The Whittaker function and alcove walks

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Outline

1. Whittaker functions
2. Tokuyama's formula
3. Example computation in $\mathfrak{sl}_2$

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Defining the Whittaker function

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$$W(t^{\lambda}) = \int_{U^-} v_K(ut^{\lambda})\psi(u)\,du$$

We evaluate over *double Iwasawa cells*

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- Thanks to Beazley-Brubaker [2] we know that

$$C_{\lambda\mu} = \bigcup_{v \in \widetilde{W}} \left(\bigcup_{w, w' \in W} U^- t^\lambda w I \cap I v I \cap U^+ t^\mu w' I \right)$$

From triple intersections to alcove walks

Thanks to Parkinson-Ram-Schwer [1] and Beazley-Brubaker [2] the points of

$$U^-t^\lambda wI \cap IvI \cap U^+t^\mu w'I$$

in G/I are in bijection with **labelled minimal walks to** v **which positively fold to** $t^\lambda w$ **and negatively fold to** $t^\mu w'$

Whittaker function in terms of alcove walks

Due to [2] we can write

$$W(t^\lambda) = \frac{1}{\mathrm{vol}(K)} \sum_{\substack{\mu \in Q^\vee \\ w,w' \in W \\ v \in \widetilde{W}}} \chi(t^\mu) \int_{U^- t^\lambda w I \cap I v I \cap U^+ t^\mu w' I} \psi(u)\, du$$

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- Don't worry, example coming soon...

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Tokuyama's formula

- Tokuyama's formula gives the Whittaker function in terms of Gelfand-Tsetlin patterns.
- Only good for A_n.

Gelfand-Tsetlin patters

- Triangular array of partitions

$$\begin{bmatrix} a_{1,1} & & \cdots & & a_{1,n} \\ & \ddots & & \iddots & \\ & & a_{n,1} & & \end{bmatrix}$$

Gelfand-Tsetlin patters

- Triangular array of partitions

$$\begin{bmatrix} a_{1,1} & & \cdots & & a_{1,n} \\ & \ddots & & \iddots & \\ & & a_{n,1} & & \end{bmatrix}$$

- Entries

$$\begin{matrix} a_{i,j} & & a_{i,j+1} \\ & a_{i+1,j} & \end{matrix}$$

satisfy $a_{i,j} \geq a_{i+1,j} \geq a_{i,j+1}$

More on GTPs

$$a_{i,j} \qquad\qquad a_{i,j+1}$$
$$a_{i+1,j}$$

- $a_{i+1,j}$ called **left-leaning** if $a_{i+1,j} = a_{i,j}$

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- $a_{i+1,j}$ called **special** otherwise
- A GTP is called **strict** if each row is strictly decreasing

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- Then [3]

$$W(t^\lambda) = \sum_{T \in \mathfrak{T}} \left(\prod_{k=1}^{n} z_k^{d_k(T) - d_{k+1}(T)} \right) t^{l(T)} (t+1)^{s(T)}$$

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- Amazing thing is that alcove walks recovers the above formula *bijectively* for $G = SL_2(\mathbb{F}_q((t)))$ [2]

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$$z_1^3 + (t+1)z_1^2 z_2 + (t+1)z_1 z_2^2 + t z_2^3$$

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- Substitute $t \leftrightarrow -1/q, z_1 \leftrightarrow 1, z_2 \leftrightarrow q z_2$
$$z + (q-1)z + q(q-1)z^2 - q^2 z^3$$

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Alcove model for $\mathfrak{sl}_2$

How to compute ψ

- Define

$$\psi_0 : \mathbb{F}_q((t)) \to \mathbb{C}^\times$$
$$\mathbb{F}_q[[t]] \mapsto 1$$
$$(c_1 + c_2\alpha_1 + \cdots + c_n\alpha_{n-1})t^{-k} \mapsto \zeta^{c_1} \cdot \zeta^{c_2} \cdots \zeta^{c_n}$$

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- Then $\psi : U^- \to \mathbb{C}^\times$ is defined as

$$\psi\left(\begin{bmatrix} 1 & & & & \\ x_1 & 1 & & & \\ & & \ddots & & \\ & & x_{n-2} & 1 & \\ & & & x_{n-1} & 1 \end{bmatrix}\right) = \psi_0\left(\sum_{i=1}^{n-1} x_i\right)$$

Example of ψ computation

Example of Whittaker function computation

Pick $\lambda = \alpha^\vee$. Let's find v that positively fold to λ, and where they can negatively fold

The rest of the computation

v	Label restrictions
$t^{\lambda} \cdot s$	$\mathbb{F}_q, \mathbb{F}_q$
$t^{\lambda} \cdot 1$	$\mathbb{F}_q, \mathbb{F}_q, \mathbb{F}_q$
$t^{-\alpha^\vee} \cdot 1$	$\mathbb{F}_q^\times, \mathbb{F}_q$
$t^{-\alpha^\vee} \cdot s$	$\mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q$
$t^{-2\alpha^\vee} \cdot 1$	$0, \mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q$
$t^{-2\alpha^\vee} \cdot s$	$0, \mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q, \mathbb{F}_q$

Table: Label restrictions from positively folding v

The rest of the computation

Original v	Where we negatively fold to	Total label restrictions
$t^{-\alpha^\vee} \cdot 1$	v	$\mathbb{F}_q^\times, \mathbb{F}_q$
$t^{-\alpha^\vee} \cdot s$	v	$\mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q$
$t^{-2\alpha^\vee} \cdot 1$	v	$0, \mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q$
$t^{-2\alpha^\vee} \cdot s$	v	$0, \mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_q, \mathbb{F}_q$
$t^{\lambda} \cdot s$	v	0
$t^{\lambda} \cdot s$	1	$\mathbb{F}_q^\times$
$t^{\lambda} \cdot 1$	v	$0, 0$
$t^{\lambda} \cdot 1$	s	$\mathbb{F}_q^\times, \mathbb{F}_q$
$t^{\lambda} \cdot 1$	1	$0, \mathbb{F}_q^\times$

Table: Total label restrictions from both types of folds

The rest of the computation

Weight	Contribution
$-2\alpha^\vee$	$z^2(-1)q^3 + z^2(-1)q^2 = -q^2(q+1)z^2$
$-\alpha^\vee$	$z(q-1)q + z(q-1)q^2 = q(q-1)(q+1)z$
0	$(q-1) + (q-1)q = (q-1)(q+1)$
λ	$z^{-1} + z^{-1} + z^{-1}(q-1) = (q+1)z^{-1}$

Table: Contributions to Whittaker coeff.

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Table: Contributions to Whittaker coeff.

- Last step is dividing by $\mathrm{vol}(K) = 1 + q$
- We get

$$z^{-1} + (q-1) + q(q-1)z + q^2 z^2$$

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References

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