xy, xz\rangle \subseteq \mathbb{C}[w,x,y,z]$

(iv) $\langle w^2+x^2, y^2+z^2, wz-xy, wy-xz\rangle \subseteq \mathbb{C}[w,x,y,z]$

(a) Use Macaulay2 to compute gmfr, betti tables, pdim, reg, deg, & codim in all examples.

(b) Show the hypotheses of EG Conj. are <u>necessary</u> by checking that each of these ideals fails the conjecture but also fails to satisfy one of the hypotheses.

Toric ideals

Fact: Every quotient of a polynomial ring is isomorphic to the quotient of a polynomial ring by a trinomial ideal.

REU Exercise 5.2: Prove this!

Prime monomial ideal $\rightsquigarrow$ $\langle x_i \mid i \in I \rangle \subsetneq \langle \underline{x} \rangle^2$

Prime binomial ideal $\rightsquigarrow$ toric ideal

$u \in \ker_{\mathbb{Z}}(A)$

$u = u_+ - u_-$

$A \in \mathbb{Z}^{d \times n}$

(w/ $\mathbb{Z}A = \mathbb{Z}^d$)

$$I_A := \langle x^{u_+} - x^{u_-} \mid Au = 0 \rangle$$

Ex:

$$A = \begin{array}{ccc} x & y & z \\ \multicolumn{3}{c}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}} \end{array} \quad u = \begin{array}{cc} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array} \rightsquigarrow I_A = \langle xz - y^2 \rangle$$

Ex:

$$A = \begin{array}{cccccc} u & v & w & x & y & z \\ \multicolumn{6}{c}{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}} \end{array} \quad \ker_{\mathbb{Z}}(A) \longleftrightarrow \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$I_A = \langle y^2 - zu,\ xu - w^2,\ v^3 - u^2 w \rangle = K$$

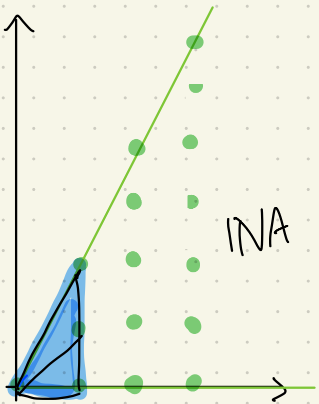
twisted cubic: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

Ex: $A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$ $\quad I_A = \langle xz - y^2 \rangle$ $\quad A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \in \mathbb{Z}^{d \times n}$

$\mathbb{N}A := \{ m_1 a_1 + \cdots + m_n a_n \mid m_i \in \mathbb{Z}_{\geq 0} \}$

semigroup generated by A

$\mathrm{val}(A) = 2$



$$\mathrm{val}(A) := d! \cdot \begin{pmatrix} \text{usual volume} \\ \text{of } \mathrm{conv}(A, \vec{0}) \end{pmatrix} = \deg(I_A)$$

Semigroup ring of A

$$\mathbb{C}[\mathbb{N}A] := \bigoplus_{a \in \mathbb{N}A} \mathbb{C} \cdot t^a \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$$

Thm: $x_i \longmapsto t^{a_i}$

$$S/I_A \cong \underbrace{\mathbb{C}[\mathbb{N}A]}_{\text{domain}} \Rightarrow I_A \text{ prime}$$

$\mathrm{codim}(I_A) = n - d$

$(\dim(S/I_A) = d)$

REU Exercise 5.3:

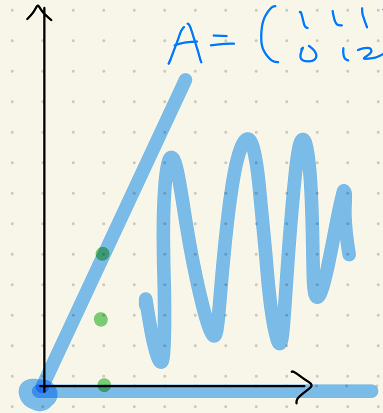
(i) Prove that a binomial $x^u - x^v$ is homogeneous if and only if the dot product $(1, \dots, 1) \cdot (u - v) = 0$.

(ii) Show $I_A \cong I_{gA}$ $\forall g \in GL_d(\mathbb{Z})$

(iii) Let $\deg(x_i) = a_i \in \mathbb{Z}^d$.

Show that the $\mathbb{Z}^d$-grd. primes in S/I_A are in one-to-one correspondence with the faces of $\mathbb{R}_{\geq 0} A$.

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$



faces: $\emptyset$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, A

Regularity of toric ideals

E-G Conj. ($\mathrm{reg}(I_A) \leq \deg(S/I_A) - \mathrm{codim}(I_A) + 1$) holds if:

- CM [Eisenbud-Goto 1984]
- $n-d=1$
- $n-d=2$ [Peeva-Sturmfels 1998]
- $d=2$ [Nitsche 2014]
- simplicial & isolated sings [Herzog-Hibi 2003]
- simplicial & seminormal [Nitsche 2012]
- simplicial & $\deg(S/I_A) \geq \mathrm{codim}(I_A) + 2$ [Hoa-Stückrad 2003]

REU Problem #5 (Toric Reg)

For which $A \in \mathbb{Z}^{d \times n}$ does $=$ hold in EG Conj,

i.e., $$\operatorname{reg}(I_A) = \deg(S/I_A) - \underbrace{\operatorname{codim}(I_A)}_{(n-d)} + 1$$

REU Problem #5 (Toric Reg)

For which $A \in \mathbb{Z}^{d \times n}$ does $=$ hold in EG Conj,

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REU Exercise 5.4:

(a) Show $=$ for $n-d=1$.

(b) Show $=$ for $d=2$, $n-d=2$, & $\boxed{\text{1 missing condition.}}$ (Find it!) ?

(c) Use Nitsche 2014 to find Veronese rings w/ $=$.

Papers to present:

2? [Peeva–Sturmfels]: Syzygies of codim 2 lattice ideals (1998)

[Nitsche]: Combinatorial proof of EG Conj for monomial curves... (2014)

[Herzog-Hibi]: C-M Regularity of simplicial semigroup rings w/ isol. sings (2003)