Problem 7(a): Augmented Bergman complexes

Elisabeth Bullock, Aidan Kelley, Kevin Ren, Gahl Shemy,
Dawei Shen, Brian Sun, Amy Tao, Joy Zhang

June 2021

Abstract

For a matroid, the augmented Bergman complex is a simplicial complex introduced in recent work of Braden, Huh, Matherne, Proudfoot and Wang [3, 4]. It may be viewed as a hybrid between two other well-studied pure shellable simplicial complexes: the independent set complex and the Bergman complex.

We show that the augmented Bergman complex itself is shellable, via two different shelling orders. We explain how the description of its homotopy type derived from the two shellings fits with a known convolution formula counting bases of the matroid. We also identify concretely the representation of the automorphism group of the matroid on the homology of the augmented Bergman complex, and generalize this description to other types of closures beyond matroid closures.

Contents

1 Introduction

Let M be a matroid on ground set E with rank function $r\colon 2^E \to \mathbb{Z}_{\geq 0}$ and closure function $\mathrm{cl}\colon 2^E \to \mathcal{F}$, where $\mathcal{F}$ denotes the lattice of flats. For a flat $F \in \mathcal{F}$, we denote the restriction of M by F as $M|_F$ and the contraction of M by F as M/F. We will also assume that M is an *ordered matroid*, that is, there is some linear ordering ω on its ground set E. For more details confer the reference by Oxley [9].

There are various well-studied shellable simplicial complexes that one can associate to M; see the survey chapter by Björner [2] as a general reference, and for terminology not defined here. Two among these complexes are the *complex of independent sets* $\mathcal{I}(M)$, and the *Bergman complex* $\mathrm{Berg}(M) := \Delta(\mathcal{F} \setminus \{\varnothing, E\})$ which is the order complex constructed from the nonempty, proper flats of the lattice of flats $\mathcal{F}$.

Here we study the following "hybrid" of these two complexes, which plays an important role in recent work of Braden, Huh, Matherne, Proudfoot and Wang [3, 4].

Definition. Given a matroid M on ground set $E = \{1, 2, \ldots, n\}$, the *augmented Bergman complex* is the abstract simplicial complex on vertex set

$$\{y_1, \ldots, y_n\} \cup \{x_F : \text{proper flats } F \subsetneq E\}$$

whose simplices are the subsets

$$\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{1}$$

for which $I \in \mathcal{I}(M)$ and $I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$. We will often abuse notation and write

$$T = I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$$

for a face T of the complex.

This complex, which we will hereafter denote as $\mathrm{AugBerg}(M)$, is pure of dimension $r(M)-1$, and contains as full-dimensional subcomplexes both

- the independent set complex $\mathcal{I}(M)$, as the simplices in (1) with $\ell = 0$, and
- the cone $\mathrm{Cone}(\mathrm{Berg}(M)) = \Delta(\mathcal{F} \setminus \{E\})$ over the Bergman complex $\mathrm{Berg}(M)$ with cone vertex $x_\varnothing$, as the simplices in (1) with $I = \varnothing$.

Example. Let M_1 be uniform matroid of rank 2 on ground set $E = \{1, 2, 3\}$, which has three bases $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Figure 1 below depicts its lattice of flats $\mathcal{F}$ on the left, and the simplicial complex $\mathrm{AugBerg}(M)$ on the right. In this case, $\mathrm{AugBerg}(M)$ is a 1-dimensional complex, that is a graph.

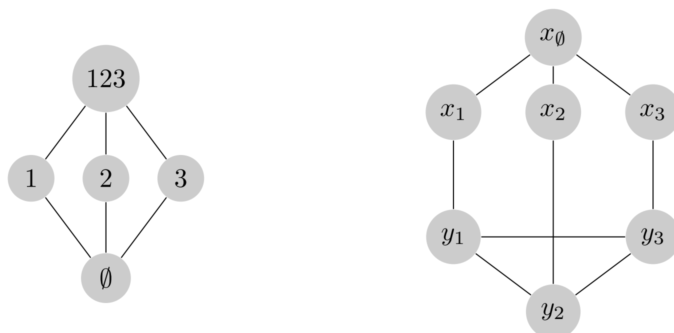


Figure 1: $\mathcal{F}(M_1)$ and $\mathrm{AugBerg}(M_1)$

We recall that a theorem of Provan and Billera [10] shows that $\mathcal{I}(M)$ is shellable, and hence homotopy equivalent to a wedge of spheres of dimension $r(M) - 1$. In fact, the number of spheres in this wedge is the evaluation $T_M(0, 1)$ of the *Tutte polynomial* $T_M(x, y)$, or the number of bases of M having *internal activity zero* with respect to ω; see Björner [2, §7.3].

Similarly, a theorem of Garsia [6] shows that $\mathrm{Berg}(M)$ is shellable, and thus homotopy equivalent to a wedge of spheres of dimension $r(M) - 2$. The number of spheres in this wedge is the Tutte polynomial evaluation $T_M(1, 0)$, or the number of bases of M having *external activity zero*; see Björner [2, §7.4, 7.6]

In Section 2, we will show that $\mathrm{AugBerg}(M)$ is shellable via at least two different orders, one order (Theorem 2.1) which shells the subcomplex $\mathrm{Cone}(\mathrm{Berg}(M))$ first and adds in the facets of $\mathcal{I}(M)$ last; the other (Theorem 2.2) shells the subcomplex $\mathcal{I}(M)$ first and adds in the facets of $\mathrm{Cone}(\mathrm{Berg}(M))$ last.

Consequently, $\mathrm{AugBerg}(M)$ is homotopy equivalent to a wedge of spheres of dimension $r(M) - 1$. In Section 3, we will show that these two shellings give rise to two different descriptions for the number of spheres in this wedge, the first shelling showing (Corollary 3.2) that the number of spheres is

$$T_M(1, 1) = \#\{\text{bases of } M\} \tag{2}$$

and the second shelling showing (Corollary 3.5) that the number of spheres is

$$\begin{aligned} &\sum_{F \in \mathcal{F}(M)} T_{M|_F}(0, 1) \cdot T_{M/F}(1, 0) \\ = &\sum_{F \in \mathcal{F}(M)} \#\{\text{bases of } M|_F \text{ of internal activity zero}\} \cdot \#\{\text{bases of } M/F \text{ of external activity zero}\}, \end{aligned} \tag{3}$$

where $M|_F$ and M/F are the matroids obtained from M by *restriction* to F and *contraction* on F, respectively. The concordance between (2) and (3) are well-known convolution formulas [5, 7]. The definitions of internal and external activity are in Section 3.

The simplicity of expression (2) for the number of $(r(M)-1)$-dimensional spheres in the homotopy type of $\mathrm{AugBerg}(M)$ is connected to a stronger statement about its equivariant structure as a representation of the automorphism group $G = \mathrm{Aut}(M)$ for the matroid M. In Section 4, we will give a simple description (Theorem 4.1) of the action of G on the integral homology group $\tilde{H}_{r(M)-1}(\mathrm{AugBerg}(M), \mathbb{Z})$: it is a signed permutation representation, in which $\mathbb{Z}$-basis elements $[B]$ indexed by the ordered bases of the matroid M permuted up to (explicit) signs.

2 Shellability of $\mathrm{AugBerg}(M)$

Recall that a pure simplicial complex Δ is shellable if there exists an ordering $T_1, T_2, \ldots, T_t$ [1] of its facets so that one of the following equivalent conditions holds:

(a) For each $j \geq 2$, the facet T_j intersects the subcomplex generated by the already shelled facets $T_1, \ldots, T_{j-1}$ in a subcomplex that is pure of codimension one inside T_j.

(b) For each pair of facets T_i, T_j where $i < j$, there is a facet T_k with $k < j$ and an element $v \in T_j$ such that $T_i \cap T_j \subseteq T_k \cap T_j = T_j - \{v\}$.

Although the first definition may seem more intuitive, we will mostly use definition (b) in our proofs.

Remark. There is a weaker condition on pure simplicial complexes than shellability, known as *gallery-connectedness* or being *connected in codimension one*. This means that any pair of facets T, T' can be connected by a sequence of facets $T = T_0, T_1, \ldots, T_m = T'$ where T_i, T_{i+1} share a codimension one face for $i = 0, 1, 2, \ldots, m-1$. It was shown that $\mathrm{AugBerg}(M)$ is connected in codimension one in [4, Prop. 2.3], leading one to speculate that it might be shellable.

We now construct a shelling order of $\mathrm{AugBerg}(M)$ that first processes the facets of $\mathrm{Cone}(\mathrm{Berg}(M))$ and finishes with the facets of $\mathcal{I}(M)$. Using the notation from (1), the general approach will be to shell the facets in increasing order based on the rank of I, where ties are broken by leveraging the existence of a shelling order for the Bergman complex of any matroid.

Theorem 2.1. *There exists a shelling of* $\mathrm{AugBerg}(M)$ *that begins with the facets of* $\mathrm{Cone}(\mathrm{Berg}(M))$ *and ends with the facets of* $\mathcal{I}(M)$*.*

Proof. We begin by defining our shelling order. As setup, fix some arbitrary ordering (e.g. lexicographical) for every collection of independent sets of the same rank. Suppose we now have two facets of $\mathrm{AugBerg}(M)$ given by

$$T_i = I \subseteq F_{i,1} \subsetneq F_{i,2} \subsetneq \cdots \subsetneq F_{i,m} \quad \text{and} \quad T_j = J \subseteq F_{j,1} \subsetneq F_{j,2} \subsetneq \cdots \subsetneq F_{j,n}. \tag{4}$$

If $\#I < \#J$, then T_i comes before T_j in the shelling order. If $\#I = \#J$ and $I \neq J$, then T_i comes before T_j if and only if I comes before J in our fixed ordering of rank $\#I$ independent sets.

We consider the final case where $I = J$. Since T_i and T_j are both facets, it must be that $F_{i,1} = F_{j,1}$ as they are closures of the same set. So, all flats of T_i and T_j are contained in the interval $[F_1, E]$ of $\mathcal{F}$, where $F_1 \coloneqq F_{i,1} = F_{j,1}$. To know when T_i is shelled before T_j and vice versa, it suffices to exhibit an ordering for the maximal chains of $[F_1, E]$ that do not contain E and have minimal flat F_1. But $\mathrm{Cone}(\mathrm{Berg}(M/F_1))$ is shellable, and since the interval $[F_1, E]$ is isomorphic to the lattice of flats of M/F_1, the shelling order for $\mathrm{Cone}(\mathrm{Berg}(M/F_1))$ can be lifted up to M to obtain the ordering we so desire. Having covered all cases, we now have an ordering $T_1, T_2, \ldots$ on all facets of $\mathrm{AugBerg}(M)$, which we now prove is a shelling order.

As indicated at the beginning of this section, we must show that for facets T_i and T_j where $i < j$, there is a facet T_k where $k < j$ and an element $v \in T_k$ such that $T_i \cap T_j \subseteq T_k \cap T_j = F_j - \{v\}$. Using the same notation as earlier in the problem, there are two cases to consider.

Case 1: $I \neq J$. Letting $K = I \cap J$, we have that $\#K < \#J$. Using the independent set exchange property, we can add elements from J to K to make a set K' such that $\#K' = \#J - 1$. We can then let

$$T_k = K' \subseteq \mathrm{cl}(K') \subsetneq F_{j,1} \subsetneq \cdots \subsetneq F_{j,n}.$$

[1] We use T for a facet instead of F to avoid confusing with flats.

Since $\#K' < \#J$, T_k comes earlier than T_j in our ordering and one can easily verify that $T_i \cap T_j \subseteq T_k \cap T_j$, and that the latter set is equal to $T_j \setminus \{y_i\}$, where i is the lone element of $J \setminus K'$.

Case 2: $I = J$. As argued previously, we have $F_1 \coloneqq F_{i,1} = F_{j,1}$. This case then falls under jurisdiction of the shelling order derived from Cone(Berg(M/F_1)). In particular, we can contract the chain of flats corresponding to both T_i and T_j by F_1. Shellability of Cone(Berg(M/F_1)) guarantees that it contains a chain of flats in M/F_1 that once lifted back up to M gives precisely the desired T_k after inserting I at the beginning of the chain.

We have thus produced a shelling order that begins by shelling the facets with an empty associated independent set, or Cone(Berg(M)), and ends by shelling the facets associated to the bases of M, or $\mathcal{I}(M)$. □

The next theorem shells the two subcomplxes of AugBerg(M) in the opposite order.

Theorem 2.2. *There exists a shelling of* AugBerg(M) *that begins with the facets of* $\mathcal{I}(M)$ *and ends with the facets of* Cone(Berg(M)).

Proof. As in the previous proof, let T_i and T_j be facets defined by (4). Now, if $\#I > \#J$, then T_i comes before T_j in our shelling order. If $\#I = \#J$ and $I \neq J$, then T_i comes before T_j if and only if I comes before J in a lexicographical ordering of rank $\#I$ independent sets based on our order ω. Again, the trickier case is when $I = J$, but fortunately we can use precisely the same rule that was used to handle this case in Theorem 2.1. We now prove that this order given by $T_1, T_2, \ldots, T_t$ is a shelling. Letting T_i and T_j be facets with $i < j$, there are three cases to handle.

Case 1: $I \neq J$ and $F_{i,1} \neq F_{j,1}$. Pick some $i \in F_{j,2} \setminus F_{j,1}$. Then we can let $T_k = J \cup \{y_i\} \subseteq F_{j,2} \subsetneq \cdots F_{j,n}$. This facet comes earlier than T_j in the shelling order since $\#(J \cup \{y_i\}) > \#J$, and satisfies $T_i \cap T_j \subseteq T_k \cap T_j$ by construction. Finally, $T_k \cap T_j = T_j \setminus \{x_{F_{j,1}}\}$.

Case 2: $I \neq J$ and $F_{i,1} = F_{j,1}$. Define $F_1 \coloneqq F_{i,1} = F_{j,1}$. It must be the case that $\#I = \#J$ as their closures are the same set, and moreover I comes before J lexicographically as $i < j$. It is known that given an order on the ground set, the lexigraphical ordering of the bases gives a shelling order of the independence complex of a matroid; see Björner [2, §7.3]. But both I and J are bases of the matroid $M|_{F_1}$, so by shellability we are given a set K such that $I \cap K \subseteq J \cap K = J \setminus \{v\}$ for some $v \in J$. Letting $T_k = K \subseteq F_1 \subsetneq F_{j,2} \subsetneq \cdots \subsetneq F_{j,n}$ finishes the job.

Case 3: $I = J$. This is handled in the exact same manner as in Theorem 2.1.

We have thus produced a shelling order that begins by by shelling the bases of M, or $\mathcal{I}(M)$, and concludes by shelling the facets with an empty independent set, or Cone(Berg(M)). □

3 Homotopy type of AugBerg(M)

Here we use the shelling from Theorems 2.1 and 2.2 to describe the homotopy type of AugBerg(M).

We start off with a quick definition that appears in [2, §7.2].

Definition. Let Δ be a shellable simplicial complex with shelling order $T_1, T_2, \ldots, T_t$ on its facets. The *restriction* of a facet T_i is defined by

$$\mathscr{R}(T_i) = \{x \in T_i : T_i \setminus \{x\} \in \Delta_{i-1}\},$$

where Δ_{i-1} is the subcomplex generated by the first $i-1$ facets in the shelling order.

Related to this definition is a useful result that relates shellability to homotopy type, which can be found in [2, §7.7].

Lemma 3.1. *Let Δ be a shellable d-dimensional simplicial complex. Suppose that there are p facets T where $\mathscr{R}(T) = T$, or equivalently, $T \subseteq \mathscr{R}(T)$. Then Δ has the homotopy type of a wedge of p copies of the d-sphere.*

Intuitively, one can interpret a facet T where $\mathscr{R}(T) = T$ as one that "caps off" a d-sphere. With that being said, we are ready to move onto our first main result of the section.

Corollary 3.2. *The complex* AugBerg(M) *is a wedge of* $T_M(1,1)$ *copies of the d-sphere, where d is its dimension and* T_M *is the Tutte polynomial of* M*.*

Proof. It is well known that $T_M(1,1)$ gives the number of bases of M. By Lemma 3.1, it suffices to show that this is equal to the number of facets T of AugBerg(M) for which $\mathscr{R}(T) = T$, where we take our shelling to be that of Theorem 2.1. We will do this by proving that facets $\{y_i\}_{i \in B}$ given by the bases B of M are precisely those that satisfy the desired condition.

Take a facet T in AugBerg(M) of the form

$$T = I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell,$$

and suppose first that I is not a basis. We need to find an element of T that is not in $\mathscr{R}(T)$; we claim that F_1 is such an element. To show this, observe that removing F_1 from T results in the chain

$$I \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell.$$

This chain must be contained in a previously shelled facet in order for x_{F_1} to be in $\mathscr{R}(T)$. But there are only two ways to complete it to a facet at all, as the chain $F_2 \subsetneq \cdots \subsetneq F_\ell$ is saturated. We can either increase the cardinality of I by one or insert a flat of rank $\#I$ sandwiched between I and F_2.

In the first case, we would have a facet whose associated independent set has larger cardinality than that of T. By our shelling order, such a facet has not yet been shelled, and hence cannot be used to show that $x_{F_1} \in \mathscr{R}(T)$. The second case is also not allowed, for the only flat of rank $\#I$ that contains I is its closure F_1, and the resulting facet would correspond to T itself. Hence $x_{F_1} \notin \mathscr{R}(T)$.

Suppose now that I is a basis. We need to show that $T \subseteq \mathscr{R}(T)$. To do so, pick any vertex $v \in T$, and notice that $v = y_i$ for some $i \in I$. Consider the facet $T' = I \setminus \{y_i\} \subseteq \mathrm{cl}(I \setminus \{y_i\})$. Since $\#(I \setminus \{y_i\}) < \#I$, the facet T' was shelled before T, and the fact that $T \setminus \{y_i\}$ is contained in T' immediately implies that $y_i \in \mathscr{R}(T)$, proving the result. □

We can prove an analogous result using the shelling order given by Theorem 2.2. An immediate corollary is an equivalent expression for the number of bases in M, which can also be found in [7, §2].

For starters, we give a few definitions. Every element $x \in B$, where B is a basis for the matroid M, gives rise to a unique *basic bond*. The element x is *internally active* in B if x is the smallest element in the bond under ω; otherwise, x is *internally passive*. Dually, for $x \in E \setminus B$, there exists a unique associated *basic circuit*. Then x is *externally active* if it is the smallest element in the circuit under ω; otherwise, it is *internally active*. More details can be found in [2, §7.3]. With this out of the way, we cite a useful lemma.

Lemma 3.3. *Let* M *be an ordered matroid. If we shell the independent set complex* $\mathcal{I}(M)$ *in lexicographic order, then* $\mathscr{R}(B) = IP(B)$ *for any basis* B*, where* $IP(B)$ *is the collection of elements that are internally passive with respect to* B*.*

This next lemma relates the facets of Berg(M) with bases that are *externally passive.*

Lemma 3.4. *Let* M *be an ordered matroid with ground set* E*. Associate to each facet* T *of* Berg(M) *a label* $\lambda(T)$ *given by the shelling order of* Berg(M) *seen in [6]. That is, for a facet*

$$T = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r(M)-1},$$

we have the labeling

$$\lambda(T) = (\min(F_1 \setminus \mathrm{cl}(\varnothing)), \min(F_2 \setminus F_1), \ldots, \min(E \setminus F_{r(M)-1})).$$

Take the elements of $\lambda(T)$ *as a set* B*, which is a basis for* M*. Then* $\lambda(T)$ *is strictly decreasing and* B *has no external activity* 0*. Moreover, all such bases of* M *with no external activity arise in this manner.*

Proof. Suppose that $\lambda(T)$ is strictly decreasing, i.e., $x_1 > x_2 > \ldots > x_r$ where $\lambda(T) = (x_1, \ldots, x_r)$. If $B = E$ then we are done, so suppose we are in the other case and pick some $x \in E \setminus B$, letting F_i is the first flat in T to contain x, extending T to E if necessary. Then the basic circuit of $B \cup x$ must contain x_i, and by the construction of $\lambda(T)$, we have $x_i < x$, and hence x is externally passive. So, B has no external activity.

Consider now a basis $B = \{x_1, \ldots, x_r\}$ with no external activity, and assume without loss of generality that $x_1 > x_2 > \cdots > x_r$. It suffices to show that there exists a facet T with label $\lambda(T) = (x_1, \ldots, x_r)$. Suppose for contradiction that as we try to construct such a facet, the label between flats F_i and F_{i+1} cannot be x_{i+1} and instead must be chosen to be $x < x_{i+1}$. Then the basic circuit of $B \cup x$ must be contained in F_{i+1}. Since $x_1 > x_2 > \cdots x_r$, we have that x is less than all of $x_1, \ldots, x_{i+1}$, showing that it is externally active, a contradiction. □

Armed with these two lemmas, we proceed to the proof of the second main theorem of this section.

Corollary 3.5. *The complex* AugBerg(M) *is a wedge of p copies of the d-sphere, where*

$$p = \sum_{\textit{flats } F \textit{ of } M} |\textit{bases } B_1 \textit{ of } M|_F \textit{ with internal activity zero}|$$
$$\times |\textit{bases } B_2 \textit{ of } M/F \textit{ with external activity zero}|$$

and d is the dimension of AugBerg(M)*.*

Proof. As with the previous proof, Lemma 3.1 allows us to reduce to problem into proving that p is equal to the number of facets T such that $\mathscr{R}(T) = T$, where in this case take our shelling order to be given by Theorem 2.2.

We can partition the facets of AugBerg(M) by the closure of their associated independent set, which is of course always a flat of M. Let's fix such a flat F, and consider which of the facets T that have the form

$$T = I \subset F \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$$

satisfy the inclusion $T \subseteq \mathscr{R}(T)$. To understand this, start by picking some vertex $v \in T$.

We first consider the case where $v = x_F$; this implies that $F \neq E$. We will show that $x_F \in \mathscr{R}(T)$, always. Start by picking some element $y \in F_2 - F$ and consider the facet T' defined by the chain of inclusions

$$T' = I \cup \{y_i\} \subseteq F_2 \subsetneq \cdots \subsetneq F_\ell.$$

Since $\#(I \cup y_i) > \#I$, the facet T' comes before T in the shelling order and contains $T \setminus \{x_F\}$.

Suppose now that $v = y_i$ for some $i \in I$. For a previous facet to contain $T \setminus \{y_i\}$, it must be of the form $T' = (T \setminus \{y_i\}) \cup v'$. Notice that v' cannot be a flat, for then T' would have an independent set of greater cardinality than that of T, and must come later in the shelling order. So, $v' = y_{i'}$ for some $i' \in F \setminus I$, and moreover, for y_i to be in $\mathscr{R}(T)$, we also require that $y_{i'} \in F \setminus I$, for otherwise the chains of flats in T and T' would not align. In other words, we can view I and $(I \setminus \{y_i\}) \cup \{y_{i'}\}$ as bases of $M|_F$. Since we lexicographically shell facets with independent sets of the same cardinality, an application Lemma 3.3 via the induced linear order on $M|_F$ from ω shows that $y_i \in \mathscr{R}(T)$ if and only if y_i is internally passive with respect to I.

Consider the final case where v is some x_{F_k} in T. Recall that for a fixed independent set I whose closure is F, we shell the facets associated to it in the order induced from the shelling of Berg(M/F_k). By the properties of this shelling given as seen in [6] and [2, §7.6], we can conclude that $x_{F_k} \in \mathscr{R}(T)$ if and only if $\min(F_k \setminus F_{k-1}) > \min(F_{k+1} \setminus F_k)$, where $F_0 = \mathrm{cl}(\varnothing)$ and $F_{\ell+1} = E$.

Putting everything together, we see that for $\mathscr{R}(T) = T$, we need I, a basis of $M|_F$, to have no internal activity, and Lemma 3.4 implies that the chain of flats $F \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$ must be uniquely associated with a basis of M/F with no external activity. Mixing and matching all possible bases for $M|_F$ and M/F for our given flat F yields the result. □

Remark. It was proven in [7] that the complex of independent sets $\mathcal{I}(M)$ has a remarkable property: when one views its simplicial boundary maps $\{\partial_i\}_{i=1,2,\ldots}$ as matrices, and forms the combinatorial Laplacian matrices $\partial_i^T \partial_i$, they will have only only *integer* eigenvalues.

One might therfore ask whether AugBerg(M) also has this property. Sadly, this fails already for the Boolean matroid M of rank 2, for which AugBerg(M) is a 5-cycle graph. One can check that its Laplacian matrix $\partial_1^T \partial_1$ has characteristic polynomial $x(x^2 - 5x + 5)^2$, whose eigenvalues are not all integers.

4 Equivariant description of the homology

We want to discuss how symmetries of the matroid M act on the homology of $\mathrm{AugBerg}(M)$.

Definition. An *automorphism* of a matroid M on ground set E is a bijection $\sigma : E \to E$ that sends independent sets $I \mapsto \sigma(I)$ to independent sets (or equivalently sends flats $F \mapsto \sigma(F)$ to flats, or equivalently sends bases $B \mapsto \sigma(B)$ to bases, etc.) The set $\mathrm{Aut}(M)$ of all such automorphisms forms a (finite) subgroup of the symmetric group $\mathfrak{S}_E$, which acts on $\mathrm{AugBerg}(M)$ via *simplicial automorphisms*, that is taking faces to faces. It therefore induces a group of homeomorphism on the geometric realization $\|\mathrm{AugBerg}(M)\|$ as a topological space, and on its homology groups with any coefficients.

We claim that the shelling in Theorem 2.1 leads to a very explicit description of the action of $G := \mathrm{Aut}(M)$ on this homology. To this end, recall that when computing simplicial homology $\tilde{H}_*(\Delta, \mathbb{Z})$ for a simplicial complex Δ using *oriented simplicial chains*[2], the d^{th} chain group $\tilde{C}_d(\Delta, \mathbb{Z})$ has the following description. One fixes for each d-dimensional simplex σ having vertex set $\{v_0, v_1, \ldots, v_d\}$ a *reference ordering* $(v_0, v_1, \ldots, v_d)$, and then $\tilde{C}_d(\Delta, \mathbb{Z})$ is a free abelian group having one $\mathbb{Z}$-basis element $[v_0, v_1, \ldots, v_d]$, called an *oriented simplex*, for each such σ, in which one considers for any permutation w in the symmetric group $\mathfrak{S}_{d+1}$ that

$$[v_{w(0)}, v_{w(1)}, \ldots, v_{w(d)}] = \mathrm{sgn}(w) \cdot [v_0, v_1, \ldots, v_d]$$

where $\mathrm{sgn}(w) \in \{+1, -1\}$ is the usual sign of the permutation w.

Note that when G is a group of symmetries of the simplicial complex Δ, if some subset $\{\sigma_1, \ldots, \sigma_t\}$ of d-simplices happens to be setwise stable under the action of G, then the $\mathbb{Z}$-span of their basis elements $\{[\sigma_1], \ldots, [\sigma_t]\}$ within $\tilde{C}_d(\Delta, \mathbb{Z})$ is a free abelian subgroup that carries a signed permutation representation of G. In other words, each element g in G permutes them up to sign: $g([\sigma]) = \pm[\sigma']$ if $g(\sigma) = \sigma'$.

Remark. The above signed permutation representation on the $\mathbb{Z}$-span of $\{[\sigma_1], \ldots, [\sigma_t]\}$ inside $\tilde{C}_d(\Delta, \mathbb{Z})$ may alternatively be viewed as a direct sum $\bigoplus_\sigma \mathrm{sgn}_\sigma \uparrow_{G_\sigma}^{G}$ of *induced representations*. Here σ runs through any choice of G-orbit representatives for $\{\sigma_1, \ldots, \sigma_t\}$, with G_σ the subgroup of G that setwise stabilizes the vertex set $\{v_0, v_1, \ldots, v_d\}$ of σ, and $\mathrm{sgn}_\sigma : G_\sigma \to \{+1, -1\}$ is the one-dimensional character sending w permuting $\{v_0, v_1, \ldots, v_d\}$ to the sign of the permutation $\mathrm{sgn}_\sigma(w)$.

Theorem 4.1. *For $\Delta = \mathrm{AugBerg}(M)$, the representation of $G = \mathrm{Aut}(M)$ on the top homology group $\tilde{H}_{r(M)-1}(\Delta, \mathbb{Z})$ is isomorphic to the signed permutation representation of G on the $\mathbb{Z}$-span inside $\tilde{C}_d(\Delta, \mathbb{Z})$ of the oriented simplices $\{[B] :$ bases B of $M\}$.*

Example. For the uniform matroid M of rank 2 on $E = \{1, 2, 3\}$ considered in Example 1, the set of bases is $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Hence $\Delta = \mathrm{AugBerg}(M)$ has $\tilde{H}^1(\Delta, \mathbb{Z}) \cong \mathbb{Z}^3$ with $\mathbb{Z}$-basis given by $\{[y_1, y_2], [y_1, y_3], [y_2, y_3]\}$, in which $[y_i, y_j] = -[y_j, y_i]$. The matroid M has automorphism group $G = \mathfrak{S}_3$ acting on the vertices via $g(y_i) = y_{g(i)}$. Theorem 4.1 then tells us, for example, that the transposition $g = (1, 3)$ in $\mathfrak{S}_3$ acts on $\tilde{H}_1(\Delta, \mathbb{Z})$ sending

$$\begin{aligned} g([y_1, y_3]) &= [y_3, y_1] = -[y_1, y_3], \\ g([y_1, y_2]) &= [y_3, y_2] = -[y_2, y_3]. \end{aligned}$$

Example. When M is the Boolean matroid of rank n, its collection of bases $\mathcal{B} = \{E\}$ contains only one element $E = \{1, 2, \ldots, n\}$. Then $\Delta = \mathrm{AugBerg}(M)$ is an $(n-1)$-sphere that turns out to be isomorphic to the boundary of the *stellohedron*; see [4, Footnote 7]. The examples with $n = 2, 3$ are depicted in Figure 2.

The set of bases $\mathcal{B} = \{E\}$ gives rise to one oriented $(n-1)$-simplex $[E]$. Here $G = \mathrm{Aut}(M)$ is the symmetric group permuting $\mathfrak{S}_n$, and it acts on the homology group $\tilde{H}_{n-1}(\Delta, \mathbb{Z}) \cong \mathbb{Z}$ via the sign representation: $g([E]) = \mathrm{sgn}(g) \cdot [E]$ for every permutaion g in $\mathfrak{S}_n$.

Remark. The explicitness of the description in Theorem 4.1 stands in stark contrast to our understanding in general of the representations of $G = \mathrm{Aut}(M)$ on the homology of $\mathcal{I}(M)$ and of $\mathrm{Berg}(M)$. Both of these homology representations have explicit descriptions only for very special cases of matroids. Their difficulty is partly reflected in the fact that the sets of bases of M having internal activity zero, or external activity zero, which indexed bases for their homology, are not setwise stable under automorphisms of M. Both of these sets require an ordering of the ground set E in order to define them.

[2]See, for example, Munkres [8, §1.5].

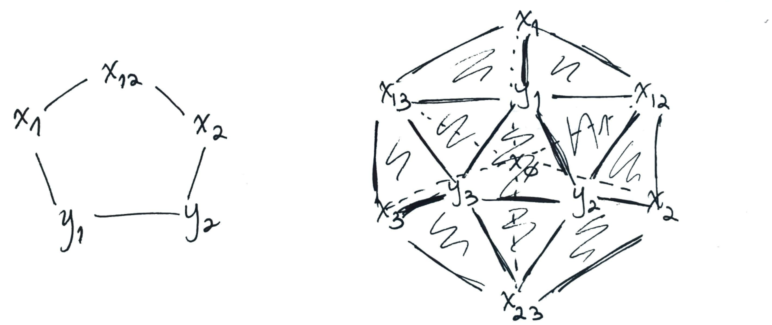


Figure 2: The stellohedra AugBerg(M) for the Boolean matroids M of rank $n = 2$ and $n = 3$.

We claim Theorem 4.1 follows immediately from the following general lemma.

Lemma 4.2. *Let Δ be a simplicial complex and $\mathcal{B}$ a collection of facets of Δ with the property that the subcomplex $\Delta' := \Delta \setminus \mathcal{B}$ obtained by removing all of the facets in $\mathcal{B}$ is contractible. Then Δ is homotopy equivalent to a wedge of spheres $\bigvee_{\sigma \in \mathcal{B}} \mathbb{S}^{\dim(\sigma)}$.*

Furthermore, for any subgroup G of simplicial automorphisms of Δ that preserves $\mathcal{B}$ setwise, the representation of G on $\tilde{H}_i(\Delta, \mathbb{Z})$ is the signed permutation representation on the $\mathbb{Z}$-span of the oriented simplices $\{[\sigma] : \sigma \in \mathcal{B}, \dim(\sigma) = i\}$ within $\tilde{C}_i(\Delta, \mathbb{Z})$.

In particular, Lemma 4.2 applies to any pure shellable simplicial complex Δ, in which $\mathcal{B}$ is the collection of facets F which are equal to their own restriction face $\mathscr{R}(F)$, that is, $\mathscr{R}(F) = F$. In this case, $\Delta' = \Delta \setminus \mathcal{B}$ is also shellable, and homotopy equivalent to a zero-fold wedge of spheres, that is, it is contractible.

Proof of Lemma 4.2. (cf. proof of [2, Theorem 7.7.2]) The homotopy-theoretic assertion is standard, and follows, for example from [1, Lemma 10.2]. For the homology assertion, start with the long exact sequence in integral homology for the pair (Δ, Δ'),

$$\cdots \to \tilde{H}_i(\Delta') \to \tilde{H}_i(\Delta) \to \tilde{H}_i(\Delta, \Delta') \to \tilde{H}_{i-1}(\Delta') \to \cdots$$

In light of the contractibility of Δ', this gives isomorphisms

$$\tilde{H}_i(\Delta) \cong \tilde{H}_i(\Delta, \Delta').$$

On the other hand, since each simplex σ in $\mathcal{B} = \Delta \setminus \Delta'$ is a *facet* of Δ, lying in no higher-dimensional faces, the boundary maps in the complex $\tilde{C}_*(\Delta, \Delta')$ computing $\tilde{H}_*(\Delta, \Delta')$ are all zero. Hence $\tilde{H}_i(\Delta, \Delta') = \tilde{C}_i(\Delta, \Delta')$ for all i. Furthermore, our assumptions on G imply that all of these isomorphisms commute with the G-action, and its action on $\tilde{C}_i(\Delta, \Delta)$ matches the signed permutation representation in the lemma. $\square$

Remark. Theorem 4.1 is also extremely closely related to a representation-theoretic decomposition of the special case $i = r(M) - 1$ in [7, Thm. 19]. In that result, each of the oriented chain groups $\tilde{C}_i$ of the independent set complex $\mathcal{I}(M)$, regarded as a representation of the automorphism group $G = \mathrm{Aut}(M)$ is decomposed into a direct sum of induced representations, in which the direct sum is indexed by pairs of flats $F \subset F'$ in which $r(F') = i + 1$. When $i = r(M) - 1$, this chain group $\tilde{C}_i$ is the same as the G-representation in Theorem 4.1, and the sum becomes a sum over all flats F. Each of the induced representations in the sum will come from a tensor product of representations for the subgroup G_F of G stabilizes the flat F, acting on the homology of $\mathcal{I}(M|_F)$ and of Berg(M/F). This description can be shown to agree with a finer analysis of the shelling in Theorem 2.2.

5 Augmented Bergman complexes for other closures

One can view a matroid M on E in terms of the *matroid closure operator* $2^E \xrightarrow{f} 2^E$ that maps a subset $A \subseteq E$ to the smallest flat $F = f(A)$ in $\mathcal{F}$ which contains A. This is an instance of a more general notion.

Definition. Given a set E, a map $2^E \xrightarrow{f} 2^E$ is called a *closure operator* on E if it satisfies three axioms: for all subsets $A, B \subseteq E$,

(C1.) $A \subseteq f(A)$

(C2.) $A \subseteq B$ implies $f(A) \subseteq f(B)$

(C3.) $f(f(A)) = f(A)$

In this context there are analogues of $\mathcal{I}(M), \mathrm{Berg}(M), \mathrm{AugBerg}(M)$, which we introduce next.

Definition. Given a closure operator f on E, define its *poset of closed sets*

$$\mathcal{F} := \{F \subseteq E : f(F) = F\}$$

partially ordered via inclusion.

Proposition 5.1. *Given a closure operator f on E, and any two closed sets F, G in $\mathcal{F}$, their intersection $F \cap G$ also lies in $\mathcal{F}$. Hence $\mathcal{F}$ always has well-defined meets $F \wedge G := F \cap G$. Whenever E is finite, $\mathcal{F}$ also has well-defined joins $F \vee G := \wedge_{H \geq F,G} H$, so that it becomes a lattice.*

Proof. We check for F, G closed that $f(F \cap G) = F \cap G$. Note $f(F \cap G) \supseteq F \cap G$ holds by Axiom C1.

For the reverse inclusion, note that $F \cap G \subseteq F, G$ implies $f(F \cap G) \subseteq f(F), f(G)$ by Axiom C2. Also $f(F) = F$ and $f(G) = G$ by Axiom C3. Hence $f(F \cap G) \subseteq F, G$, and therefore $f(F \cap G) \subseteq F \cap G$. $\square$

Definition. Given a closure operator f on E, define a subset $I \subseteq E$ to be *independent* if $f(I \setminus \{i\}) \subsetneq f(I)$ for all $i \in I$. Let $\mathcal{I}(f)$ denote the collection of all independent subsets $I \subseteq E$.

The empty set $\varnothing$ is vacuously independent. We next check that $\mathcal{I}(f)$ always forms a simplicial complex.

Proposition 5.2. *For any closure operator f on E, if $J \subseteq I \subseteq E$ has I independent, then J is independent.*

Proof. Suppose there exists j in J with $f(J \setminus \{j\}) = f(J)$. We show this gives $f(I \setminus \{j\}) = f(I)$, contradicting our assumption that I is independent. To see this, note Axiom C2 gives one of the inclusions $f(I \setminus \{j\}) \subseteq f(I)$, so we only need argue the reverse inclusion. Since $I \setminus \{j\} \supseteq I \setminus J, J \setminus \{j\}$, Axiom C2 implies

$$f(I \setminus \{j\}) \supseteq f(I \setminus J) \supseteq I \setminus J$$

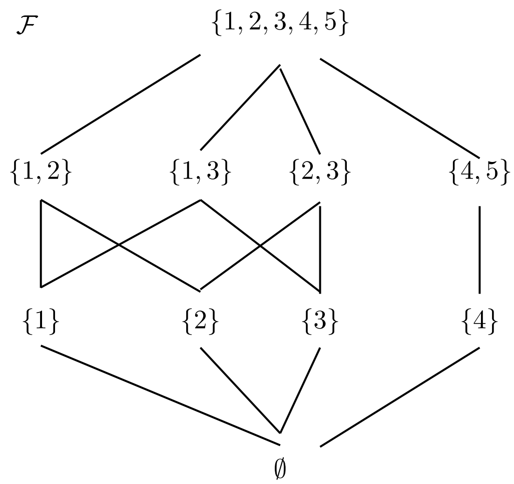
using Axiom C1 for the last inclusion, and also

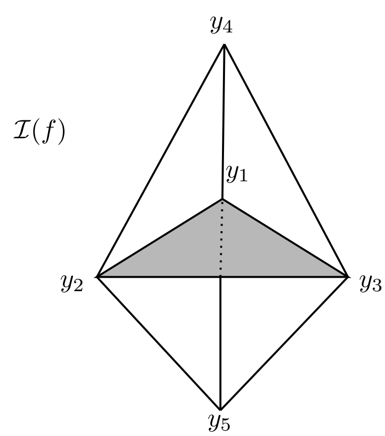
$$f(I \setminus \{j\}) \supseteq f(J \setminus \{j\}) = f(J) \supseteq J$$

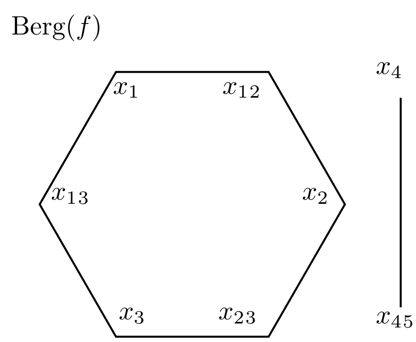
using our assumption about j for the above equality. Hence $f(I \setminus \{j\}) \supseteq I \setminus J, J$, and therefore $f(I \setminus \{j\}) \supseteq I$. But then using Axiom C3 followed by Axiom C2, one obtains the desired reverse inclusion:

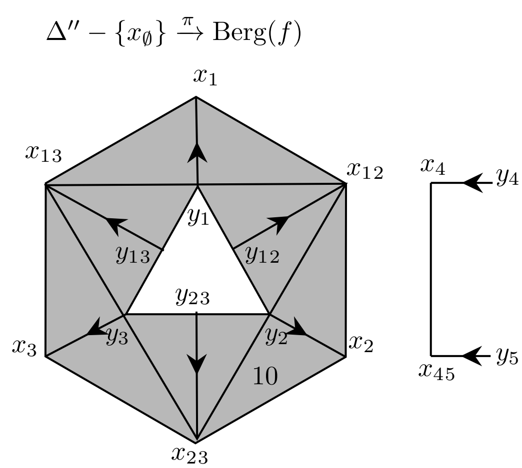
$$f(I \setminus \{j\}) = f(f(I \setminus \{j\})) \supseteq f(I). \qquad \square$$

In light of Propositions 5.1 and 5.2, it seems reasonable, given any closure relation f on a finite set E, to regard the order complex $\mathrm{Berg}(f) := \Delta(\mathcal{F} \setminus \{f(\varnothing), E\})$ of the proper part of the lattice $\mathcal{F}$ as a reasonable generalization of the Bergman complex of a matroid, and to regard the simplicial complex $\mathcal{I}(f)$ as a reasonable generalization of the independent sets complex of a matroid. However, in neither case should one expect these complexes to be pure, nor shellable, nor even homotopy equivalent to wedges of spheres. It is not hard to show that $\mathrm{Berg}(f)$ is not always pure, and can have the homotopy type of any finite simplicial complex. Small examples show that $\mathcal{I}(f)$ also can be non-pure and non-shellable.


$\mathcal{F}$
$\{1,2,3,4,5\}$
$\{1,2\}$
$\{1,3\}$
$\{2,3\}$
$\{4,5\}$
$\{1\}$
$\{2\}$
$\{3\}$
$\{4\}$
$\emptyset$


$\mathcal{I}(f)$
y_4
y_1
y_2
y_3
y_5


Berg(f)
x_1
x_{12}
x_{13}
x_2
x_3
x_{23}
x_4
x_{45}


$\Delta'' - \{x_\emptyset\} \xrightarrow{\pi} \mathrm{Berg}(f)$
x_1
x_{13}
x_{12}
y_1
y_{13}
y_{12}
y_{23}
y_3
y_2
x_3
x_2
x_{23}
x_4
y_4
x_{45}
y_5

Example. Consider the closure operator $f : 2^E \to 2^E$ with $E = [5] = \{1,2,3,4,5\}$ whose poset of closed sets $\mathcal{F}$ is depicted in Figure 5(a). The closure $f(A)$ for any set A can be inferred from $\mathcal{F}$, since $f(A)$ is the smallest closed set containing A, or the intersection of all closed sets containing A. Parts (b),(c) of the figure depict, repsectively, the complex of independent sets $\mathcal{I}(f)$ on vertex set $\{y_i\}_{i\in E}$, and the Bergman complex $\mathrm{Berg}(f)$ on vertex set $\{x_F\}_{F\in\mathcal{F}\setminus\{\varnothing,E\}}$.

Definition. Given a closure operator f on a finite set $E = \{1,2,\ldots,n\}$, define $\mathrm{AugBerg}(f)$ to be the simplicial complex on vertex set

$$\{y_1,\ldots,y_n\} \cup \{x_F : F \in \mathcal{F}\setminus\{E\}\}$$

whose simplices are the subsets

$$\{y_i\}_{i\in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{5}$$

for which $I \in \mathcal{I}(f)$ and $I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$.

This complex $\mathrm{AugBerg}(f)$ need not be pure. Nevertheless, it does contain contains as subcomplexes both

- the independent set complex $\mathcal{I}(f)$, as the simplices in (5) with $\ell = 0$, and
- the cone $\mathrm{Cone}(\mathrm{Berg}(f)) = \Delta(\mathcal{F}\setminus\{E\})$ over the Bergman complex $\mathrm{Berg}(f)$ with cone vertex $x_{f(\varnothing)}$, as the simplices in (5) with $I = \varnothing$.

In spite of all of our caveats about bad behavior of $\mathrm{Berg}(f)$ and $\mathcal{I}(f)$, we claim that Theorem 4.1 about the topology of $\mathrm{AugBerg}(f)$ has a simple generalization. To state it, we introduce two further definitions.

Definition. For any closure f on E, define the set $\mathcal{B}$ of *bases*

$$\mathcal{B} := \{B \in \mathcal{I}(f) : f(B) = E\}.$$

Also define the *automorphism group* $\mathrm{Aut}(f)$ to be the subgroup of permutations g in the symmetric group $\mathfrak{S}_E$ that commute with the closure f, meaning $f(g(A)) = g(f(A))$ for all $A \subseteq E$.

Theorem 5.3. *Let f be a closure operator on a finite set E, and let $\Delta = \mathrm{AugBerg}(f)$.*

Then Δ is homotopy equivalent to a wedge of spheres $\bigvee_{\text{bases } B\in\mathcal{B}} \mathbb{S}^{\#B-1}$.

Furthermore, the group $G = \mathrm{Aut}(f)$ acting on any homology group $\tilde{H}_i(\Delta,\mathbb{Z})$ is isomorphic to the signed permutation representation of G on the $\mathbb{Z}$-span inside $\tilde{C}_i(\Delta,\mathbb{Z})$ of the oriented simplices

$$\{[B] : \text{bases } B \in \mathcal{B} \text{ with } \#B - 1 = i\}.$$

Proof. In light of Lemma 4.2, it suffices to show that $\Delta = \mathrm{Berg}(f)$ has the subcomplex $\Delta' := \Delta \setminus \mathcal{B}$ contractible. Our strategy introduces another simplicial complex Δ'', and shows it has these two properties:

(a) Δ'' is a subdivision of Δ', and hence homeomorphic to it.

(b) Δ'' is homotopy equivalent to the subcomplex $\mathrm{Cone}(\mathrm{Berg}(f)) = \Delta(\mathcal{F}\setminus\{E\})$ inside $\mathrm{AugBerg}(f)$.

Since cones are contractible, this would suffice to show that Δ'' (and hence also Δ') is contractible.

We define Δ'' as the simplicial complex on vertex set

$$\{y_I : I \in \mathcal{I}(f)\setminus\mathcal{B}\} \cup \{x_F : F \in \mathcal{F}\setminus\{E\}\}$$

whose simplices are the subsets

$$\{y_{I_1}, y_{I_2}, \cdots, y_{I_k}\} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{6}$$

for which $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_k \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell (\subsetneq E)$.

One can check assertion (a) as follows. The typical simplex of Δ' from (5) is subdivided in Δ'' by all of the simplicies from (6) which have the property that $I_k \subseteq I \subseteq F_1$.

One can check assertion (b) as follows. Define a simplicial map $\Delta'' \xrightarrow{\pi} \Delta(\mathcal{F}\setminus\{E\})$ via the following map on vertices:

$$\begin{array}{rcll} x_F & \longmapsto & x_F & \text{for } F \in \mathcal{F}\setminus\{E\} \\ y_I & \longmapsto & x_{f(I)} & \text{for } I \in \mathcal{I}(f)\setminus\mathcal{B} \end{array}$$

It is not hard to check that this induces a well-defined simplicial map, that is, it carries simplices to simplices. One can also check that, for every element F in the poset $\mathcal{F} \setminus \{E\}$ the inverse image of the order complex of the principal order ideal $\mathcal{F}_{\leq F}$ is the star of the vertex x_F within Δ'', and hence contractible. Thus by Quillen's Fiber Lemma [1, (10.5)(i)], the map π induces a homotopy equivalence. $\square$

Remark. Note that $\mathrm{Cone}(\mathrm{Berg}(M)) = \Delta(\mathcal{F} \setminus \{E\})$ can be identified with the subcomplex of Δ'' induced on the vertex subset $\{x_F : F \in \mathcal{F} \setminus \{E\}\}$. Since these vertices are all pointwise fixed by π, the same is true for this subcomplex, so that the map π is actually a homotopy inverse to the inclusion map $\mathrm{Cone}(\mathrm{Berg}(M)) \hookrightarrow \Delta''$, showing that π is a *deformation retraction.*

It should also be noted that since the cone vertex $x_{f(\varnothing)}$ of $\mathrm{Cone}(\mathrm{Berg}(M))$ is fixed by π, the map π actually restricts to a deformation retraction

$$\Delta'' \setminus \{x_{f(\varnothing)}\} \xrightarrow{\pi} \mathrm{Berg}(f). \tag{7}$$

Example. We illustrate some of the preceding proof for the closure operator $f : 2^E \to 2^E$ with $E = [5] = \{1, 2, 3, 4, 5\}$ in Example 5. Part (d) of the figure depicts the subcomplex $\Delta' - \{x_\varnothing\} = \mathrm{AugBerg}(f) \setminus \mathcal{B} \setminus \{x_\varnothing\}$ obtained from Δ' by deleting the vertex $x_\varnothing$. Lastly, part (e) dentotes the homeomorphic subdivided complex $\Delta'' \setminus \{x_\varnothing\}$, along with a depiction via arrows along certain edges, of the simplicial retraction map $\Delta'' \setminus \{x_\varnothing\} \xrightarrow{\pi} \mathrm{Berg}(f)$ from (7).

Acknowledgements

This project was partially supported by NSF RTG grant DMS-1148634, as well as NSF grant DMS-1949896. It was supervised as part of the University of Minnesota School of Mathematics Summer 2021 REU program. The authors would like to extend their gratitude to Profesor Vic Reiner for introducing the problem and suggesting methods of approaching it. We would also like to thank Trevor Karn and Sasha Pevzner for presenting background information and providing valuable edits on this report.

References

[1] A. Björner, *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier Sci. B. V., Amsterdam, 1995, pp. 1819–1872. MR 1373690

[2] Anders Björner, *The homology and shellability of matroids and geometric lattices*, Matroid applications, Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 226–283. MR 1165544

[3] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, *Singular hodge theory for combinatorial geometries*, `arXiv:2010.06088` (2010).

[4] ———, *A semi-small decomposition of the chow ring of a matroid*, `arXiv:2002.03341` (2020).

[5] Gwihen Etienne and Michel Las Vergnas, *External and internal elements of a matroid basis*, Discrete Math. **179** (1998), no. 1-3, 111–119. MR 1489076

[6] Adriano M. Garsia, *Combinatorial methods in the theory of Cohen-Macaulay rings*, Adv. in Math. **38** (1980), no. 3, 229–266. MR 597728

[7] W. Kook, V. Reiner, and D. Stanton, *Combinatorial Laplacians of matroid complexes*, J. Amer. Math. Soc. **13** (2000), no. 1, 129–148. MR 1697094

[8] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR 755006

[9] James G. Oxley, *Matroid theory (oxford graduate texts in mathematics)*, Oxford University Press, Inc., USA, 2006.

[10] J. Scott Provan and Louis J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576–594. MR 593648