(K)not detecting boundary slopes via intersections in the character variety arising from epimorphisms

Isidora Bailly-Hall, Karina Dovgodko, Akash Ganguly, Jiachen Kang, and Jishi Sun

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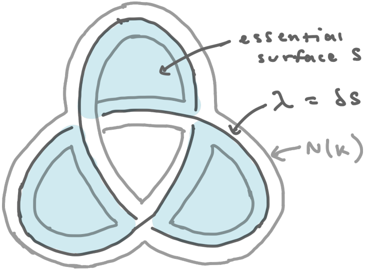
Definition

- A *knot* K is an embedding of S^1 into S^3.
- The *knot complement* of K is the 3-manifold $M(K) = S^3 \setminus N(K)$.
- The *knot group* of K is $\Gamma_K = \pi_1(M(K))$.



Goal

To find essential surfaces in the complement of two-bridge knots.

A Bird's Eye View

Two-bridge knots

$\downarrow$

Character varieties

$\downarrow$ epimorphisms

Intersection points

$\downarrow$ Farey recursion

Algebraic non-integral representations

$\downarrow$ SL_2-trees

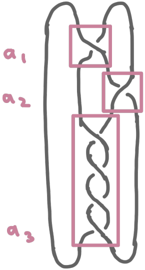
Detected essential surfaces

Two-Bridge Knots

Definition

A *two-bridge knot* is a knot with diagram having two local maxima.

- Every two-bridge knot can be associated to a reduced fraction $q/p \in (0, 1)$ with p, q both odd, called its *two-bridge normal form*.
- q/p is given by the continued fraction expansion $[a_1, \ldots, a_k] = q/p$



$$[a_1, a_2, a_3] = [1, 1, 4]$$

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{4}}} = \frac{5}{9}$$

Presentation of Two-Bridge Knot Groups

Theorem (Maylands, 1974)

Given a two-bridge knot $K = (p, q)$, $\Gamma_{q/p}$ has the following canonical presentation:

$$\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$$

where w is determined by p and q, and a and b are conjugate.

Example

For $q/p = [1, 1, 4] = 5/9$ we have

$$w = ab^{-1}a^{-1}bab^{-1}a^{-1}b$$

with

$$\Gamma_{q/p} = \langle a, b \mid ab^{-1}a^{-1}bab^{-1}a^{-1}ba = bab^{-1}a^{-1}bab^{-1}a^{-1}b \rangle$$

Representations of Two-Bridge Knot Groups

Corollary

Every irreducible representation $\rho : \Gamma_{q/p} \to \mathrm{SL}_2(\mathbb{C})$ *is determined by* $\rho(a)$ *and* $\rho(b)$*, which (up to conjugation) has the form*

$$\rho(a) = \begin{bmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{bmatrix} \text{ and } \rho(b) = \begin{bmatrix} \alpha & 0 \\ t & 1/\alpha \end{bmatrix}$$

Therefore every representation ρ *of* $\Gamma_{q/p}$ *corresponds to a point* $(\alpha, t) \in \mathbb{C}^2$ *that satisfies* $\rho(wa) = \rho(bw)$*.*

Character Varieties

We can rewrite the polynomial relation $\rho(wa) = \rho(bw)$ in terms of the *traces* of $\rho(a)$ and $\rho(ab^{-1})$: we define

$$x := \mathrm{tr}(\rho(a)) = \alpha + 1/\alpha$$

$$y := \mathrm{tr}(\rho(ab^{-1})) = 2 - t$$

Definition

The algebraic set $X(\Gamma_{q/p})$ in $\mathbb{C}^2$ defined by this polynomial in x and y is called the *character variety* of $\Gamma_{q/p}$.

Example

The defining polynomial of $X(\Gamma_{1/3})$ is $x^2 - y - 1 = 0$.

Epimorphisms onto the trefoil knot

Definition

The rational number

$$q/p = [\underbrace{3, 2, \ldots, 3, 2}_{n-\text{many 2's}}, 3k]$$

is the two-bridge normal form of a knot whenever $n + k$ is odd. We denote this knot by $\mathcal{K}(n, k)$.

Theorem (Ohtsuki-Riley-Sakuma, 2008)

For all $n, k > 0$ there exists an epimorphism

$$\Gamma_{\mathcal{K}(n,k)} \twoheadrightarrow \Gamma_{1/3}$$

where $\Gamma_{1/3}$ is the knot group of the trefoil knot.

Intersection Points

Given an epimorphism $\Gamma_{\mathcal{K}(n,k)} \twoheadrightarrow \Gamma_{1/3}$, every representation $\Gamma_{1/3} \to \mathrm{SL}_2(\mathbb{C})$ will induce a representation $\Gamma_{\mathcal{K}(n,k)} \to \mathrm{SL}_2(\mathbb{C})$. This implies the following:

Corollary

$X(\mathcal{K}(n,k))$ always contain an irreducible component $x^2 - y - 1 = 0$, which corresponds to $X(\Gamma_{1/3})$.

Goal

To describe the intersection points between $x^2 - y - 1 = 0$ and other components of $X(\mathcal{K}(n,k))$.

However this is **HARD!**

Horrific Example

Character variety of $\mathcal{K}(1,2) = [3,2,6]$

```
(-x^2 + y + 1)^2 * (-x^30*y^6 + 12*x^30*y^5 + 15*x^28*y^7 - 60*x^30*y^4 - 168*x^28*y^6 - 105*x^26*y^8 + 160*x^30*y^3 +
756*x^28*y^5 + 1092*x^26*y^7 + 455*x^24*y^9 - 240*x^30*y^2 - 1680*x^28*y^4 - 4347*x^26*y^6 - 4368*x^24*y^8 -
1365*x^22*y^10 + 192*x^30*y + 1680*x^28*y^3 + 7469*x^26*y^5 + 15015*x^24*y^7 + 12012*x^22*y^9 + 3003*x^20*y^11 - 64*x^30 -
2030*x^26*y^4 - 16827*x^24*y^6 - 34398*x^22*y^8 - 24024*x^20*y^10 - 5005*x^18*y^12 - 1344*x^28*y - 10360*x^26*y^3 -
19494*x^24*y^5 + 12462*x^22*y^7 + 54054*x^20*y^9 + 36036*x^18*y^11 + 6435*x^16*y^13 + 768*x^28 + 10976*x^26*y^2 +
57960*x^24*y^4 + 107985*x^22*y^6 + 38566*x^20*y^8 - 57057*x^18*y^10 - 41184*x^16*y^12 - 6435*x^14*y^14 + 784*x^26*y -
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- 3808*x^26 - 40560*x^24*y^2 - 74324*x^22*y^4 + 162178*x^20*y^6 + 454905*x^18*y^8 + 262647*x^16*y^10 - 24024*x^12*y^14 -
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17199*x^8*y^15 + 1092*x^6*y^17 + 105*x^4*y^19 - 12224*x^22 + 219952*x^20*y^2 + 1071495*x^18*y^4 + 545415*x^16*y^6 -
1313964*x^14*y^8 - 1050924*x^12*y^10 - 73788*x^10*y^12 + 60835*x^8*y^14 + 7098*x^6*y^16 - 168*x^4*y^18 - 15*x^2*y^20 +
114752*x^20*y - 222910*x^18*y^3 - 1977927*x^16*y^5 - 1682604*x^14*y^7 + 618436*x^12*y^9 + 730548*x^10*y^11 +
87140*x^8*y^13 - 16866*x^6*y^15 - 1890*x^4*y^17 + 12*x^2*y^19 + y^21 + 2656*x^20 - 465300*x^18*y^2 - 379585*x^16*y^4 +
2180256*x^14*y^6 + 2256912*x^12*y^8 + 13290*x^10*y^10 - 344300*x^8*y^12 - 46395*x^6*y^14 + 2838*x^4*y^16 + 297*x^2*y^18 -
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2645292*x^12*y^6 + 210342*x^10*y^8 + 1011818*x^8*y^10 + 165768*x^6*y^12 - 19830*x^4*y^14 - 2495*x^2*y^16 + y^18 -
70384*x^16*y - 855714*x^14*y^3 + 1073835*x^12*y^5 + 2582828*x^10*y^7 + 379570*x^8*y^9 - 353964*x^6*y^11 - 59812*x^4*y^13 +
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- 116*x^2*y - 274*y^3 - 12*x^2 + 54*y^2 + 12*y - 1)^2
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Horrific Example

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- 116*x^2*y - 274*y^3 - 12*x^2 + 54*y^2 + 12*y - 1)^2
```

Moral of the story

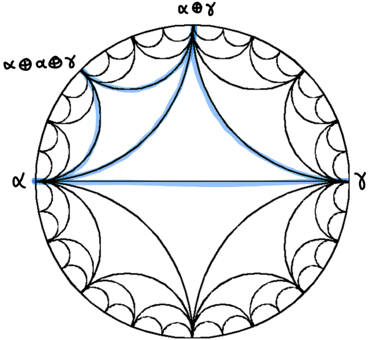
This sucks. New approach needed.

Farey Recursion

Definition

- For any p/q, $r/s \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, we call them a *Farey pair* if $ps - qr = \pm 1$;
- For any Farey pair $(p/q, r/s)$, we define their *Farey sum* to be $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$.

This operation has a geometric explanation on the *Farey graph*:



Farey Recursion

Definition

Let R be any commutative ring. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is called a *Farey recursive function* if for every Farey pair (α, γ) we have

$$\mathcal{F}(\gamma \oplus \alpha \oplus \alpha) = -\mathcal{F}(\gamma) + \mathcal{F}(\alpha)\mathcal{F}(\gamma \oplus \alpha)$$

Cool stuff! (Chesebro 2019)

The defining polynomial of $X(\Gamma_{\mathcal{K}(n,k)})$ can be generated recursively using Farey recursion.

Farey Recursion

Example

If we substitute $y = x^2 - 1$ into the rest of the defining polynomial of $\mathcal{K}(n,k)$, we get a polynomial $\tilde{p}(x)$ that describes the intersection points:

Knot	$\tilde{p}(x)$
$\mathcal{K}(1,2)$	$4x^2 - 15$
$\mathcal{K}(1,4)$	$8x^2 - 29$
$\mathcal{K}(1,6)$	$12x^2 - 43$
$\mathcal{K}(2,1)$	$4x^4 - 32x^2 + 63$
$\mathcal{K}(2,3)$	$12x^4 - 92x^2 + 173$
$\mathcal{K}(2,5)$	$20x^4 - 152x^2 + 283$

Upshot

Using Farey recursion, we found a general formula for $\tilde{p}(x)$; it follows that for all $\mathcal{K}(n,k)$, all coefficients of $\tilde{p}(x)$ but the constant term are even.

$\mathcal{P}$-adic valuation

Definition

Let F be a number field, and let $\mathcal{O}_F$ denote the ring of integers of F. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_F$.
A discrete valuation $v_{\mathcal{P}}$ on F as follows:

- For any $x \in \mathcal{O}_F$, let $v_{\mathcal{P}}(x) = \max\{n \in \mathbb{Z}_{\geq 0} : x \in \mathcal{P}^n\}$;
- For $x \in F - \mathcal{O}_F$, write $x = a/b$ where $a, b \in \mathcal{O}_F$, and define $v_{\mathcal{P}}(x) = v_{\mathcal{P}}(a) - v_{\mathcal{P}}(b)$.

The discrete valuation $v_{\mathcal{P}}$ is called the $\mathcal{P}$-adic valuation on F.

Example

For $F = \mathbb{Q}$ we have $\mathcal{O}_F = \mathbb{Z}$ consider $\mathcal{P} = 2\mathbb{Z}$, then

$$v_2(2) = 1,\ v_2\left(\frac{4}{5}\right) = 2,\ v_2(5) = 0,\ v_2\left(\frac{1}{2}\right) = -1$$

Algebraic non-integral representations

Definition

Let $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ be a representation of Γ_K where F is a number field. We call ρ an *algebraic non-integral (ANI)* representation if there exists some $\gamma \in \Gamma_K$ such that $\mathrm{tr}(\rho(\gamma))$ is not an algebraic integer. That is, there is a $\mathcal{P}$-adic valuation $v_{\mathcal{P}}$ such that $v_{\mathcal{P}}(\mathrm{tr}(\rho(\gamma))) < 0$.

Fact (Culler-Shalen, 1983)

Every ANI-representation of Γ_K can detect essential surfaces in the knot complement of K (via SL_2-tree actions from Bass-Serre theory).

Algebraic non-integral representations

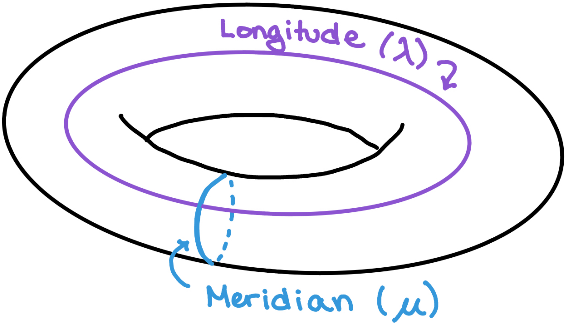
This leads to our first main theorem:

Theorem (B-D-G-K-S, 2023+)

For every two-bridge knot $\mathcal{K}(n,k)$, and every $(x_0, y_0) \in \mathbb{C}^2$ that is an intersection point between $x^2 - y - 1 = 0$ and another component of $X(\Gamma_{\mathcal{K}(n,k)})$, every $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\Gamma_{\mathcal{K}(n,k)}$ corresponding to (x_0, y_0) is an ANI-representation.

In other words, every intersection point will detect essential surfaces for $\mathcal{K}(n,k)$.

Boundary slope



Definition

- A *slope* of K is an element $a/b \in \mathbb{Q} \cup \{\infty\}$, which corresponds to the element $\mu^a \lambda^b \in \pi_1(\partial M(K))$.
- A *boundary slope* of K is a slope that appears in ∂S for an essential surface S in $M(K)$.

Detecting boundary slopes

Although the detected essential surfaces may not be unique, their boundary slope is unique:

Theorem (Schanuel-Zhang, 2001)

Let $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ be an ANI-representation of Γ_K with respect to a $\mathcal{P}$-adic valuation $v_{\mathcal{P}}$. Then there exists a unique boundary slope γ of K such that $v_{\mathcal{P}}(\mathrm{tr}(\rho(\gamma))) \geq 0$, and γ is the detected boundary slope.

For a fixed two-bridge knot K, (Hatcher-Thurston, 1985) gives an explicit description of all the boundary slopes of K, so we can calculate their traces and find the unique one with integral trace.

Detecting boundary slopes

Proposition (B-D-G-K-S, 2023+)

The set of all boundary slopes of $\mathcal{K}(n,k)$ is

$$\{6k+6a+10b \mid a+b \leq n\} \cup \{6a+10b \mid a+b \leq n, 0 < a\} \cup \{0\}.$$

Example

The knot $\mathcal{K}(1,k)$ (where k is even) has exactly 5 boundary slopes:

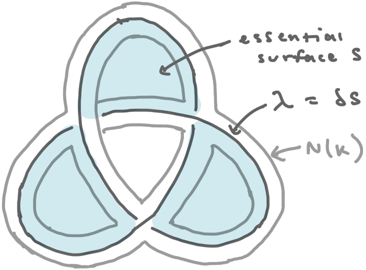
$$0, 6, 6k, 6k+6, 6k+10$$

Detecting boundary slopes

Our second main theorem:

Theorem (B-D-G-K-S, 2023+)

For $\mathcal{K}(n,k)$, the detected boundary slope is $6n+6k$. That is, $\mu^{6(n+k)}\lambda$ is a loop in the boundary corresponding to a detected essential surface.



Thank you!

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