Infinite Frieze Patterns and Dissections on Annuli

Jiuqi (Lena) Chen
Mentor: Esther Banaian
TA: Libby Farrell

08/07/20

Outline

Frieze Patterns

Definition (Frieze Pattern)

- First row consists of all zeroes
- Second row consists of all ones.
- The first non trivial row is called the *quiddity row*.
- All diamonds $\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$ satisfy the diamond condition $bc - ad = 1$.

$$
\begin{array}{ccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & m_{-1,1} & & m_{0,2} & & m_{1,3} & & m_{2,4} & & m_{3,5} & \cdots \\
\cdots & & m_{-1,2} & & m_{0,3} & & m_{1,4} & & m_{2,5} & & \cdots \\
\cdots & m_{-2,2} & & m_{-1,3} & & m_{0,4} & & m_{1,5} & & m_{2,6} & \cdots \\
\cdots & & m_{-2,3} & & m_{-1,4} & & m_{0,5} & & m_{1,6} & & \cdots \\
 & & & \iddots & & m_{-1,5} & & \ddots & & &
\end{array}
$$

Previous Findings on Frieze Patterns

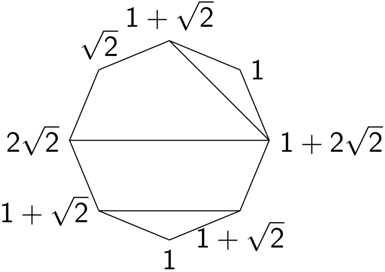
- Conway-Coxeter:
  Finite frieze of positive integers $\underset{Bijection}{\Longleftrightarrow}$ Triangulations of polygon
- Baur-Parsons-Tschabold:
  Infinite frieze of positive integers $\underset{Bijection}{\Longleftrightarrow}$ Triangulations of annuli
- Holm-Jørgensen:
  Finite frieze of type $\Lambda_{p_1,\ldots,p_s}$ $\underset{Injection}{\Longleftarrow}$ Dissections of polygon

Definition

Friezes with entries in the ring of algebraic integers of the field $\mathbf{Q}(\lambda_{p_1},\ldots,\lambda_{p_s})$ are called friezes of type $\Lambda_{p_1,\ldots,p_s}$, where $\lambda_{p_i} = 2\cos(\pi/p_i)$.

Frieze Patterns arise from Dissections-Example

If a vertex is adjacent to polygons of size $p_1, \ldots, p_n$, we associate to it the weighted count $\sum_{i=1}^{n} \lambda_{p_i}$.



p	λ_p
3	1
4	$\sqrt{2}$
5	$\frac{1+\sqrt{5}}{2}$
6	$\sqrt{3}$

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1

$1+\sqrt{2}$ 1 $1+2\sqrt{2}$ $1+\sqrt{2}$ 1 $1+\sqrt{2}$ $2\sqrt{2}$ $\sqrt{2}$

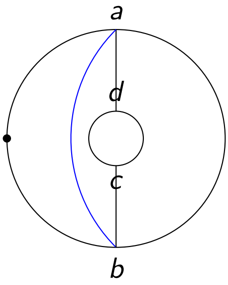
REU Problem 6.3

Would there be any correspondence between polygon dissections on annuli and infinite friezes of type $\Lambda_{p_1,\ldots,p_s}$?

Outline

Dissection on Annuli

- Let $A_{n,m}$ be an annulus with n outer vertices and m inner vertices. If $m = 0$, it's a punctured disc.
- A *peripheral arc* connects 2 outer vertices or 2 inner vertices. A dissection that does not contain any peripheral arc is called *skeletal*.
- A *bridging arc* connects an outer vertex and an inner vertex.



Realization of Type $\Lambda_{p_1,\ldots,p_s}$ Friezes

Definition

A frieze pattern that has a dissected annulus interpretation is called *realizable*. If a vertex is adjacent to polygons of size $p_1, \ldots, p_n$, we associate to it the weighted count $\sum_{i=1}^{n} \lambda_{p_i}$ in the quiddity row.

Proposition (Realizability test)

In a quiddity row, $m_{i-1,i+1} = \sum_{p \in A_i} \lambda_p$. If there exists a pair of neighboring quiddity entries that doesn't share a common λ_p in their sums, this frieze pattern is not realizable.

Reduced Infinite Friezes

Entries in the quiddity row record information of subgons that incident with vertices, but some of them are trivial.

$$(\sum_{q\in A}\lambda_q+\lambda_p,\ \lambda_p,\ \lambda_p,\ \lambda_p+\sum_{q\in B}\lambda_q)$$

Definition

A frieze pattern whose quiddity row does not contain terms in the form of λ_p is said to be *reduced*.

The Reduction Algorithm

Let $q = (a_1, a_2, \ldots, a_n)$ be a quiddity sequence.

Definition (cut)

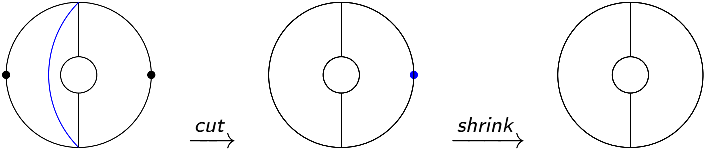
We can delete $p-2$ consecutive λ_p from q through a *cut*.

$$(a_i, \lambda_p, \ldots, \lambda_p, a_{i+p-1}) \longrightarrow (a_i - \lambda_p, a_{i+p-1} - \lambda_p)$$

Definition (shrink)

We can delete $k < p-2$ consecutive λ_p from q through a *shrink*.

$$(a_i, \lambda_p, \ldots, \lambda_p, a_{i+k-1}) \longrightarrow (a_i - \lambda_p + \lambda_{p-k}, a_{i+k-1} - \lambda_p + \lambda_{p-k})$$



Realization of Reduced Frieze

Lemma

If a frieze is realizable, then its reduced frieze is realizable.

Note: the converse is not true.

Proposition

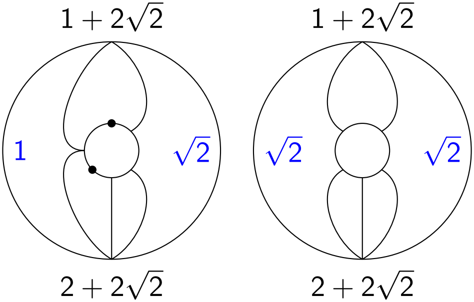
Let $\mathcal{F}$ be a reduced frieze, $m_{i-1,i+1} = \sum_{p \in A_i} \lambda_p, i \in \mathbb{Z}$ be its quiddity row. $\mathcal{F}$ is realizable if and only if there exists a sequence of n numbers $p_1, p_2, ..., p_n$ such that

- $p_i \in A_i \cap A_{i+1}$ *for all* $i \in [n]$;
- *If* $p_{i-1} = p_i$ *numerically, there are at least two copies of* p_i *in* A_i.

In plain words, $\mathcal{F}$ is realizable iff we can determine all the shapes of its outer subgons.

Realization of Reduced Frieze-Example

Dissection of annulus corresponding to $q = \ldots 1 + 2\sqrt{2}, 2 + 2\sqrt{2}, \ldots$



Frieze Realizable by Quotient Dissection

Period 2 quiddity sequence:

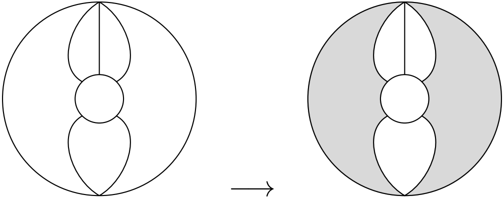
$$q = \ldots\ 1+\sqrt{3},\ 1+\sqrt{2},\ 1+\sqrt{3},\ 1+\sqrt{2},\ 1+\sqrt{3},\ \ldots$$

- Neighboring entries share 1, but only one copy of it.
- It passes the realizability test but is not realizable by a normal dissection on annulus.

Quotient Dissection on annulus

Definition

A quotient dissection on an annulus can be obtained by identifying neighbouring outer subgons of same shapes on the normal dissection.



Frieze Patterns Realizable by Quotient Dissections

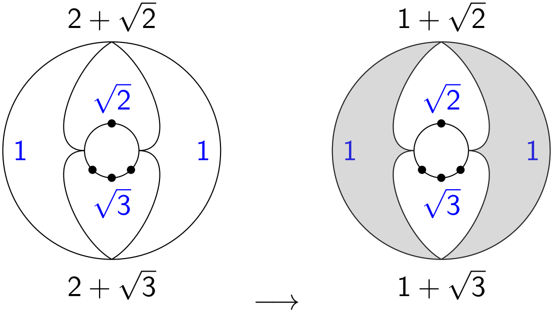
Proposition

Let $\mathcal{F}$ be a reduced frieze. If $\mathcal{F}$ passes the realizability test, then $\mathcal{F}$ is realizable either by a normal dissection or by a quotient dissection on annulus.

Basic idea: Add copies of λ_p to the quiddity row until it becomes realizable and then identify the added subgons to obtain a quotient dissection.

Example of Friezes Realizable by Quotient Dissection

- Period 2 frieze:
  $q = \ldots\ 1+\sqrt{3},\ 1+\sqrt{2},\ 1+\sqrt{3},\ 1+\sqrt{2},\ 1+\sqrt{3},\ \ldots$
- Neighboring entries share 1, but only one copy of it.
- $q' = \ldots\ 2+\sqrt{3},\ 2+\sqrt{2},\ 2+\sqrt{3},\ 2+\sqrt{2},\ 2+\sqrt{3},\ \ldots$



Outline

Polygon Paths on Dissections of Annuli

Definition

A *path from vertex i to vertex j*, $w_{i,j}$ would be a sequence of subgons: $w_{i,j} = (p_i, p_{i+1}, p_{i+2}, ..., p_{j-1}, p_j)$ such that p_i incident with vertex v_i on the universal cover of the annulus.

We will use $Path_{i,j}$ to denote the set of all path from v_i to v_j.

- Path of length 0 would have weight 1. $wt((p_i)) = \lambda_{|p_i|}$;
- $wt((p_i, ..., p_j)) = \begin{cases} \lambda_{|p_i|} wt((p_{i+1}, ..., p_j)) & \text{if } p_i \neq p_{i+1} \\ \lambda_{|p_i|} wt((p_{i+1}, ..., p_j)) - wt((p_{i+2}, ..., p_j)) & \text{if } p_i = p_{i+1} \end{cases}$
- Chebyshev polynomial: $U_{k+1}(x) = U_1(x)U_k(x) - U_{k-1}(x)$.
- $wt(w) = \prod_{\text{Distinct } p \in w} U_k(p)$

Nontrivial Entries in a Realizable Frieze

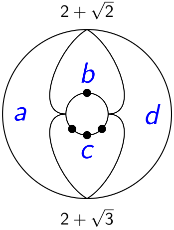
Theorem

Every nontrivial entry in a realizable infinite frieze pattern satisfies that

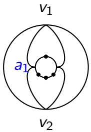
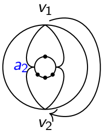
$$m_{i-1,j+1} = \sum_{w \in Path_{i,j}} wt(w)$$

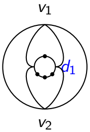
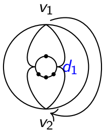
- $m_{\text{i-1,j+1}} = m_{\text{i-1,i+1}} m_{\text{i,j+1}} - m_{\text{i+1,j+1}}$;
- $\sum_{w \in Path_{i,j}} wt(w) = \sum_{w \in Path_{i,i}} wt(w) \sum_{w \in Path_{i+1,j}} wt(w) - \sum_{w \in Path_{i+2,j}} wt(w)$

Nontrivial Entries in a Realizable Frieze-Example



…	0			0		…
…		1			1	…
…	$2+\sqrt{2}$			$2+\sqrt{3}$		…
…		$3+2\sqrt{2}+2\sqrt{3}+\sqrt{6}$			$3+2\sqrt{2}+2\sqrt{3}+\sqrt{6}$	…
		⋰			⋱	

v_1	v_2	$wt(w)$
		$U_1(\lambda_3)U_1(\lambda_3)=1$
a_1	c	$\sqrt{3}$
a_1	d	1
b	a_2	$\sqrt{2}$

		$U_2(\lambda_3)=0$
b	c	$\sqrt{6}$
b	d	$\sqrt{2}$
d	a_2	1
d	c	$\sqrt{3}$

Positivity of Realizable Skeletal Frieze

We can only choose a p-gon in skeletal dissection up to $p-1$ consecutive times.
For $k \leq p-1$, $U_k(\lambda_p) \geq 0$.

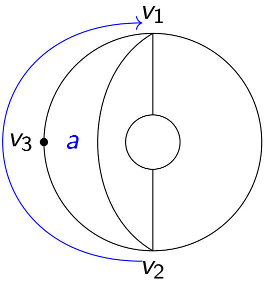
Lemma

Polygon paths on skeletal dissections of annuli always have non-negative weights.

Corollary

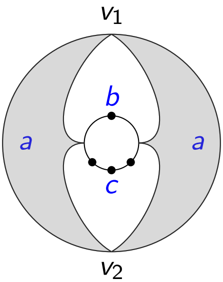
All nontrivial entries in a realizable skeletal frieze are positive.

We can have negatively-weighted path in non-skeletal dissections.



$$(a_1, a_1, a_1) \in Path_{2,4} \;\; wt((a_1, a_1, a_1)) = -1$$

Nontrivial Entries in Friezes Realizable by Quotient Dissections



$$w = (a, a, a, \dots, a, a, a) \in Path_{1,n}$$
$$wt(w) = U_n(1)$$

Outline

The Growth Coefficients

Theorem (Growth Coefficient Theorem, BFPT)

Given an infinite frieze pattern with period n, the growth coefficient $s_k := m_{i,i+kn+1} - m_{i+1,i+kn}$ is constant for each $k \geq 1$.

$$
\begin{array}{ccccccccc}
\cdots & 0 & & 0 & & 0 & \cdots \\
\cdots & & 1 & & 1 & \cdots & \\
\cdots & & & \vdots & & & \cdots \\
\cdots & m_{i,i+kn+1} - s_k & & m_{i+1,i+kn+2} - s_k & & m_{i+2,i+kn+3} - s_k & \cdots \\
\cdots & & m_{i,i+kn} & & m_{i+1,i+kn+1} & & \cdots \\
\cdots & m_{i,i+kn+1} & & m_{i+1,i+kn+2} & & m_{i+2,i+kn+3} & \cdots
\end{array}
$$

Annulus Weights and The Principle Growth Coefficient

Definition

The *annulus weight* of a path is defined as: $wt_A(w) = \prod_{p \in U} U_{N(p)}(\lambda_{|p|})$ where U is the set of all distinct subgons in w, $N(p)$ is the number of times where p is used in w.

Theorem

Let $\mathcal{F}$ be a realizable skeletal infinite frieze pattern of period n and let s_1 denote its principle growth coefficient, then

$$s_1 = \sum_{w \in Path_{i+1,i+n}} wt_A(w)$$

Outline

Positivity and Realizabity

Corollary

A realizable skeletal frieze is positive.

Conjecture

In realizable non-skeletal friezes, for every negatively weighted path, we can find a corresponding positively weighted path that cancels the negative terms. Hence, every realizable infinite frieze is positive.

Conjecture

A frieze that fails the realizability test would contain negative entries.

Realizability $\xrightarrow{?}$ Positivity; Unrealizability $\xrightarrow{?}$ Negativity
What about the infinite frieze that are realizable by quotient dissections?

Acknowledgment

Much appreciation to

Esther Banaian
Libby Farrell
REU at University of Minnesota
National Science Foundation(DMS-1745638)

Thank you!