Shelling AugBerg and the Weak Lefschetz Property

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- *Independent sets*: sets of linearly independent vectors. *Flats*: closed under linear span
- A matroid can be equiv. defined by its independent sets or by its flats

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2. $I(\mathcal{M})$ is a simplicial complex in which faces correspond to independent sets of $\mathcal{M}$

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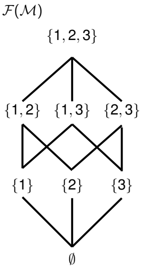
- Simplices are given by $\{y_i\}_{i \in I} \cup \{x_{F_1}, \ldots x_{F_k}\}$ where $I \in \mathcal{I}(\mathcal{M})$ and $I \subseteq F_1 \subset F_2 \subset \ldots \subset F_k$

AugBerg Example



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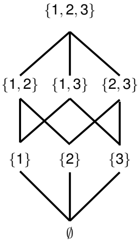

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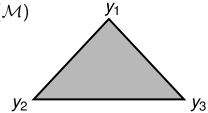

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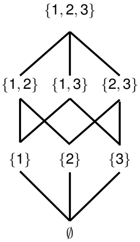

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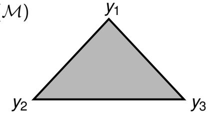

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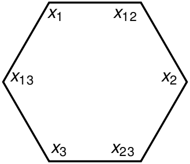

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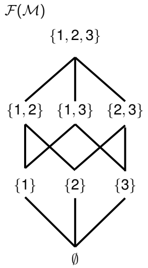

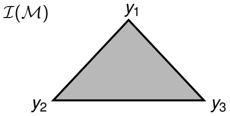

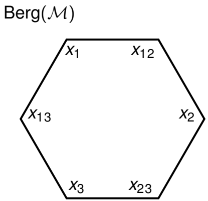

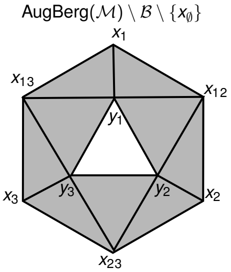

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- Also known that AugBerg is gallery connected, a weaker property than shellable [1]

A Natural Question

Is AugBerg shellable?

Shelling AugBerg

Theorem

AugBerg(M) is shellable. Furthermore, we have

- a shelling that shells Cone(Berg(M)) first and $I(M)$ last.
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Idea

We leverage the following two well-known facts.

- For the "base case," apply the lexicographic shelling of I(M)
- For the "inductive step," apply the lexicographic shelling of Berg(M') for some "quotient" of M

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The Shelling Order

Shell in increasing order based on rank of independent set.

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1. If $\#I < \#J$, order T_i before T_j.
2. If $\#I = \#J$ but $I \neq J$,
   Apply the lexicographic order on I and J.
3. If $I = J$, then $F_1^i = F_1^j = \text{span}\{I\} =: F$
   Define the *contraction matroid*
   $M/F = (E \setminus F, \{I : I \cup F \in I(M)\})$.
   Then $\{\text{Flats in } M \text{ containing } F\} \leftrightarrow \{\text{Flats in } M/F\}$.
   Apply the shelling order on Berg(M/F).

Shelling AugBerg: I(M) to Cone

The Shelling Order

Shell in *decreasing* order based on rank of independent set!

Homotopy Type of AugBerg

Let M be a matroid of rank $r(M)$. Recall the **Tutte Polynomial**:

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}$$

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- $Cone(Berg(M))$ is homotopy equiv. to a wedge of $T_M(1, 0)$ spheres of dimension $r(M) - 2$ (Garsia [2])

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- $Cone(Berg(M))$ is homotopy equiv. to a wedge of $T_M(1,0)$ spheres of dimension $r(M)-2$ (Garsia [2])

Our Result

$AugBerg(M)$ is homotopy equiv. to a wedge of $T_M(1,1)$ spheres of dimension $r(M)-1$.

Moving on...

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Now introducing:

the Weak Lefschetz Property

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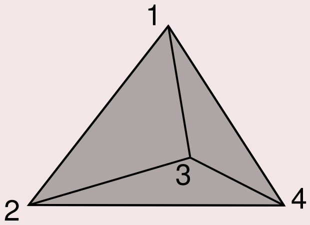

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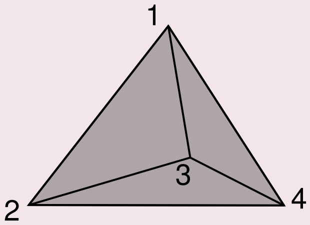

Taking Δ to be the boundary of a tetrahedron, we have $K[\Delta] = K[x_1, x_2, x_3, x_4]/(x_1 x_2 x_3 x_4)$.

Linear Systems of Parameters

Definition

A linear system of parameters (**LSOP**) $\underline{\theta}$ is a set of $\theta_i \in K[\Delta]$ that are linear in the x_j's such that $K[\Delta]/(\underline{\theta})$ is finite dimensional over K

M($\underline{\theta}$)

$$M(\underline{\theta}) == \begin{bmatrix} - & \theta_1 & - \\ \vdots & \vdots & \vdots \\ - & \theta_r & - \end{bmatrix}$$

Fact

If Δ is the boundary of a simplicial polytope, then we can get an LSOP as follows: $M(\underline{\theta}) = \begin{bmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{bmatrix}$

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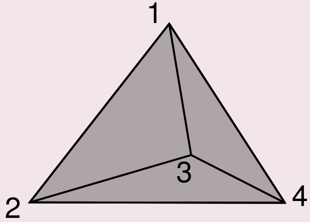

$$M(\underline{\theta}) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{aligned} \theta_1 &= x_1 - x_4 \\ \theta_2 &= x_2 - x_4 \\ \theta_3 &= x_3 - x_4 \end{aligned}$$

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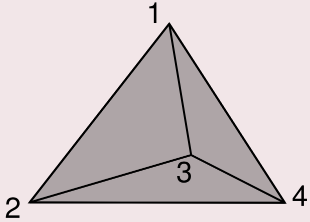

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Definition

Given an $\ell \in A_1$, we say that ℓ is Weak-Lefschetz (WL) if and only if the multiplication by ℓ map $(\cdot\ell)$ from A_i to A_{i+1} is full rank for all $i \in \{0, \ldots, d-1\}$.

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In particular, if Δ is the boundary of a convex simplicial polytope, then ℓ is WL iff $\cdot\ell$ from A_i to A_{i+1} is injective for $i < r/2$ and surjective otherwise, since the dimensions of the A_i's are symmetric and unimodal.

What do we want to know?

Big Question

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Matroidal

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Does WL property depend on minors of $\hat{M}(\underline{\theta}, \ell)$?

Reduction to Middle Map

Proposition

- If d odd, ℓ is WL $\iff A_{\frac{d-1}{2}} \xrightarrow{\cdot \ell} A_{\frac{d+1}{2}}$ is injective.
- If d even, ℓ is WL $\iff A_{\frac{d}{2}-1} \xrightarrow{\cdot \ell} A_{\frac{d}{2}}$ is injective

  $\iff A_{\frac{d}{2}} \xrightarrow{\cdot \ell} A_{\frac{d}{2}+1}$ is surjective.

Reduction to Even Dimensions

Bipyramid Construction

For a polytope P, let P', its bipyramid, be the polytope with vertex set $\{x_1 \cdots x_n\} \bigcup \{x_{n+1} x_{n+2}\}$, where

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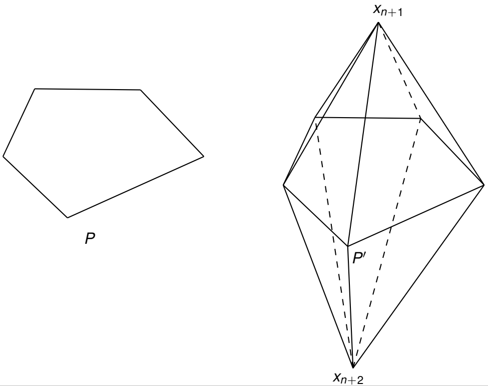

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Proposition

Let d be odd.
$\sum_{i=1}^{n} \alpha_i x_i \in A_1$ is WL in $A \Longleftrightarrow \sum_{i=1}^{n} \alpha_i x_i \in A'_1$ is WL in A'.

Stacked Polytopes

Stacking Construction

Let P be a polytope and $F \in \mathcal{F}(P)$.
To obtain P' from P, add in a new vertex x_{n+1} "close enough" to F on the outside.

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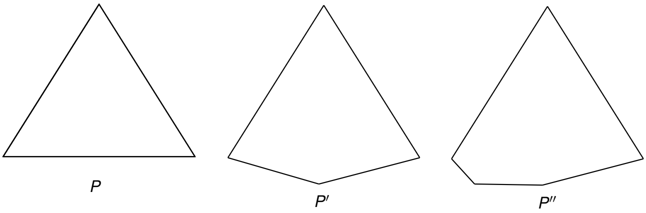

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$$\sum_{i=1}^{n+1} \alpha_i x_i \in A_1' \text{ is WL in } A' \iff \begin{cases} \sum_{i=1}^{n} \alpha_i x_i \in A_1 \text{ is WL in } A \\ \alpha_{n+1} \neq 0 \end{cases}$$

Cyclic Polytopes

Definition

$C(n,d)$, the d-dimensional polytope on n vertices is the convex hull of any n points on the moment curve

$$t \mapsto \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$$

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- Let d even. ℓ is WL $\iff \ell \neq 0$
- Let d odd. ℓ is WL $\iff$ all minors of $M((\underline{\theta}), \ell)$ with columns indexed by $\{x_1, x_{i_1}, x_{i_2}, \cdots x_{i_{d-1}}, x_n\}$ are *L.I.*, where $\{x_1, x_{i_1}, x_{i_2}, \cdots x_{i_{d-1}}\}$ runs through all facets not containing x_n.

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Proposition

Let $\ell = \sum_{i=1}^{n} c_i x_i \in K[\Delta]/(\underline{\theta})$.

- If n is odd, ℓ is WL if and only if $c_i \neq 0$ for all i.
- If n is even, ℓ is WL if and only if $c_i = 0$ for at most one i.

Counterexample

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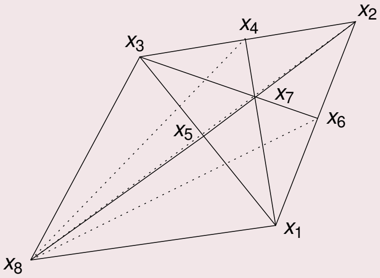

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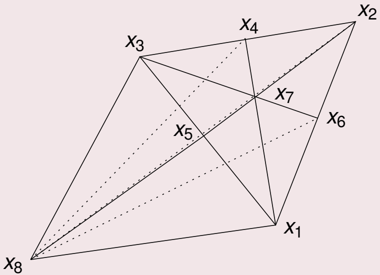
Consider the following Δ:



with vertex LSOP: $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 \end{bmatrix}$

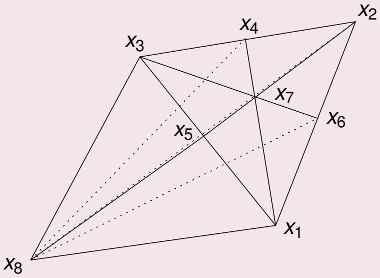
Counterexample

What We Found

Is the WL property matroidal in general? **No!**

Boundary of a Tetrahedron Counterexample

Consider the following Δ:



with vertex LSOP: $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 \end{bmatrix}$

Claim: The rank of $\cdot\ell : A_1 \to A_2$ is not det. by minors of $\hat{M}(\underline{\theta}, \ell)$.

Thank You Slide

Thank you for watching and thank you to all the REU staff who were super thoughtful and encouraging throughout the research process, and especially to Vic for providing team 7 with a great problem to work on, and to Sasha and Trevor for their guidance!

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