F-POLYNOMIAL RATIOS IN THE r-KRONECKER

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ABSTRACT. We provide an algebraic proof for the limit of the ratio of consecutive F-polynomials of the 2-Kronecker. We do this by expanding a proof by Reading [4] which involves cluster scattering diagrams. Furthermore we use this result to recursively define the limit for general r-Kronecker.

CONTENTS

1. INTRODUCTION

This paper explores an infinite limit of ratios of F-polynomials obtained from the cluster algebra associated with the r-Kronecker quiver, which we define as

$${}_rG_{Q_1,Q_2}(y_1, y_2) \coloneqq \lim_{i\to\infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}},$$

where F denotes a F-polynomial and the Q_i sequence satisfies $Q_n = rQ_{n-1} - Q_{n-2}$. F-polynomials are one piece of data that encode information about the Laurent expansions of cluster variables [2]. As such, they inherit the Laurent phenomenon and positivity.

For the cluster algebra associated to the 2-Kronecker quiver, it's known that the limit $\lim_{i\to\infty} F_{i+1}/F_i$ converges. Reading obtained a functional equation for this limit using cluster scattering diagrams. In this report, we develop a distinct functional equation through purely algebraic methods. We hope to further generalize this technique to stable ratios of F-polynomials for other values of r. Our approach to finding the limit of consecutive F-polynomials involves expressing $\lim_{i\to\infty} F_{i+1}^{Q_i}/F_i^{Q_{i+1}}$ as an infinite product and then using that product to develop a functional equation.

Following necessary background information, given in Section 2, this report proves a recurrence identity for F-polynomials which is then used to demonstrate convergence of this limit. This recurrence, and other useful technical results, appear in Section 3. In Section 4, we then go on to express $\lim_{i\to\infty}\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$, for any sequence Q_i which satisfies the recurrence $Q_n = rQ_{n-1} - Q_{n-2}$, as the following infinite product.

Proposition 4.4: *The sequence $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ converges as a formal power series, namely*

$$\lim_{i\to\infty}\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{k=2}^{\infty}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}.$$

In Section 5 we then use this form to develop the following functional equation for ${}_rG_{Q_1,Q_2}(y_1,y_2)$.

Proposition 5.4: *The function $\mathcal{N}(y_1,y_2) := {}_2G_{1,1} = \lim_{i\to\infty}\frac{F_{i+1}}{F_i}$ satisfies the functional equation*

$$\mathcal{N}(y_1,y_2) = \mathcal{N}\left(\frac{y_1^2y_2}{(1+y_1)^2},\frac{1}{y_1}\right)\cdot(1+y_1).$$

Finally, in Section 5.3, we generalize this functional equation and find a closed form for ${}_2G_{0,1}$. Because a closed form is already known for ${}_2G_{1,1}$ and any sequence Q_i which satisfies the recurrence $Q_{n+2} = 2Q_{n+1} - Q_n$ is a linear combination of those sequences, this completely solves the $r = 2$ case.

Theorem 5.7: *We have*

$${}_2G_{0,1}(y_1,y_2) = \lim_{i\to\infty}\frac{F_{i+1}^{i-1}}{F_i^i} = \frac{(1+y_1+y_1y_2)^2-4y_1y_2+(-1+y_1+y_1y_2)\sqrt{(1+y_1+y_1y_2)^2-4y_1y_2}}{2y_1}.$$

2. Background

2.1. Cluster Algebras.

Cluster algebras are a subclass of commutative algebras with distinguished generators, introduced by Fomin and Zelevinsky in 2000 [1]. They function as a concrete combinatorial framework to investigate dual canonical bases and total positivity within semisimple groups.

Definition 2.1. *Suppose $\mathcal{F}$ is a field of rational functions in n independent variables with coefficients in $\mathbb{QP}$, where $(\mathbb{P},\bigoplus,\cdot)$ is an arbitrary semifield. A **cluster seed** is a triple $(\vec{x},\vec{y},B)$ such that: $\vec{x} = (x_1,...,x_n)$ is a free generating set of $\mathcal{F}$, $\vec{y} = (y_1,...,y_n)$ is an n-tuple with elements in the semifield $(\mathbb{P},\oplus,\cdot)$, and $B = [b_{ij}]_{i,j=1}^n$ is a non skew-symmetrizable matrix with entries in $\mathbb{Z}$. $\vec{x}$ is the **cluster**, $x_1,...,x_n$ are the **cluster variables**, $\vec{y}$ is the **coefficient tuple**, $y_1,...,y_n$ are the **coefficient variables**, and B is the **exchange matrix**.*

In this paper, we use the **tropical semifield**, where $a\cdot b = a+b$ and

$$y_1^{t_1}y_2^{t_2}\cdots y_n^{t_n}\oplus y_1^{t_1'}y_2^{t_2'}\cdots y_n^{t_n'} := y_1^{min(t_1,t_1')}y_2^{min(t_2,t_2')}\cdots y_n^{min(t_n,t_n')}.$$

Definition 2.2. *The **cluster algebra** associated with initial seed $(\vec{x},\vec{y},B)$ is the subalgebra of $\mathcal{F}$ generated by the cluster $\vec{x} = (x_1,...x_n)$ and all clusters $\vec{x'} = (x_1',...,x_n')$ of seeds which are reachable from the initial seed via a sequence of mutations. The set of reachable clusters could be finite or infinite. A **cluster algebra with principal coefficients** comes from an extended exchange matrix.*

Definition 2.3. *A mutation in direction k, denoted μ_k, is defined by the relations*

$$x_i' = \begin{cases}\frac{1}{x_k}\left(\frac{1}{1\oplus y_k}\right)\left(y_k\prod_{j=1}^n x_j^{[b_{jk}]_+}\prod_{j=1}^n x_j^{[-b_{jk}]_+}\right) & \textit{if } i=k\\ x_i & \textit{otherwise}\end{cases}$$

$$y_i' = \begin{cases}\frac{1}{y_k} & \textit{if } i=k\\ y_iy_k^{[b_{ki}]_+}(1\oplus y_k)^{b_{ki}} & \textit{otherwise}\end{cases}$$

$$b_{ij}' = \begin{cases}-b_{ij} & \textit{if } i=k \textit{ or } j=k\\ b_{ij}+([-b_{ik}]_+b_{kj}+b_{ik}[b_{kj}]_+) & \textit{otherwise}\end{cases}$$

where $[\cdot]_+ = \max(\cdot,0)$.

Cluster algebras have the Laurent phenomenon, which means that any cluster variable can be written as a Laurent polynomial in terms of any choice of cluster. We refer to this expression as the Laurent expansion.

Definition 2.4. *The ℓ-th **F-polynomial** is the Laurent expansion of the cluster variable x_ℓ in terms of some fixed initial cluster $(x_1, \ldots, x_n)$ with all the x_i specialized to 1.*

The remainder of this paper deals specifically with the r-Kronecker, a rank 2 cluster algebra associated with the initial seed: $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $B_0 = \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}$. We are going to be using the mutation sequence consisting of alternating μ_1 and μ_2.

Definition 2.5. *Let $y_{k,s}$ be the principal coefficient of position k (zero indexed), at mutation s, where $k \in \{0, 1\}, s \in \mathbb{Z}_{\geq 0}$. Let $(x_1, x_2, \ldots)$ be the sequence of cluster variables generated using the above mutation sequence. Then, for $k \equiv s + 1 \mod 2$, mutation for the r-Kronecker gives*

$$\begin{aligned}
x_{s+2}x_s &= \left(\frac{1}{1 \oplus y_{k,s-1}}\right)\left(y_{k,s-1}\prod_{j=0}^{1} x_{s+j}^{[b_{jk}]_+} + \prod_{j=0}^{1} x_{s+j}^{[-b_{jk}]_+}\right) \\
y_{k,s} &= \frac{1}{y_{1-k,s-1}} \\
y_{1-k,s} &= y_{1-k,s-1}y_{k,s-1}^{[b_{k(1-k)}]_+}\left(1 \oplus y_{k,s-1}\right)^{-b_{k(1-k)}} \\
b'_{ij} &= \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + ([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+) & \text{otherwise} \end{cases}
\end{aligned}$$

Example 2.6. *In the following table, we give a choice of initial seed for the 2-Kronecker and 3-Kronecker. We also show several F-polynomials, calculated using Definition 2.3 and Definition 2.4 and the mutation sequence $\mu_2\mu_1$.*

	2-Kronecker	***3-Kronecker***
$\vec{x}$	(x_1, x_2)	(x_1, x_2)
$\vec{y}$	(y_1, y_2)	(y_1, y_2)
exchange matrix	$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$
F-polynomials	$F_1 = F_2 = 1$ $F_3 = 1 + y_1$ $F_4 = (1 + y_1)^2 + y_1^2 y_2$	$F_1 = F_2 = 1$ $F_3 = 1 + y_1$ $F_4 = (1 + y_1)^3 + y_1^3 y_2$

2.2. **Reading and the 2-Kronecker.** In Reading's work [4] on the wall-crossing automorphism on the limiting ray, he examines the ratio between consecutive F-polynomials of the 2-Kronecker. He uses the notation $\mathcal{N}(y_1, y_2) := \lim_{i\to\infty} \frac{F_{i+1}}{F_i}$ for this ratio, and derives the closed form of this function using an algebraic geometry tool called scattering diagrams.

Definition 2.7. *For $i, j \geq 0$, the **Narayana number** $\operatorname{Nar}(i, j)$ is defined as*

$$\operatorname{Nar}(i, j) := \begin{cases} \frac{1}{i}\binom{i}{j}\binom{i}{j-1} & \text{if } i, j > 0 \\ 1 & \text{if } i = j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.8 (Reading, [4])**.** *The limit of consecutive F-polynomials in the 2-Kronecker is*

$$\mathcal{N}(y_1, y_2) = 1 + y_1 \sum_{i,j\geq 0} (-1)^{i+j} \operatorname{Nar}(i, j) y_1^i y_2^j.$$

The known generating function of the Narayana numbers (see [3] for further details) can then be used to simplify $\mathcal{N}(y_1, y_2)$ into the following closed form function.

Corollary 2.9 (Reading, Corollary 3.11)**.**

$$\mathcal{N}(y_1, y_2) = \frac{1 + y_1 + y_1y_2 + \sqrt{(1 + y_1 + y_1y_2)^2 - 4y_1y_2}}{2}.$$

In section 5.2, we provide an alternate proof of the above theorem without the use of scattering diagrams.

3. F-polynomial Recurrence

Recall our notation for the limit of consecutive F-polynomials of the r-Kronecker, ${}_rG_{Q_1,Q_2}(y_1, y_2)$. We now lay the groundwork for the infinite product form of ${}_rG_{Q_1,Q_2}$. In particular, we prove Proposition 3.1 which is used in the inductive step of Lemma 4.1 and will allow us to recursively calculate F-polynomials.

Proposition 3.1. *We have the following recurrence for F-polynomials:*

$$F_{s+1}F_{s-1} = F_s^r + y_1^{a_s} y_2^{a_{s-1}}.$$

Where $\{a_i\}$ is defined by $a_1 = 0, a_2 = 1, a_n = ra_{n-1} - a_{n-2}$ for $n \geq 3$.

Note that the $\{a_i\}$ sequence is nonnegative and increasing. This recurrence can be used to calculate the first few F-polynomials:

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= 1 + y_1 \\
F_4 &= (1 + y_1)^r + y_1^r y_2 \\
F_5 &= \frac{((1 + y_1)^r + y_1^r y_2)^r + y_1^{r^2-1} y_2^r}{1 + y_1}
\end{aligned}$$

To prove Proposition 3.1, we require two intermediate lemmas.

Lemma 3.2. *At mutation s, the exchange matrix is $B_s = (-1)^s \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$.*

Proof. Recall that the initial exchange matrix for the r-Kronecker is $B_0 = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$, where we use the convention of zero indexing matrix entries. Using the mutation formula 2.3 for the matrix B, it is easy to see that at mutation s, the exchange matrix is $B_s = (-1)^s \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$. □

Lemma 3.3. *On mutation s, where $s \equiv k \mod 2$, for $s > 0$,*

$$y_{k,s} = y_1^{a_{s+2}} y_2^{a_{s+1}}$$

$$y_{k,s+1} = \frac{1}{y_1^{a_{s+2}} y_2^{a_{s+1}}}$$

where a_i is as in Proposition 3.1: $a_1 = 0$, $a_2 = 1$ and $a_n = ra_{n-1} - a_{n-2}$ for $n \geq 3$.

Proof. We proceed by induction on s. Here, our base case is:

$$y_{0,0} = y_1, \quad y_{1,0} = y_2$$

$$y_{0,1} = \frac{1}{y_1}, \quad y_{1,1} = y_1^r y_2$$

$$y_{1,2} = \frac{1}{y_1^r y_2}$$

By induction, assume that Lemma 3.3 holds for $s - 1$. Then for $1 - k \equiv s + 1 \mod 2$, Definition 2.3 and Lemma 3.2 give us

$$\begin{aligned} y_{1-k,s+1} &= y_{1-k,s} y_{k,s}^{[b_{k(1-k)}]_+} (1 \oplus y_{k,s})^{-b_{k(1-k)}} \\ &= y_{1-k,s} y_{k,s}^{[b_{ki}]_+} \\ &= y_{1-k,s} y_{k,s}^{r}. \end{aligned}$$

Note that because the $\{a_i\}$ sequence is nonnegative, $1 \oplus y_{k,s} = 1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned} y_{1-k,s+1} &= \frac{1}{y_1^{a_{s+1}} y_2^{a_s}} \cdot y_1^{ra_{s+2}} y_2^{ra_{s+1}} \\ &= y_1^{ra_{s+2}-a_{s+1}} y_2^{ra_{s+1}-a_s} \\ &= y_1^{a_{s+3}} y_2^{a_{s+2}}, \end{aligned}$$

as desired.

□

Lemma 3.4. *For $k \equiv s \mod 2$, $1 \oplus y_{k,s} = 1$*

Proof. From Lemma 3.3, we know that $y_{y_{k,s}}$ has nonnegative exponents, as the $\{a_i\}$ sequence is nonnegative. □

Proof of Proposition 3.1. Recall that the recurrence for the next cluster variable is

$$x_{s+2} x_s = \left(\frac{1}{1 \oplus y_{k,s-1}} \right) \left(y_{k,s-1} \prod_{j=0}^{n-1} x_{s+j}^{[b_{jk}]_+} + \prod_{j=0}^{n-1} x_{s+j}^{[-b_{jk}]_+} \right)$$

where $s-1 \equiv k \mod 2$, and $y_1^i y_2^j \oplus y_1^{i'} y_2^{j'} = y_1^{min(i,i')} y_2^{min(j,j')}$. By Lemma 3.4, $1 \oplus y_{k,s-1} = 1$, and our recurrence is reduced to

$$x_{s+2} x_s = y_{k,s-1} \prod_{j=0}^{1} x_{s+j}^{[b_{jk}]_+} + \prod_{j=0}^{1} x_{s+j}^{[-b_{jk}]_+}.$$

By inspection of the exchange matrix B_s at mutation s from Lemma 3.2, $\prod_{j=0}^{1} x_{s+j}^{[b_{jk}]_+} = 1$, and $\prod_{j=0}^{1} x_{s+j}^{[-b_{jk}]_+} = x_{s+1}^r$. Thus, we are left with

$$x_{s+2} x_s = y_{k,s-1} + x_{s+1}^r = x_{s+1}^r + y_1^{a_{s+1}} y_2^{a_s}$$

by Lemma 3.3, and it follows that the specialization $x_1 = x_2 = 1$ gives us

$$F_{s+2} F_s = F_{s+1}^r + y_1^{a_{s+1}} y_2^{a_s}.$$

□

4. Rewriting ${}_rG_{Q_1,Q_2}$ as an Infinite Product

Before we begin studying ${}_rG_{Q_1,Q_2} = \lim_{i \to \infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ we will first find an expression for $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ in Lemma 4.1, and then take a limit of this expression as a formal power series. In the following section, the infinite product will be used to find a functional equation for ${}_rG_{Q_1,Q_2}$.

Lemma 4.1. *For all sequences Q_i satisfying $Q_n = rQ_{n-1} - Q_{n-2}$, we have the equality*

$$\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{k=2}^{i} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} = \prod_{k=2}^{i} \left(\frac{F_{k+1} F_{k-1}}{F_k^r} \right)^{Q_k}.$$

This proposition follows from of the identity $F_{k+1} F_{k-1} = F_k^r + y_1^{a_k} y_2^{a_{k-1}}$ (Proposition 3.1). Each factor in the product is simply the right side of proposition 3.1 divided by F_k^r, and can therefore be re-written $\frac{F_{k+1} F_{k-1}}{F_k^r}$. This last expression appears after inductively breaking up $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$.

Proof. The equality

$$\frac{F_3^{Q_2}}{F_2^{Q_3}} = (1+y_1)^{Q_2} = \prod_{k=2}^{2}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}$$

follows directly from our computations in section 3 and forms the base case of our induction. Then for $i \geq 3$, we can compute

$$\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \frac{F_{i+1}^{Q_i}}{F_i^{rQ_i - Q_{i-1}}} = F_i^{Q_{i-1}}\left(\frac{F_{i+1}}{F_i^r}\right)^{Q_i}\left(\frac{F_{i-1}^{Q_i}}{F_{i-1}^{Q_i}}\right) = \left(\frac{F_{i+1}F_{i-1}}{F_i^r}\right)^{Q_i}\frac{F_i^{Q_{i-1}}}{F_{i-1}^{Q_i}}$$

Finally, we conclude by applying the inductive hypothesis and Proposition 3.1:

$$\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \left(\frac{F_{i+1}F_{i-1}}{F_i^r}\right)^{Q_i}\prod_{k=2}^{i-1}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k} = \prod_{k=2}^{i}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}.$$

□

We now proceed to take the limit of $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ as a formal power series. Intuitively, the $\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}$ term in each factor has larger and larger minimal degree, in the sense that for each pair $(b_1, b_2) \in \mathbb{N}$, there is an i so that $\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}$ contains no terms $y_1^{c_1}y_2^{c_2}$ with $c_1 < b_1$ or $c_2 < b_2$. This degree eventually surpasses each (b_1, b_2), which causes the coefficient of $y_1^{b_1}y_2^{b_2}$ to be equal to some finite product.

Before we prove that $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ converges as a formal power series, we require two technical lemmas.

Lemma 4.2. *All of the F_i polynomials have constant term 1 and include only terms $c_{i,j}y_1^iy_2^j$ such that $i > j$ (condition A). Let $H_i = 1 - F_i$. Then H_i has only terms involving $y_1^iy_2^j$ such that $i > j$ (condition B).*

Proof. We can easily compute the first few F-polynomials using the recurrence relation for the r-Kronecker (see Proposition 3.1), $F_{i+1}F_{i-1} = F_i^r + y_1^{a_i}y_2^{a_{i-1}}$.

$$F_1 = F_2 = 1, \qquad F_3 = 1 + y_1^{a_2}y_2^{a_1}, \qquad F_4 = (1 + y_1^{a_2}y_2^{a_1})^r + y_1^{a_3}y_2^{a_2}.$$

Note that the sequence (a_n) is strictly increasing. We then proceed by induction on i. By assumption, we know F_{i-1} has constant term 1, so $\frac{1}{F_{i-1}} = \frac{1}{1-H_{i-1}} = \sum_{j=0}^{\infty} H_{i-1}^j$. Using the recurrence relation for F-polynomials, we can now write

$$F_{i+1} = \left(F_i^r + y_1^{a_i}y_2^{a_{i-1}}\right)\left(\frac{1}{F_{i-1}}\right) = \left(F_i^r + y_1^{a_i}y_2^{a_{i-1}}\right)\left(\sum_{j=0}^{\infty} H_{i-1}^j\right).$$

By our inductive hypothesis, F_i and H_{i-1} satisfy conditions A and B respectively. Since both conditions are preserved under multiplication and addition, $\sum_{j=1}^{\infty} H_{i-1}^j$ also satisfies condition B, and so $\sum_{j=0}^{\infty} H_{i-1}^j$ satisfies condition A. Thus, F_{i+1} is a product of two polynomials satisfying condition A, and so also satisfies condition A. It follows that H_{i+1} satisfies condition B.

□

Lemma 4.3. $\lim_{i\to\infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ *has constant term 1 and its other terms are of the form $c_{i,j}y_1^iy_2^j$ where $i > j$ (condition A).*

Proof. Recall from Proposition 4.4 that,

$$\lim_{i\to\infty}\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{i=2}^{\infty}\left(1+\frac{y_1^{a_i}y_2^{a_{i-1}}}{F_i^r}\right)^{Q_i}.$$

Consider a single factor in this infinite product. Using the notation from the previous section $F_i = 1 - H_i$. Thus, the factor expands into

$$1 + \frac{y_1^{a_i} y_2^{a_{i-1}}}{F_i^r} = 1 + y_1^{a_i} y_2^{a_{i-1}} \left(\frac{1}{1 - H_i} \right)^r = 1 + y_1^{a_i} y_2^{a_{i-1}} \left(\sum_{j=0}^{\infty} H_i^j \right)^r .$$

This satisfies condition A, since Lemma 4.2 asserts H_i contains only terms involving $y_1^i y_2^j$, where $i > j$ and the sequence a_i is increasing. Thus, every term in the infinite product satisfies condition A, which is closed under polynomial multiplication. So, the limit also satisfies condition A.

□

Proposition 4.4. *The sequence* $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ *converges as a formal power series, namely*

$$\lim_{i \to \infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{k=2}^{\infty} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} .$$

Proof. As in lemma 4.2, let $H_k := 1 - F_k$. Then H_k has no constant term and we can write

$$\begin{aligned} 1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} &= 1 + y_1^{a_k} y_2^{a_{k-1}} \frac{1}{(1 - H_k)^r} \\ &= 1 + y_1^{a_k} y_2^{a_{k-1}} \left(\sum_j H_k^j \right)^r \end{aligned}$$

Every term in $y_1^{a_k} y_2^{a_{k-1}} \left(\sum_j H_k^j \right)^r$ has degree at least (a_k, a_{k-1}). Take $(b_1, b_2) \in \mathbb{N}^2$. We will show that the coefficient of $y_1^{b_1} y_2^{b_2}$ in $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ eventually (as $i \to \infty$) stabilizes to the corresponding coefficient in

$$\prod_{k=2}^{\infty} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} ,$$

thereby proving the lemma. Since $r \geq 2$, there exists some M such that $a_{M-2} > b_1$ and $a_{M-3} > b_2$. Because

$$\prod_{k=M+1}^{i} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} = \prod_{k=M+1}^{i} \left(1 + y_1^{a_k} y_2^{a_{k-1}} \left(\sum_{j \geq 0} H_k^j \right)^r \right)^{Q_i}$$

and the expression $y_1^{a_k} y_2^{a_{k-1}} \left(\sum_j H_k^j \right)^r$ has only terms with degree strictly larger than (b_1, b_2), we can rewrite the above product as

$$\prod_{k=M+1}^{i} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} = 1 + I(y_1, y_2)$$

where $I(y_1, y_2)$ has only terms with degree larger than (b_1, b_2).

Let $i \geq M$. By Lemma 4.1, we know that

$$\begin{aligned} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} &= \prod_{k=2}^{i} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} \\ &= \left(\prod_{k=2}^{M} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} \right) (1 + I(y_1, y_2)) \\ &= \prod_{k=2}^{M} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} + I(y_1, y_2) \left(\prod_{k=2}^{M} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r} \right)^{Q_k} \right) \end{aligned}$$

The second summand in the above expression has only terms of degree larger than (b_1, b_2). Then the coefficient of $y_1^{b_1}y_2^{b_2}$ in $\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}$ is exactly the coefficient of $y_1^{b_1}y_2^{b_2}$ in $\prod_{k=2}^{M}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}$, which is independent of i. This is true for all (b_1, b_2), so

$$\lim_{i\to\infty}\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}}=\prod_{k=2}^{\infty}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}$$

as desired.

□

5. A Functional Equation for ${}_rG_{Q_1,Q_2}$

5.1. **General r.** We find a functional equation for ${}_rG_{Q_1,Q_2}$. This functional equation arises from the substitution given in the following proposition.

Proposition 5.1. *For $y_1^\circ = (\frac{y_1}{1+y_1})^r y_2, y_2^\circ = \frac{1}{y_1}$, we have*

$$\frac{(y_1^\circ)^{a_k}(y_2^\circ)^{a_{k-1}}}{F_k(y_1^\circ, y_2^\circ)^r}=\frac{y_1^{a_{k+1}}y_2^{a_k}}{F_{k+1}(y_1,y_2)^r}.$$

Proof. We use induction on k, with base cases $k = 2, 3$. Notice $F_2(y_1, y_2) = 1$, $F_3(y_1, y_2) = y_1 + 1$, and $F_4(y_1, y_2) = (y_1 + 1)^r + y_1^r y_2$. Then one easily sees that

$$\frac{(y_1^\circ)^{a_2}(y_2^\circ)^{a_1}}{F_2(y_1^\circ, y_2^\circ)^r}=y_1^\circ=\frac{y_1^r y_2}{(1+y_1)^r}=\frac{y_1^{a_3}y_2^{a_2}}{F_3(y_1,y_2)^r},$$

and also, with some additional effort,

$$\frac{(y_1^\circ)^{a_3}(y_2^\circ)^{a_2}}{F_3(y_1^\circ, y_2^\circ)^r}=\frac{(y_1^\circ)^r y_2^\circ}{(1+y_1^\circ)^r}=\frac{y_1^{r^2-1}y_2^r}{((1+y_1)^r+y_1^r y_2)^r}=\frac{y_1^{a_4}y_2^{a_3}}{F_4(y_1,y_2)^r}.$$

Thus, we have established the base case for our inductive argument. Assuming that the equality holds for $k-1$ and $k-2$, and using Proposition 3.1, we get

$$\begin{aligned}
F_k(y_1^\circ, y_2^\circ)^r &= \left(\frac{F_{k-1}(y_1^\circ, y_2^\circ)^r + (y_1^\circ)^{a_{k-1}}(y_2^\circ)^{a_{k-2}}}{F_{k-2}(y_1^\circ, y_2^\circ)}\right)^r \\
&= \left(\frac{(y_1^\circ)^{a_{k-1}}(y_2^\circ)^{a_{k-2}}}{F_{k-2}(y_1^\circ, y_2^\circ)}\right)^r \cdot \left(1+\frac{F_k(y_1,y_2)^r}{y_1^{a_k}y_2^{a_{k-1}}}\right)^r \\
&= \left(\frac{(y_1^\circ)^{a_{k-1}}(y_2^\circ)^{a_{k-2}}}{F_{k-2}(y_1^\circ, y_2^\circ)}\right)^r \cdot \left(\frac{F_{k-1}(y_1,y_2)F_{k+1}(y_1,y_2)}{y_1^{a_k}y_2^{a_{k-1}}}\right)^r \\
&= \frac{((y_1^\circ)^{a_{k-1}}(y_2^\circ)^{a_{k-2}})^r}{(y_1^\circ)^{a_{k-2}}(y_2^\circ)^{a_{k-3}}} \cdot \frac{y_1^{a_{k-1}}y_2^{a_{k-2}}}{F_{k-1}(y_1,y_2)^r} \cdot \left(\frac{F_{k-1}(y_1,y_2)F_{k+1}(y_1,y_2)}{y_1^{a_k}y_2^{a_{k-1}}}\right)^r \\
&= (y_1^\circ)^{a_k}(y_2^\circ)^{a_{k-1}} \cdot \frac{F_{k+1}(y_1,y_2)^r}{y_1^{a_{k+1}}y_2^{a_k}},
\end{aligned}$$

as desired.

□

Note that the $\circ$ operation "shifts the factors" of the infinite product, producing the following immediate corollary.

Corollary 5.2. *With y_1°, y_2° as above, we have*

$$\prod_{k=2}^{\infty}\left(1+\frac{(y_1^\circ)^{a_k}(y_2^\circ)^{a_{k-1}}}{F_k(y_1^\circ, y_2^\circ)^r}\right)^{Q_k}=\prod_{k=3}^{\infty}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_{k-1}}.$$

Using the recurrence $Q_k = rQ_{k-1} - Q_{k-2}$ for $k \geq 3$, we obtain the following additional corollary.

Corollary 5.3. *With y_1°, y_2° as above and $y_1^{\circ\circ} = (\frac{y_1^\circ}{1+y_1^\circ})^r y_2^\circ, y_2^{\circ\circ} = \frac{1}{y_1^\circ}$, we have*

$$ {}_rG_{Q_1,Q_2}(y_1,y_2) = \frac{{}_rG_{Q_1,Q_2}(y_1^\circ, y_2^\circ)^r}{{}_rG_{Q_1,Q_2}(y_1^{\circ\circ}, y_2^{\circ\circ})} \cdot (1+y_1)^{Q_2} \cdot \left(1 + \frac{y_1^r y_2}{(1+y_1)^r}\right)^{Q_3 - rQ_2}. $$

Using the substitution $y_1 \mapsto \frac{1}{y_2}$, $y_2 \mapsto (y_2+1)^r y_1$, we obtain the following alternative form:

$$ \begin{aligned} &{}_rG_{Q_1,Q_2}(y_1,y_2)^r \left(1 + \frac{1}{y_2}\right)^{Q_2} (1+y_1)^{Q_3 - rQ_2} \\ &\quad = {}_rG_{Q_1,Q_2}\left(\frac{1}{y_2}, (y_2+1)^r y_1\right) \cdot {}_rG_{Q_1,Q_2}\left(\frac{y_1^r y_2}{(1+y_1)^r}, \frac{1}{y_1}\right). \end{aligned} $$

Although it may not be immediately obvious, the latter form is often a simplification. For example, in the $r=3$ case with $Q_2 = 1, Q_3 = 4$ the equation from Corollary 5.2 is

$$ G(y_1,y_2) = \frac{G\left(\frac{y_1^3 y_2}{(1+y_1)^3}, \frac{1}{y_1}\right)^3}{G\left(\frac{y_1^8 y_2^3}{(y_1^3 y_2 + (1+y_1)^3)^3}, \frac{(1+y_1)^3}{y_1^3 y_2}\right)} \cdot (1+y_1) \cdot \left(1 + \frac{y_1^3 y_2}{(1+y_1)^3}\right). $$

Using the alternative form, the above equation is equivalent to

$$ G(y_1,y_2)^3 \left(1 + \frac{1}{y_2}\right)(1+y_1) = G\left(\frac{1}{y_2}, (y_2+1)^3 y_1\right) \cdot G\left(\frac{y_1^3 y_2}{(1+y_1)^3}, \frac{1}{y_1}\right). $$

This procedure works for $r=2$ as well. In fact, because in the $r=2$ we have $Q_i = 1$ for all i, we don't need to do the above "2-deep" substitution and only need to substitute once. Then (using Reading's notation for $\mathcal{N}$ as the limit of ratio of F-polynomials) we get the following simpler form of our functional equation.

Proposition 5.4. *The function $\mathcal{N}(y_1,y_2) := {}_2G_{1,1} = \lim_{i\to\infty} \frac{F_{i+1}}{F_i}$ satisfies the functional equation*

$$ \mathcal{N}(y_1,y_2) = \mathcal{N}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) \cdot (1+y_1). $$

Proof. Using Corollary 5.2, we get

$$ \begin{aligned} \mathcal{N}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) &= \prod_{k=2}^{\infty} \left(1 + \frac{(y_1^\circ)^{a_k} (y_2^\circ)^{a_{k-1}}}{F_k(y_1^\circ, y_2^\circ)^2}\right)^1 \\ &= \prod_{k=3}^{\infty} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^2}\right)^1 \\ &= \frac{\mathcal{N}(y_1,y_2)}{1+y_1}. \end{aligned} $$

□

5.2. **Power Series for $\mathcal{N}(y_1,y_2)$.** As mentioned in Section 5.2, Reading provides a proof of the closed form limit of F-polynomial ratios in the 2-Kronecker. To do so, he utilizes a functional equation derived from properties of scattering diagrams. In this section, we use our own functional equation (Proposition 5.4) to provide an alternate proof of Reading's Theorem 3.10 without scattering diagrams.

Theorem 5.5.

$$ \mathcal{N}(y_1,y_2) = 1 + y_1 \sum_{i,j\geq 0} (-1)^{i+j} \operatorname{Nar}(i,j) y_1^i y_2^j $$

Proof. We first write $\mathcal{N}(y_1, y_2)$ as a power series with unknown coefficients,

$$\mathcal{N}(y_1, y_2) = \sum_{i,j\geq 0} n_{ij} y_1^i y_2^j,$$

and rewrite the functional equation of Proposition 5.4 in terms of this power series expansion:

$$\begin{aligned}
\mathcal{N}(y_1, y_2) &= \mathcal{N}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) \cdot (1+y_1) \\
\sum_{i,j\geq 0} n_{ij} y_1^i y_2^j &= (1+y_1) \sum_{i,j\geq 0} n_{ij} \left(\frac{y_1^2 y_2}{(1+y_1)^2}\right)^i \left(\frac{1}{y_1}\right)^j \\
\sum_{\ell,m\geq 0} n_{\ell m} y_1^\ell y_2^m &= \sum_{i,j\geq 0} n_{ij} y_1^{2i-j} y_2^i (1+y_1)^{-2i+1}.
\end{aligned}$$

Now, we can extract the coefficient of $y_1^\ell y_2^m$ from both sides of our functional equation to get an explicit form for each $n_{\ell m}$:

$$n_{\ell m} = \sum_{k=2m-\ell}^{2m} n_{mk} \binom{-2m+1}{\ell-2m+k} = \sum_{k=0}^{m} n_{mk} \binom{-2m+1}{\ell-2m+k}. \tag{1}$$

The second equality changes the limits of summation but does not change the sum in any substantial way. The lower index can be written as $k = 0$ because $n_{mk} = 0$ for all $k < 0$, and $\binom{-2m+1}{\ell-2m+k} = 0$ for all $k < 2m-\ell$, so the only terms either added or removed are zeroes. For the upper limit we use Lemma 4.3, which asserts that $n_{mk} = 0$ for all $k > m$.

Notice that whenever both $n_{\ell m}$ and n_{mk} are nonzero, then either $\ell = 0$ or $\ell > 0$ and $\ell > m \geq k$, so we must have $\ell > k$. Thus, Equation (1) may be read as an expression for n_{ij} in terms of n_{00} and other nonzero $n_{i'j'}$ such that $j' < j$.

We claim that the power series coefficients can be written as

$$n_{ij} = (-1)^{(i-1)+j}\,\mathrm{Nar}(i-1, j).$$

Recall the definition of the **rising factorial**: $(x)_{j-1} := (x)(x+1)\cdots(x+j-2)$ for any real x and positive integer j; for $j \leq 0$ we write $(x)_{j-1} = 0$. Using this notation, for all $i > j \geq 1$, we can rewrite our claimed formula for the power series coefficients as

$$\begin{aligned}
n_{ij} &= (-1)^{(i-1)+j}\,\mathrm{Nar}(i-1, j) \\
&= (-1)^{i+j+1}\left(\frac{1}{i-1}\right)\binom{i-1}{j}\binom{i-1}{j-1} \\
&= (-1)^{i+j+1}\frac{(2-i)_{j-1}(1-i)_{j-1}}{(2)_{j-1}(j-1)!}.
\end{aligned}$$

We prove this formula via induction on j. For the base cases $j = 0$ and $j = 1$, note that $n_{00} = 1$ follows from Proposition 4.3. Combining Equation (1) with Proposition 4.3 yields that $n_{i0} = n_{00}\binom{1}{i}$, and that

$$n_{i1} = \sum_{k=0}^{1} n_{1k}\binom{-1}{i+k-2} = n_{10}\binom{-1}{i-2} + n_{11}\binom{-1}{i-1} = n_{10}(-1)^{i-2} = n_{10}\binom{-1}{i}.$$

This completes the base case. We now prove the claimed formula for $n_{\ell m}$ with $m \geq 2$, given that it holds for all n_{ij} with $j < m$. In other words, we want to show that

$$n_{\ell m} = (-1)^{\ell+m+1}\frac{(2-\ell)_{m-1}(1-\ell)_{m-1}}{(2)_{m-1}(m-1)!}.$$

First, apply Equation (1) and the inductive hypothesis:

$$\begin{aligned} n_{\ell m} &= \sum_{k=2m-\ell}^{m} n_{mk} \binom{-2m+1}{\ell-2m+k} \\ &= \sum_{k=2m-\ell}^{m} (-1)^{m+k+1} \frac{(2-m)_{k-1}(1-m)_{k-1}}{(2)_{k-1}(k-1)!} \binom{-2m+1}{\ell-2m+k}. \end{aligned}$$

Now observe the following rewriting of the binomial coefficient $\binom{-2m+1}{\ell-2m+k}$:

$$\binom{-2m+1}{\ell-2m+k} = (-1)^{k-1} \binom{-2m+1}{\ell-2m+1} \frac{(\ell)_{k-1}}{(\ell-2m+2)_{k-1}}.$$

Using this, we manipulate the series for $n_{\ell m}$ into a standard hypergeometric form.

$$\begin{aligned} n_{\ell m} &= \sum_{k=2m-\ell}^{m} (-1)^{m+k+1} \frac{(2-m)_{k-1}(1-m)_{k-1}}{(2)_{k-1}(k-1)!} \binom{-2m+1}{\ell-2m+k} \\ &= \sum_{k=2m-\ell}^{m} (-1)^{m+k+1}(-1)^{k-1} \binom{-2m+1}{\ell-2m+1} \frac{(2-m)_{k-1}(1-m)_{k-1}(\ell)_{k-1}}{(2)_{k-1}(\ell-2m+2)_{k-1}(k-1)!} \\ &= (-1)^{m} \binom{-2m+1}{\ell-2m+1} \sum_{k=1}^{\infty} \frac{(2-m)_{k-1}(1-m)_{k-1}(\ell)_{k-1}}{(2)_{k-1}(\ell-2m+2)_{k-1}(k-1)!}, \end{aligned}$$

where the last equality holds because the terms in the summation equal zero outside of the interval $[2m-\ell, m]$. We can see that this infinite summation is the following hypergeometric series:

$$(-1)^{m} \binom{-2m+1}{\ell-2m+1} {}_3F_2 \left(\begin{matrix} 2-m & \ell & 1-m \\ 2 & \ell-2m+2 & \end{matrix} ; 1 \right).$$

This ${}_3F_2$ is suitable for transformation under Saalschütz's Theorem for hypergeometric series, and then we obtain the desired formula after routine manipulations:

$$\begin{aligned} n_{\ell m} &= (-1)^{m} \binom{-2m+1}{\ell-2m+1} \frac{(m)_{m-1}(2-\ell)_{m-1}}{(2)_{m-1}(m-\ell)_{m-1}} \\ &= \left((-1)^{m} \frac{(2-\ell)_{m-1}}{(2)_{m-1}} \right) (-1)^{\ell-2m+1} \binom{\ell-1}{\ell-2m+1} \frac{(m)_{m-1}}{(m-\ell)_{m-1}} \\ &= \left((-1)^{m} \frac{(2-\ell)_{m-1}}{(2)_{m-1}} \right) (-1)^{\ell+1}(-1)^{m-1} \frac{(\ell-1)(\ell-2)\cdots(\ell-m+1)}{(m-1)!} \\ &= (-1)^{\ell+m+1} \frac{(2-\ell)_{m-1}(1-\ell)_{m-1}}{(2)_{m-1}(m-1)!}. \end{aligned}$$

□

5.3. **Power series for generalized F-polynomial ratios in the 2-Kronecker.** It turns out that we can find a closed form for ${}_2G_{Q_1,Q_2}$ in general. Notice that an arbitrary Q sequence satisfying $Q_{n+2} = 2Q_{n+1} - Q_n$ is a linear combination of the sequences $(1, 1, 1, \dots)$ and $(0, 1, 2, 3, \dots)$ (since it is specified by two parameters), and also that

$${}_2G_{Q_1,Q_2} = {}_2G_{1,1}^{Q_1} \cdot {}_2G_{0,1}^{Q_2-Q_1}.$$

Since we have already determined ${}_2G_{1,1}$, it suffices to determine ${}_2G_{0,1}$. In this case, $Q_i = i-1$ and so

$${}_2G_{0,1}(y_1, y_2) = \lim_{i\to\infty} \frac{F_{i+1}^{i-1}}{F_i^{i}} = \left(1 + \frac{y_1}{F_2^2}\right) \left(1 + \frac{y_1^2 y_2}{F_3^2}\right)^2 \left(1 + \frac{y_1^3 y_2^2}{F_4^2}\right)^3 \cdots.$$

Making the substitution $y_1^\circ = \frac{y_1^2 y_2}{(1+y_1)^2}, y_2^\circ = \frac{1}{y_1}$ yields the following proposition.

Proposition 5.6. *We have the functional equation*

$$ {}_2G_{0,1}\left(\frac{y_1^2y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) = \left(1+\frac{y_1^2y_2}{F_3^2}\right)\left(1+\frac{y_1^3y_2^2}{F_4^2}\right)^2\left(1+\frac{y_1^4y_2^3}{F_5^2}\right)^3\cdots = \frac{{}_2G_{0,1}(y_1,y_2)}{\mathcal{N}(y_1,y_2)}. $$

Solving this functional equation yields the following theorem.

Theorem 5.7. *We have*

$$ {}_2G_{0,1}(y_1,y_2) = \lim_{i\to\infty}\frac{F_{i+1}^{i-1}}{F_i^i} = \frac{(1+y_1+y_1y_2)^2-4y_1y_2+(-1+y_1+y_1y_2)\sqrt{(1+y_1+y_1y_2)^2-4y_1y_2}}{2y_1}. $$

Proof. Note that we can rearrange our functional equation, Proposition 5.6, to

$$ {}_2G_{0,1}(y_1,y_2) = {}_2G_{0,1}\left(\frac{y_1^2y_2}{(1+y_1)^2}, \frac{1}{y_1}\right)\mathcal{N}(y_1,y_2). $$

We first write the left hand side, ${}_2G_{0,1}$, as a power series with unknown coefficients:

$$ {}_2G_{0,1}(y_1,y_2) = \sum_{i,j\geq 0} g_{ij}y_1^iy_2^j. $$

Then, the right hand side can be written in terms of these coefficients as follows:

$$ \begin{aligned} {}_2G_{0,1}\left(\frac{y_1^2y_2}{(1+y_1)^2}, \frac{1}{y_1}\right)\mathcal{N}(y_1,y_2) &= \sum_{i,j\geq 0} g_{ij}\left(\frac{y_1^2y_2}{(1+y_1)^2}\right)^i\left(\frac{1}{y_1}\right)^j\mathcal{N}(y_1,y_2) \\ &= \sum_{i,j\geq 0} g_{ij}y_1^{2i-j}y_2^i(1+y_1)^{-2i}\mathcal{N}(y_1,y_2). \end{aligned} $$

By Proposition 4.3, we know that $g_{00}=1$, and that in order for other g_{ij} to be non-zero, we must have $i>j$ or equivalently $i<2i-j$. In each term on the right hand side, the factors $(1+y_1)^{-2i}\mathcal{N}(y_1,y_2)$ only increase the degree of the y_1 and y_2 relative to the indices of the coefficients, as we can write both $(1+y_i)^{-2i}$ and $\mathcal{N}(y_1,y_2)$ as power series. Then comparing coefficients of each $y_1^iy_2^j$ between ${}_2G_{0,1}(y_1,y_2)$ and $\sum_{i,j\geq 0} g_{ij}y_1^iy_2^j$, we get g_{ij} on the right side and a linear combination of g_{cd} on the right side for $c<a$ and $d<b$ (except in the $y_1^0y_2^0$ case, where we already know $g_{00}=1$). Then our functional equation uniquely specifies the coefficients of ${}_2G_{0,1}$: we can recursively construct the g_{ij} from the bottom up.

Therefore there is a unique power series solution to the functional equation satisfying condition A. Since the formula given satisfies the functional equation, we are done. $\square$

An arbitrary Q sequence satisfying $Q_{n+2}=2Q_{n+1}-Q_n$ is a linear combination of the sequences $(1,1,1,\dots)$ and $(0,1,2,3,\dots)$, so using the above theorem completely solves the $r=2$ case, as we have

$$ {}_2G_{Q_1,Q_2} = {}_2G_{1,1}^{Q_1}\cdot {}_2G_{0,1}^{Q_2-Q_1}. $$

5.4. **Power Series Coefficients for the r-Kronecker.** For $r\geq 3$, the polynomial equation $x^2=rx-1$ has two distinct roots, which we have denoted λ_+ and λ_-. Then the sequence $(Q_1,Q_2,\dots)$ is a linear combination of the sequences $(r-\lambda_+,1,\lambda_+,\lambda_+^2,\dots)$ and $(r-\lambda_-,1,\lambda_-,\lambda_-^2,\dots)$. For $\alpha\in\{\lambda_+,\lambda_-\}$, we have a simpler functional equation for ${}_rG_{1,\alpha}$.

Proposition 5.8. *For $y_1^\circ = (\frac{y_1}{1+y_1})^ry_2, y_2^\circ = \frac{1}{y_1}$, the functional equation*

$$ {}_rG_{1,\alpha}(y_1,y_2) = {}_rG_{1,\alpha}(y_1^\circ,y_2^\circ)^\alpha\cdot(1+y_1) $$

holds.

Proof. It is easy to see by induction that $Q_k = \alpha^{k-1}$. Then using Corollary 5.2, we get

$$\begin{aligned}
{}_rG_{1,\alpha}(y_1^\circ, y_2^\circ) &= \prod_{k=2}^{\infty}\left(1 + \frac{(y_1^\circ)^{a_k}(y_2^\circ)^{a_{k-1}}}{F_k(y_1^\circ, y_2^\circ)^2}\right)^{\alpha^{k-1}} \\
&= \prod_{k=3}^{\infty}\left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^2}\right)^{\alpha^{k-2}} \\
&= \left(\frac{{}_rG_{1,\alpha}(y_1, y_2)}{1 + y_1}\right)^{1/\alpha}.
\end{aligned}$$

□

Definition 5.9. *Note that taking a power series with constant term 1 to the power α is a well-defined notion in the ring of power series, due to the equality*

$$(1 + x)^\alpha = \sum x^i \binom{\alpha}{i}.$$

Solving the above functional equation for $\alpha = \lambda_+, \lambda_-$ yields a solution for general ${}_rG_{Q_1,Q_2}$.

6. Another Functional Equation for ${}_rG_{Q_1,Q_2}$

Inspired by Reading, we demonstrate a functional equation for ${}_rG_{Q_1,Q_2}$ that does not require writing it as an infinite product, though it presupposes the existence of ${}_rG_{Q_1,Q_2}$ as a limit.

Theorem 6.1. *We have the functional equation*

$$x^{g_iQ_{i-1} - g_{i-1}Q_i}\, {}_rG_{Q_1,Q_2}(x_2^{-2}, x_1^2) = \left(\frac{x_1^r + 1}{x_2}, x_1\right)^{g_{i+1}Q_{i-1} - g_iQ_i}\, {}_rG_{Q_0,Q_1}\left(x_1^{-2}, \left(\frac{x_1^r + 1}{x_2}\right)^2\right).$$

Proof. We have

$$\lim_{i\to\infty} \frac{x_i^{Q_{i-1}}}{x_{i-1}^{Q_i}} = x^{g_iQ_{i-1} - g_{i-1}Q_i} \lim_{i\to\infty} \frac{F_i(\hat{y})^{Q_{i-1}}}{F_{i-1}(\hat{y})^{Q_i}} = x^{g_iQ_{i-1} - g_{i-1}Q_i}\, {}_rG_{Q_1,Q_2}(\hat{y}).$$

Writing $\tilde{x}_i = x_{i-1}$, we have that $\tilde{x}_1, \tilde{x}_2, \ldots$ forms a cluster algebra (without principal coefficients). If we set $y_1 = y_2 = 1$, then the cluster algebra with cluster variables $\tilde{x}_1, \tilde{x}_2, \ldots$ has the same relations as the cluster algebra with cluster variables $x_1, x_2, \ldots$. Under the condition $y_1 = y_2 = 1$, we have $\hat{y}_1 = x_2^{-2}, \hat{y}_2 = x_1^2$, so

$$\lim_{i\to\infty} \frac{x_i^{Q_{i-1}}}{x_{i-1}^{Q_i}} = x^{g_iQ_{i-1} - g_{i-1}Q_i}\, {}_rG_{Q_1,Q_2}(x_2^{-2}, x_1^2).$$

By translating from x to $\tilde{x}$, we get

$$\lim_{i\to\infty} \frac{x_i^{Q_{i-1}}}{x_{i-1}^{Q_i}} = \lim_{i\to\infty} \frac{\tilde{x}_{i+1}^{Q_{i-1}}}{\tilde{x}_i^{Q_i}} = \tilde{x}^{g_{i+1}Q_{i-1} - g_iQ_i} \lim_{i\to\infty} \frac{F_{i+1}(\tilde{x}_2^{-2}, \tilde{x}_1^2)^{Q_{i-1}}}{F_i(\tilde{x}_2^{-2}, \tilde{x}_1^2)^{Q_i}} = \tilde{x}^{g_{i+1}Q_{i-1} - g_iQ_i}\, {}_rG_{Q_0,Q_1}(\tilde{x}_2^{-2}, \tilde{x}_1^2).$$

Using $\tilde{x}_2 = x_1, \tilde{x}_1 = x_0 = \frac{1 + x_1^r}{x_2}$, we get the result. □

7. Conjectures

A natural direction for future work would be to investigate a closed form for the coefficients of ${}_rG_{1,\alpha}$ 5.5, and eventually for the power series as a whole. Currently, we can recursively calculate these coefficients through a method similar to 5.5.

Additionally, we hope to expand our proof of infinite product convergence to literal convergence of infinite products for positive y_1, y_2.

Somewhat tangentially, it appears that the coefficients of unspecialized F-polynomials of the 3-Kronecker follow a pattern related to rows of the arithmetic triangle, briefly discussed below.

$F_1 = F_2 = 1$, $F_3 = 1 + y_1$, $F_4 = 1 + 3y_1 + 3y_1^2 + y_1^3 + y_1^3 y_2$. Notice that the first four coefficients of F_4 are the third row of the arithmetic triangle.

$F_5 = 1 + 8y_1 + 28y_1^2 + 56y_1^3 + 70y_1^4 + 56y_1^5 + 28y_1^6 + 8y_1^7 + y_1^8 + 3y_1^3 y_2 + 15y_1^4 y_2 + 30y_1^5 y_2 + 30y_1^6 y_2 + 15y_1^7 y_2 + 3y_1^8 y_2 + 3y_1^6 y_2^2 + 6y_1^7 y_2^2 + 3y_1^8 y_2^2 + y_1^8 y_2^3$. Notice that the coefficients are rows of the arithmetic triangle scaled by a row of the arithmetic triangle. Particularly, 1(row 8), 3(row 5), 3(row 2), 1(row -1).

For $i = 4$ and $i = 5$, we can more specifically write this observation as

$$F_i = \sum_k \binom{\deg_1 F_{i-1}}{k} (1 + y_1)^{\deg_1 F_i - 3k} (y_1^3 y_2)^k,$$

where $\deg_a F_i$ denotes the highest degree of y_a present in F_i. It appears that the pattern of the coefficients of F_6 initially follows the above summation before becoming exceedingly more complicated. This tangent has been abandoned.

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