Virtual Resolutions of Monomial Ideals (REU)

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Monomial Ideals

Definition

A monomial ideal is an ideal that can be generated by monomials.

Example

- $\langle x - y, y \rangle = \langle x, y \rangle$
- $\left\langle x^2, y^2, z^2 \right\rangle$

Staircase Diagrams

Staircase diagrams are a pictorial way to characterize monomial ideals, they rely on the following facts.

REU Exercise (8.1)

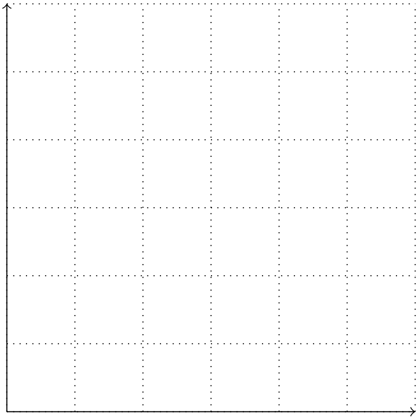
Show the following facts about monomial ideals

1. A monomial ideal is uniquely characterized by the set of monomials it contains. i.e. if two monomial ideals containing the same monomials, they are the same ideal.
2. Every monomial ideal has a unique minimal set of monomial generators.

Staircase Diagrams

Example

Let $I = \langle x^3y, xy^2, y^3 \rangle$



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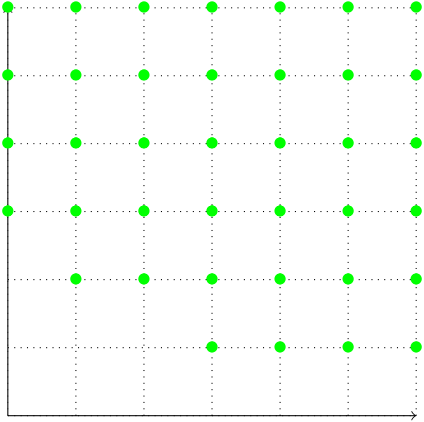

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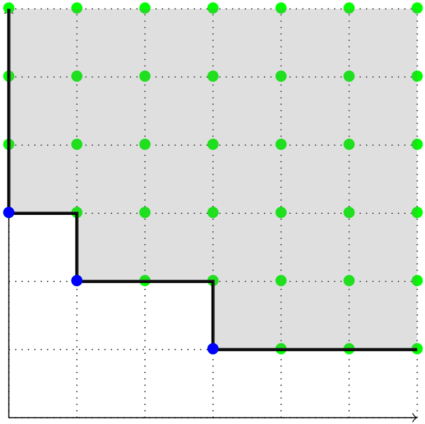


Squarefree Monomial Ideals

Squarefree monomial ideals are a special case of monomial ideas where none of the variables show up in a generator with degree higher than 2.

Definition (Stanley-Reisner Correspondence)

For a simplicial complex Δ on n vertices, define $I_\Delta \subset k[x_1, \ldots, x_n]$ to be the ideal generated by the minimal non-faces.

Theorem (Hochster's Formula)

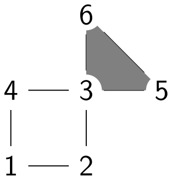
For Δ a simplicial complex and I_Δ the associated Stanley-Reisner Ideal

$$\beta_{i,j}(S/I_\Delta) = \sum_{|\alpha|=j} \dim \widetilde{H}_{i-j-1}(\Delta|_\alpha)$$

Example

	6	
4	3	5
1	2	

Example



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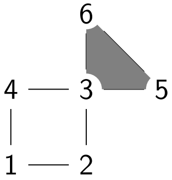

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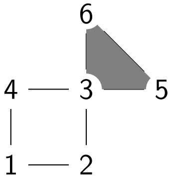

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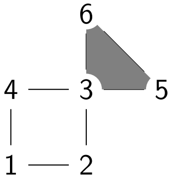

$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$

Randomness

Two sides to randomness:

- Use Randomness to sample to space of possible outcomes
- Prove facts about certain distributions of ideals to

Same Starting Point: Construct a model of a "Random Monomial Ideal".

Random Graphs

The main inspiration for all of this is the theory of Random Graphs.

Theorem (Erdős-Rényi 1976)

Choose a random graph G with $M(n)$ edges on n vertices uniformly. Then for $\epsilon > 0$ as $n \to \infty$ if $M(n) \geq (1+\epsilon)n \log n$, then asymptotically almost surely, the graph is connected. Conversely, if $M(n) \leq (1-\epsilon)n \log n$, then asymptotically almost surely the graph is disconnected.

Existing models of random (monomial) ideals

1. Erdős-Rényi type Random monomial ideals.
2. Random Stanely Reisner Ideals via Random Flag Complexes
3. "RandomIdeals" package in Macaulay 2

Erdos-Renyi type Random Monomial Ideals

First described by De Loera, Petrović, Silverstein, Stasi, and Wilburne in 2018. This model uses 3 parameters, n for the number of variables, D for the maximum degree, and p for the probability (of taking a particular monomial). Consider $n = 2$,$D = 6$,$p = 0.1$

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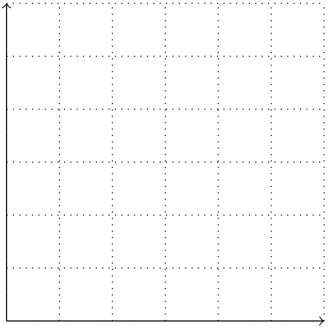

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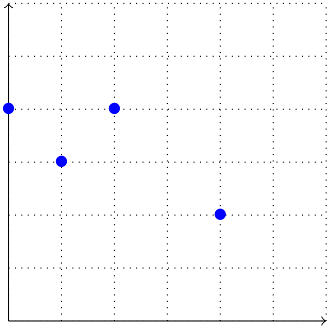

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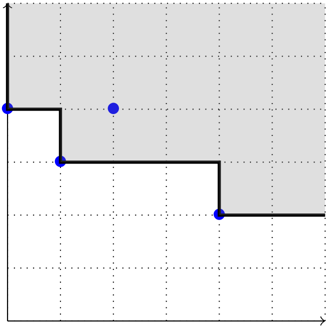
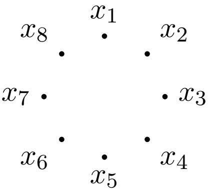


Random Squarefree Monomial Ideals

First described in a joint paper with Daniel Erman also in 2018. This model has two parameters, n for the number of variables, and p for an "attaching probability". Choose a random graph, then create the largest simplicial complex with these edges.

Example

n=8 and p=0.4



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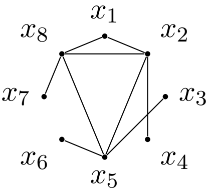

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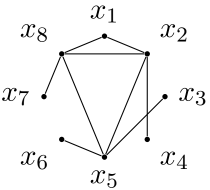

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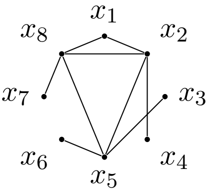

$$I = \langle x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2x_3, x_2x_6, x_2x_7, x_3x_4, x_3x_6, x_3x_7, x_3x_8, x_4x_5, x_4x_6, x_4x_7, x_4x_8, x_5x_7, x_6x_7, x_6x_8\rangle$$

Random Syzygies

Theorem (Erman-Y. 2018)

Fix some $r \geq 1$*. Let* $\Delta \sim \Delta(n,p)$ *with* $\frac{1}{n^{1/r}} \ll p \ll \frac{1}{n^{2/(2r+1)}}$*, then asymptotically almost surely* $r+1 \leq \operatorname{reg}(S/I_\Delta) \leq 2r$*.*

Product of Projective Spaces

Goal: Work with syzygies over a product of projective spaces (or more generally a Toric Variety).

- For $\mathbf{n} \in \mathbb{N}^r$ we write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$.
- Need to define a "Coordinate Ring" for $\mathbb{P}^{\mathbf{n}}$.
- Need to figure out what syzygies should look like.

Multigraded Polynomial Rings

Definition

We say the polynomial ring $k[x_1, \ldots, x_n]$ is $\mathbb{Z}^r$-graded if $\deg(x_i)$ is an element of $\mathbb{Z}^r$

Example

The polynomial ring $k[x_1, \ldots, x_n]$ with the "standard grading" is $\mathbb{Z}$-graded, with $\deg(x_i) = 1$.

Example

Consider the polynomial ring $k[x_0, x_1, y_0, y_1, y_2]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Then the degrees of the following monomials are

- $\deg(x_0 x_1) = (2, 0)$
- $\deg(x_1^2 y_1 y_2) = (2, 2)$

Polynomial Rings for Products of Projective Spaces

Example

The polynomial ring $k[x_0, x_1, y_0, y_1, y_2]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$ is the homogeneous coordinate ring for the space $\mathbb{P}^1 \times \mathbb{P}^2$

More generally, for $\mathbb{P}^{\mathbf{n}}$ the homogeneous coordinate ring is

$$S := k[x_{1,0}, x_{1,1}, \ldots, x_{1,n_1}, x_{r,0}, \ldots, x_{r,n_r}]$$

with $\deg(x_{i,j}) = e_i$ where e_i is the i-th standard basis vector in $\mathbb{Z}^r$ and the irrelevant ideal is $B = \bigcap_{i=1}^r \langle x_{i,0}, \ldots, x_{i,n_i} \rangle$
This is a special case of a more general theory of **Toric Varieties**.

Geometry of a Product of Projective Spaces

$\mathbb{P}^n$ is a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$. In particular, we write a coordinate as $[a_0 : a_1 : \cdots : a_n]$ where we require a_i not all be 0 and two coordinates represent the same point if they differ by a non-zero constant. i.e. $[a_0 : a_1 : \cdots : a_n]$ and $[\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n]$ for $\lambda \neq 0$ are the same point.

Example

Now consider $\mathbb{P}^1 \times \mathbb{P}^2$. The coordinates are of the form $([a_0 : a_1], [b_0 : b_1 : b_2])$ where not all a_i are 0 and not all b_i are 0. Finally $([a_0 : a_1], [b_0 : b_1 : b_2])$ and $([\lambda_1 a_0 : \lambda_1 a_1], [\lambda_2 b_0 : \lambda_2 b_1 : \lambda_2 b_2])$ represent the same point for $\lambda_1, \lambda_2 \neq 0$

Irrelevant Ideal

Remark

The irrelevant ideal corresponds to the coordinates that don't have any geometric realization in $\mathbb{P}^{\mathbf{n}}$. That is to say, it corresponds to the "invalid coordinates".
For example, for $\mathbb{P}^1 \times \mathbb{P}^2$ the irrelevant ideal is $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$
But if $f \in B$, then f is zero on the coordinates where a_0 and a_1 are 0 or where b_0, b_1, and b_2 are all zero.

Homogeneous Polynomial

Definition

A polynomial f in a $\mathbb{Z}^r$-graded polynomial ring is homogeneous if the degree of every term is the same.

Proposition

If f is homogeneous, then $\lambda \in \mathbb{C}^r$ with $\lambda_i \neq 0$, $f(\ldots, \lambda_i \cdot x_{i,j}, \ldots) = 0$ if and only if $f(\mathbf{x}) = 0$

Remark

This is exactly the condition that we need to be able to tell if a polynomial is zero at a point in $\mathbb{P}^{\mathbf{n}}$.

Ideals in a Product of Projective Space

Proposition

Subvarieties of a product of projective spaces correspond to homogeneous B-saturated radical ideals in the homogeneous coordinate ring

$$\{\textit{Varieties in } \mathbb{P}^{\mathbf{n}}\} \leftrightarrow \{\textit{homogeneous } B\textit{-saturated radical ideals}\}$$

Remark

All monomial ideals are homogeneous and a monomial ideal is radical if and only if it is squarefree.

Saturation

Definition

The <u>saturation</u> of an ideal I by an ideal B is given by

$$I : B^\infty := \left\{ r \in S : r \cdot B^k \subset I \text{ for } k \text{ sufficiently large} \right\}$$

Geometrically, the saturation "removes the component corresponding to B"

Proposition

$$V(I : B^\infty) = \overline{V(I) \setminus V(B)}$$

Saturation Example

Example

$$I = \left\langle x_0^2, x_0 * y_0, x_1 * y_0 \right\rangle$$
$$B = \langle x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1 \rangle$$
$$I : B^\infty = \left\langle x_0^2, y_0 \right\rangle$$

REU Exercise (8.2)

1. Given the monomial ideal $\langle x_0x_1^2y_0, y_0y_1^2 \rangle$, compute its saturation with respect to $\langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle$ (You may assume that the saturation of a monomial ideal is a monomial ideal)
2. Check your answer using Macaulay 2
3. Try computing the saturation of some square free monomial ideals. Can you give a geometric method for computing the saturation of a squarefree monomial ideal by another squarefree monomial ideal?

Free Resolutions

Recall the main features of a minimal free resolution

Definition

A complex $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} \cdots$ is a minimal free resolution if

1. C_i are free modules,
2. It is minimal,
3. $d_{i+1} \circ d_i = 0$ for $i > 0$,
4. $\operatorname{img} d_{i+1} = \ker d_i$ for $i > 0$

Virtual Resolutions (for a product of projective spaces)

Definition

A complex $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} \cdots$ is a virtual resolution if

1. C_i are free modules,
2. $d_i \circ d_{i-1} = 0$ for $i > 0$,
3. $H_i(C_\bullet) := \ker d_{i-1} / \operatorname{img} d_i$ is irrelevant for $i > 0$.

Why Virtual Resolutions

Remark

Over $\mathbb{P}^{\mathbf{n}}$ minimal free resolutions don't accurately reflect the geometry.

Theorem (Hilbert Syzygy Theorem)

If I is a non-maximal $\mathbb{Z}^r$-graded ideal on $\mathbb{P}^n$, then S/I has a free resolution of length at most n

Theorem (Berkesch-Erman-Smith, 2017)

Every finitely generated $\mathbb{Z}^r$-graded B-saturated module on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ has a virtual resolution of length at most $n_1 + \cdots + n_r$

Example of a Virtual Resolution

Example

This example is taken from the BES 2017 paper. For I the ideal corresponding to a specific curve in $\mathbb{P}^1 \times \mathbb{P}^2$, we have that the minimal free resolution of I is

$$S^1 \leftarrow \begin{array}{c} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{array} \leftarrow \begin{array}{c} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{array} \leftarrow \begin{array}{c} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{array} \leftarrow S(-3,-7)^1 \leftarrow 0\,.$$

Example of a Virtual Resolution

Example

This example is taken from the BES 2017 paper. For I the ideal corresponding to a specific curve in $\mathbb{P}^1 \times \mathbb{P}^2$, we have that the minimal free resolution of I is

$$S^1 \leftarrow S^8 \leftarrow S^{12} \leftarrow S^6 \leftarrow S^1 \leftarrow 0.$$

However there is a virtual resolution of the form

$$S^1 \leftarrow \begin{array}{c} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \end{array} \leftarrow S(-3,-3)^3 \leftarrow 0\,.$$

What is known?

- Virtual resolutions in a product of projective spaces have length $\leq \sum_i n_i$ (BES 2017)
- Virtual resolution of a pair $(M, \mathbf{b})$ where $\mathbf{b} \in \operatorname{reg}(M)$. (BES 2017)
- Monomial ideals on a toric variety X have virtual resolutions of that have length $\leq \dim X$ (Y. 2019)
- Conditions for points in $\mathbb{P}^1 \times \mathbb{P}^1$ to be virtual complete intersections (Gao, Li, Loper, Mattoo 2020)
- Certain 1-dimensional monomial ideals have length $\dim X - 1$ virtual resolutions. (Work in progress)

Special Case of Virtual Resolutions

Lemma

If I is a B-saturated ideal, and $J : B^{\infty} = I$ then a minimal free resolution of S/J is a virtual resolution of S/I.

Multigraded Regularity

See the paper " Multigraded Castelnuovo-Mumford Regularity" by Diane Maclagan and Greg Smith for a definition. In the case of $\mathbb{P}^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{N}^r$ we have the following properties:

1. $\text{reg}(M) \subset \mathbb{N}^r$.
2. If $\mathbf{b} \in \text{reg}(M)$ then $\mathbf{b} + \mathbb{N}^r \in \text{reg}(M)$.
3. In the case of $\mathbb{P}^n$, $\min(\text{reg}(M))$ is the usual regularity.
4. Macaulay2 can compute it.

Resolution Regularity

Definition (Sidman-Van Tuyl 2006)

For a module M, given a minimal free resolution $F_0 \leftarrow F_1 \leftarrow \cdots$ of M define the resolution regularity denoted $\text{res-reg}(M) \in \mathbb{N}^r$ given by

$$\text{res-reg}(M)_l = \max\{\mathbf{a}_l : \mathbf{a} + i \cdot e_l \text{ is the degree of a generator in } F_i\}$$

Remark

The resolution regularity gives a bound on the multigraded regularity. But in general, it does not give the whole multigraded regularity.

Resolution Regularity

$$\text{res-reg}(M)_l = \max\{\mathbf{a}_l : \mathbf{a} + i \cdot e_l \text{ is the degree of a generator in } F_i\}$$

Example

$$S^1 \leftarrow \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{matrix} \leftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{matrix} \leftarrow S(-3,-7)^1 \leftarrow 0\,.$$

$$\text{res-reg}(S/I) = (2,7)$$

REU Exercise (8.3)

- Use the VirtualResolutions package in Macaulay2 to compute some examples of multigraded regularity
- Write code to compute the resolution regularity

REU Problem

REU Problem

Use random methods to characterize the virtual resolutions of monomial ideals that are given by free resolutions of monomial ideals.

1. Which multidegrees show up as twists in virtual resolutions.
2. What can we say about the "virtual resolution regularities", do they still give bounds on the multigraded regularity?
3. Is there any structure to the set of virtual resolutions coming from monomial ideals?
4. What about monomial modules