Generalities on alcove walks

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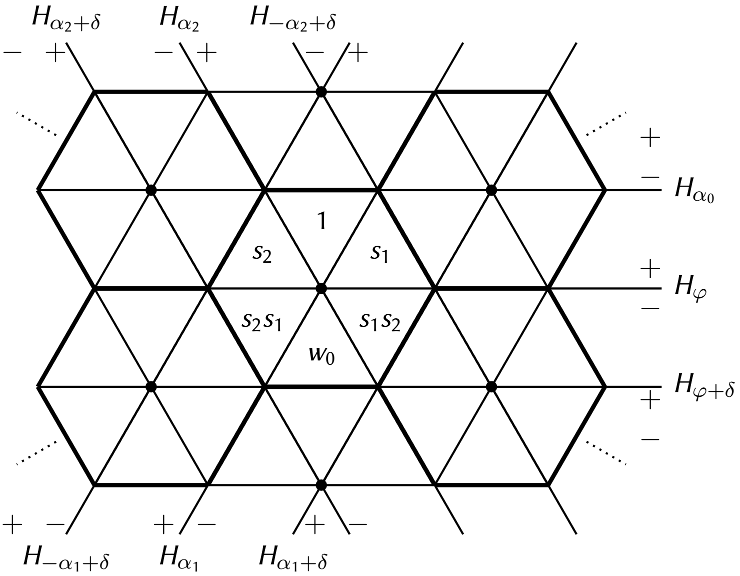
UMN REU final presentation

Introduction

We will outline four results on alcove walks true for higher rank root systems.

1. Reflection property
2. Sign change in folded walk
3. Maximum number of folds
4. Independence of choice of reduced expression in folding

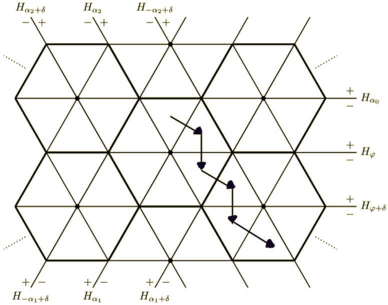
The Alcove model for Sl_3



1. Reflection property of alcove walks

Lemma

If w is a walk with a fold at step k across the hyper-plane $H_{\pm\alpha_i+j\delta}$, then the folded walk w_f is obtained from w by introducing a folded step at step k and reflecting the tail across $H_{\pm\alpha_i+j\delta}$.



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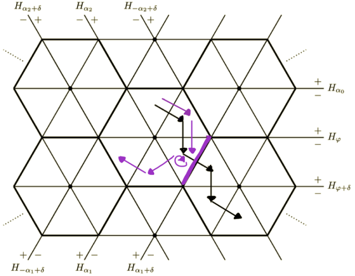

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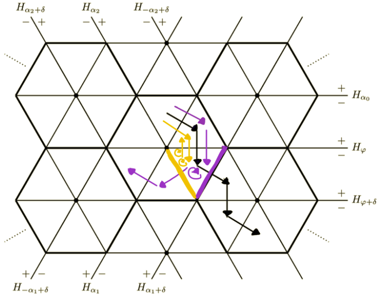
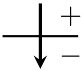


2. Signs of crossings

Steps which cross hyperplanes of type 0 called shapes of type 0 are



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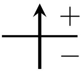
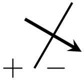


Shapes of type 1 are



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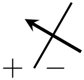
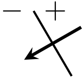


and shapes of type 2 are



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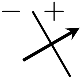
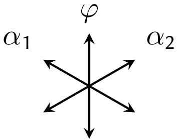


2.Signs of crossings in walk continued

In the Sl_3 case our finite root system is of type A_2.
$\Delta = \{\alpha_1, \alpha_2\}$ are the simple roots
The root system takes the form



Thus, the shapes on the previous slide correspond to positive or negative roots from above. To each walk we associate a set of roots $\{\pm\varphi, \pm\alpha_1, \pm\alpha_2\}$ indicating the types of allowed shapes.

2.Sign change in folded walk

Proposition

Let R be the root system of a simple Lie algebra $\mathfrak{g}$.
Each step in our alcove walk looks like a root.
Steps that look like positive roots cross hyper planes from negative to positive and steps that look like negative roots cross hyper planes from positive to negative.
If we fold at a hyperplane parallel to H_α for some $\alpha \in R^+$, then apply s_α to each step (root) in the tail, where s_α acts by reflection as usual.

Corollary for Sl_3

Let w walk that positively (resp. negatively) folds at step k across a hyperplane $H_{\pm\alpha_i+j\delta}$. Then the positive (resp. negative) crossings in the tail that cross hyperplanes of type i become negative (resp. positive) crossings in w_f. . The other crossings either reverse signage or do not depending on if $H_{\pm\alpha_i+j\delta}$ is of type 0 or not.

Maximum number of folds background

Remark

Steps in a minimal length walk will cross hyper planes of type i exactly one way, either from positive to negative or vice versa. Positive folding only occurs at steps whose shape is a negative root since these are the steps that cross a hyperplane from $+$ to $-$. Similarly, negative folding occurs at steps whose shape is a positive root.

Lemma

A minimal length walk to w contains k negative roots if and only if a reduced expression for w has k letters.
(This follows from Proposition 5.6 in [2]).

Theorem

a)Suppose v is a labeled walk. Then the maximum number of folds that can occur in the positive (resp. negative) folded walk is $\ell(w_0)$.

b) Weyl chambers for the finite Weyl group W are in bijection with the elements of W. If an alcove walk ends in a chamber corresponding to w, then the maximum number of positive folds for that walk is $l(w)$

Thus, in the Sl_3 case we can fold any alcove walk a maximum of 3 times as the longest word $s_1 s_2 s_1$ has length 3.

4.Independence of choice of reduced expression up to labels

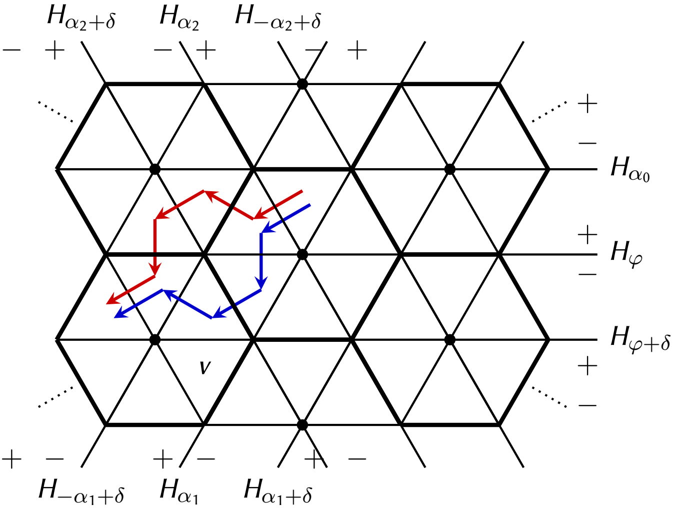
Proposition

If w is a labeled walk that positively (resp. negatively) folds to some alcove, then any other reduced expression w' positively (resp. negatively) folds using the same number of folds to the same alcove for a possibly different labeling.

Example

Consider the two walks $w = s_2 s_0 s_1 s_0 s_2$ and $w' = s_2 s_1 s_0 s_1 s_2$.
They are walks to the same alcove, since they differ by the braid $s_0 s_1 s_0 = s_1 s_0 s_1$.
Write the labels for w be $(c_1, c_2, 0, 0, c_5)$ with $c_1, c_2, c_5 \in \mathbb{F}_q^*$.
w' folds to the same alcove as w with labels $(c_1, c_2, 0, 0, c_5)$.

We have colored w and w' by red and blue respectively.



References

J. Parkinson, A. Ram, C. Schwer: *Combinatorics in affine flag varieties* (2008), arXiv:0801.0709v1.

J. Humphreys: *Reflection groups and Coxeter groups.* Cambridge University Press, 1990.

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