Financial Mathematics

Completing the square

Collecting like terms

We do not typically write

$$\underline{4x^2} + \underline{3} + \underline{6x} + \underline{5} + \underline{2x} + \underline{x^2} \underline{- 10} + \underline{8} \underline{- x}$$

Instead:

$$(4+1)x^2 \quad + \quad (6+2-1)x \quad + \quad (3+5-10+8)$$

$$= 5x^2 + 7x + 6 \quad \text{(decreasing degree)}$$

or: $6 + 7x + 5x^2$ (increasing degree)

SKILL: Collecting like terms

Problem: Collect like terms on

$$ax + 4x^2 - 2x + 15x^3 + bx - c + 7 + rx^2 + k$$

Sol'n:

$$15x^3 + (4+r)x^2 + (a-2+b)x - 2x + (-c+7+k)$$

General problem:

Graph some equation in x and y.

Given a number a.

Replace x by $x-a$ in the equation.

Graph the new equation.

Example:

Graph $x^2 = 9 - y^2$.

Replace x by $x-2$.

Graph $(x-2)^2 = 9 - y^2$.

$\Leftrightarrow x^2 + y^2 = 9$

$\Leftrightarrow \sqrt{(x-0)^2 + (y-0)^2} = 3$

$\Leftrightarrow \text{dist}((x,y),(0,0)) = 3$

In $\mathbb{R}$: $\text{dist}(a,s) = |s-a| = \sqrt{(s-a)^2}$

In $\mathbb{R}^2$: $\text{dist}((a,b),(s,t)) = \sqrt{(s-a)^2 + (t-b)^2}$

In $\mathbb{R}^3$: $\text{dist}((a,b,c),(s,t,u))$

$$= \sqrt{(s-a)^2 + (t-b)^2 + (u-c)^2}$$

Example:

Graph $x^2 = 9 - y^2$. $\Leftrightarrow$ dist$((x,y),(0,0)) = 3$

Replace x by $x - 2$.

Graph $(x-2)^2 = 9 - y^2$.

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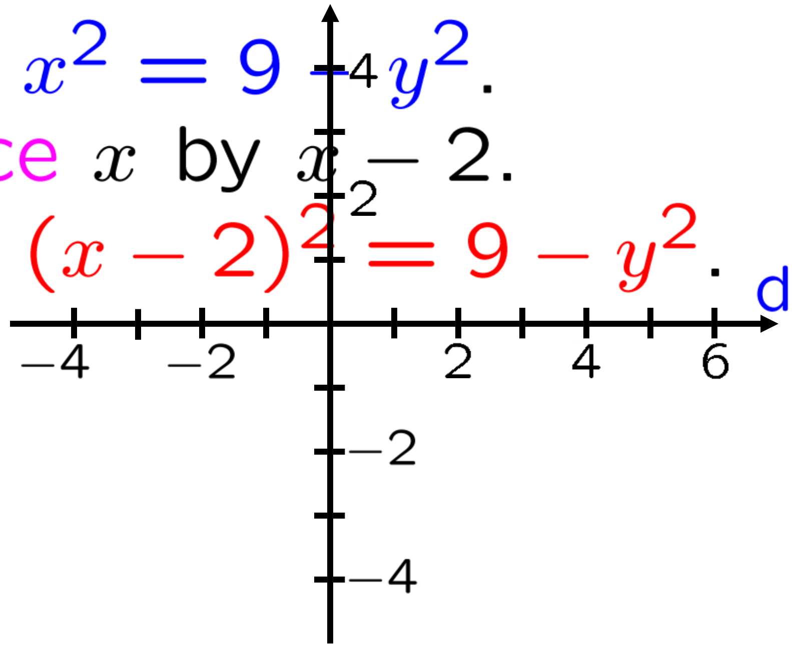


Translation

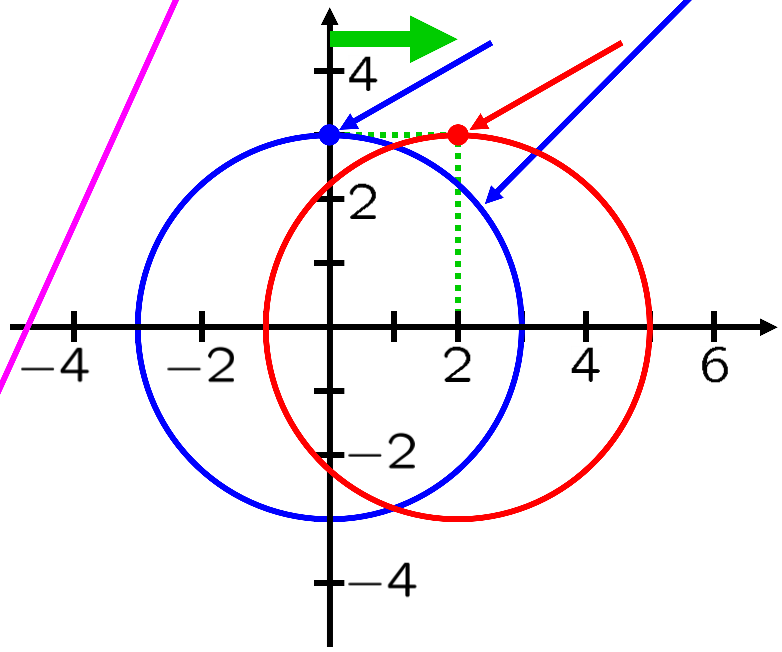
Example: $x :\to 0, y :\to 3$ $\Leftrightarrow$ $\text{dist}((x,y),(0,0)) = 3$

Graph $x^2 = 9 - y^2$.

Replace x by $x - 2$.

Graph $(x-2)^2 = 9 - y^2$.

$x :\to 2, y :\to 3$

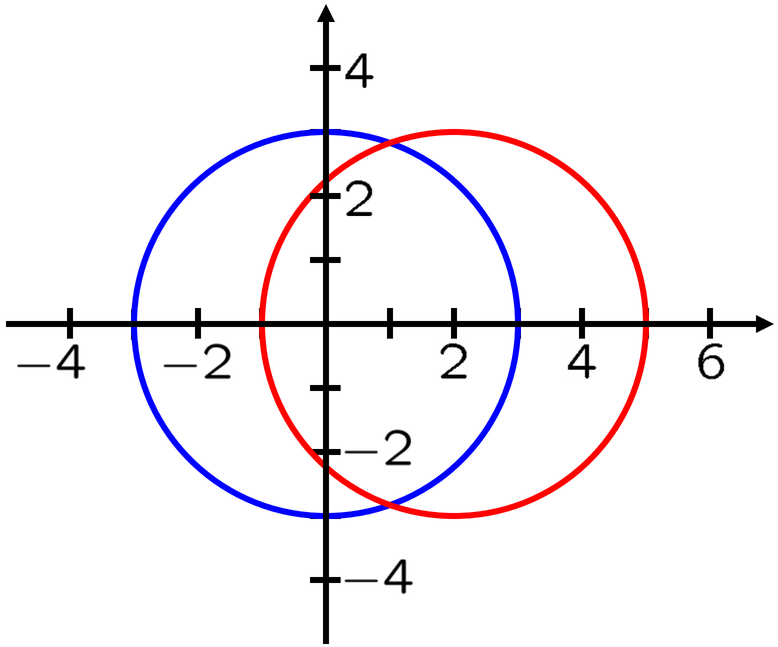


Shift old graph 2 units to right to get new graph.

Translation

General problem:

Graph some equation in x and y.
Given a number a.
Replace $\boxed{x}$ by $\boxed{x-a}$ in the equation.
Graph the new equation.

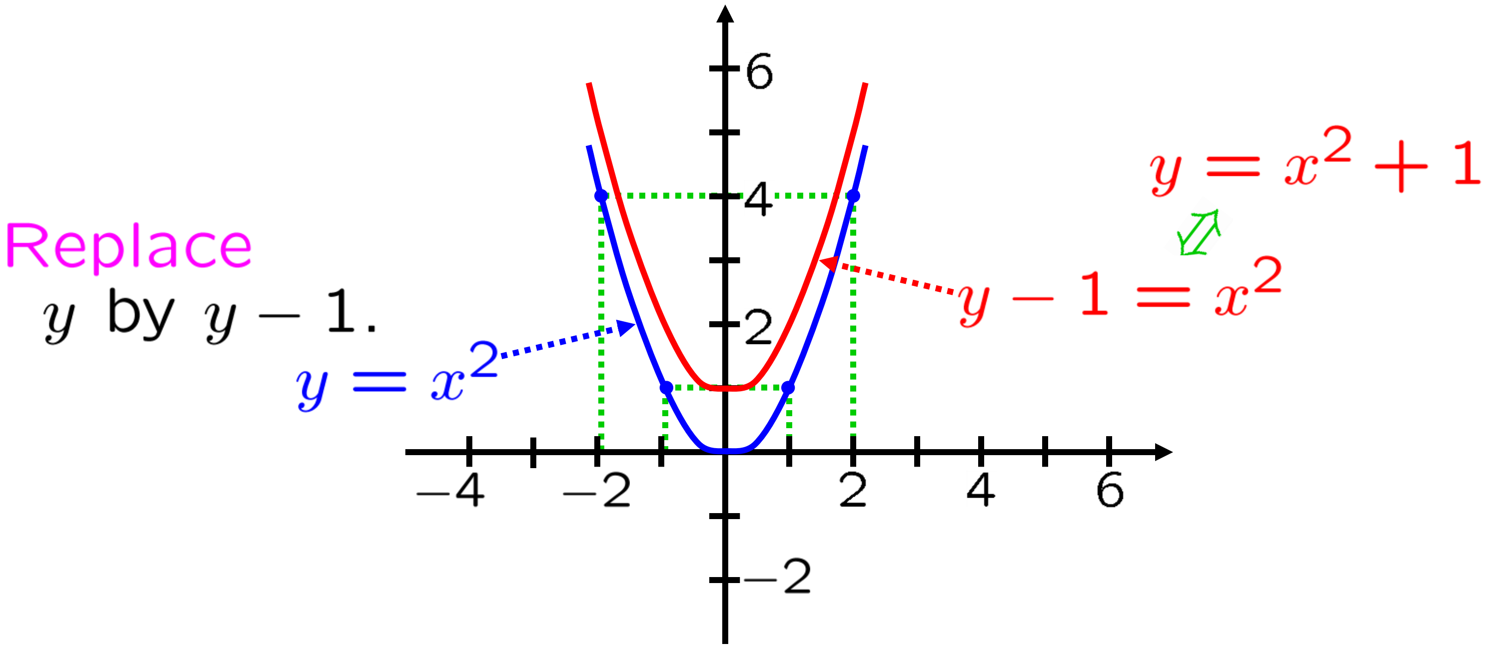


Shift old graph a units to right
to get new graph.

Translation

General problem:

Graph some equation in x and y.
Given a number a.
Replace $\boxed{y}$ by $\boxed{y-a}$ in the equation.
Graph the new equation.



Shift old graph a units upward
to get new graph.

Dilation

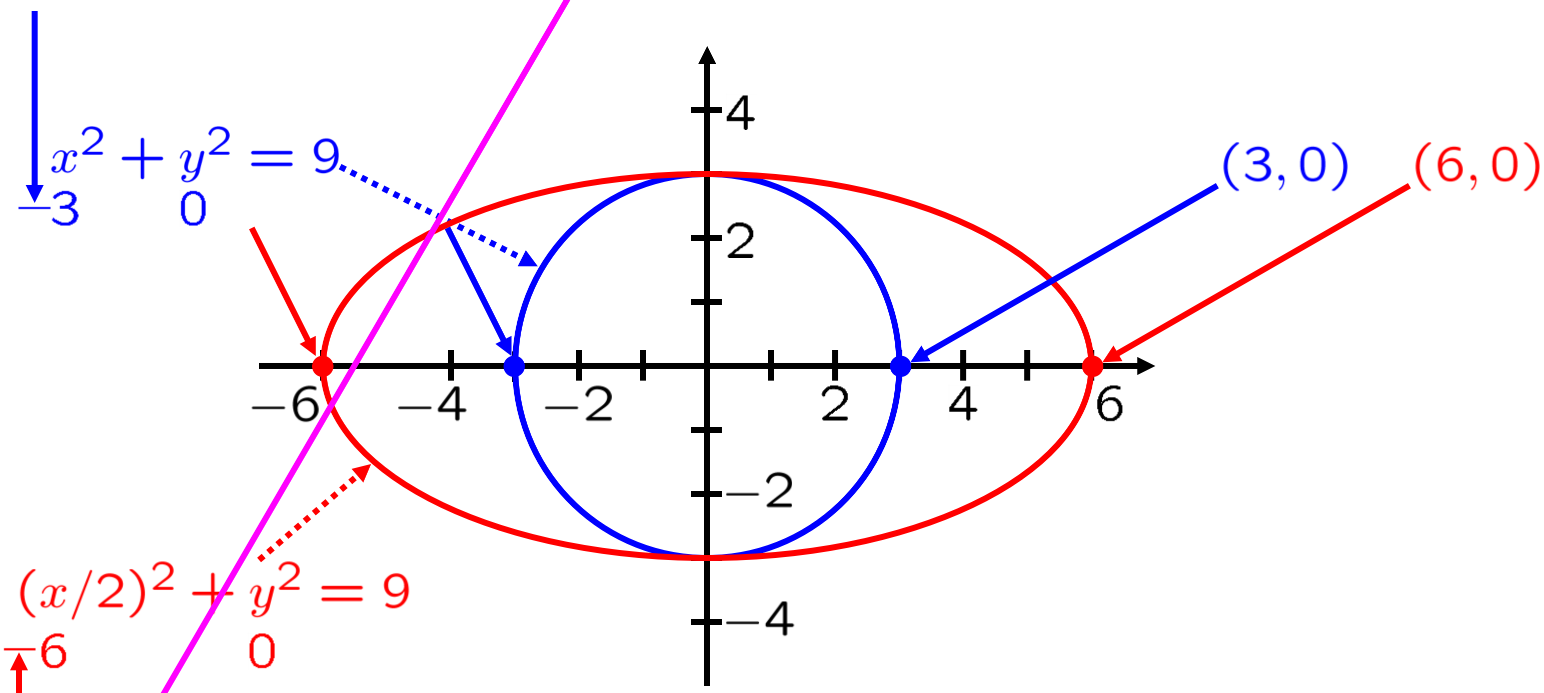
General problem:

Graph some equation in x and y.

Given a number $a > 0$.

Replace x by x/a in the equation.

Graph the new equation.

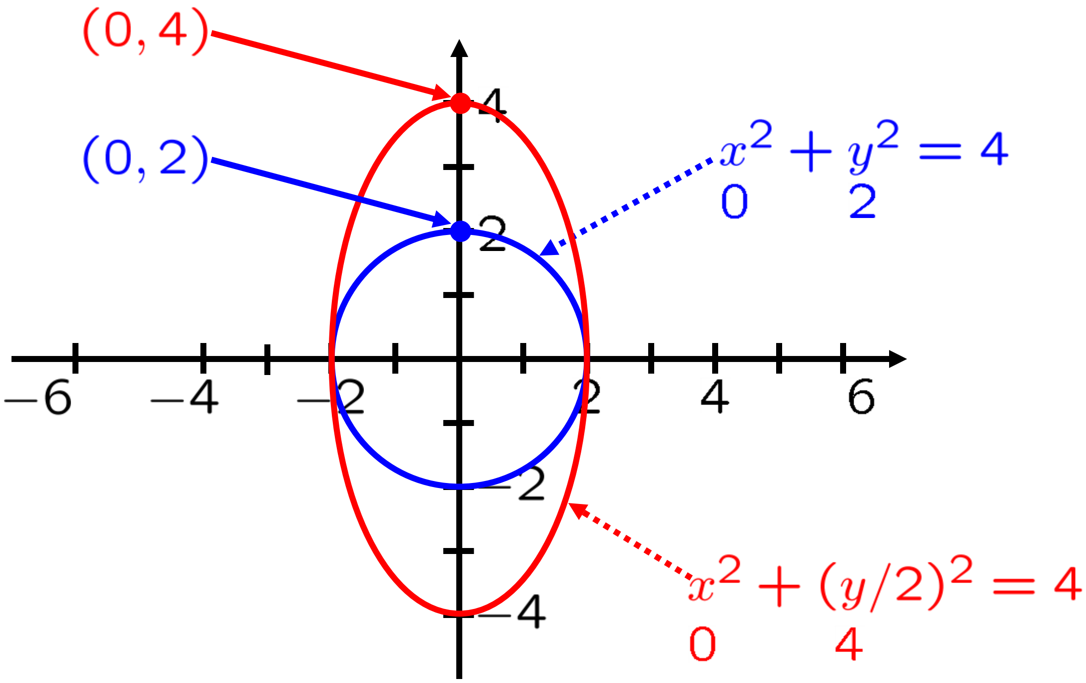


Stretch old graph by a factor of a in x-direction to get new graph.

Dilation

General problem:

Graph some equation in x and y.
Given a number $a > 0$.
Replace y by y/a in the equation.
Graph the new equation.



Stretch old graph by a factor of a in y-direction to get new graph.

Goal: Eliminate the linear term.

Problem: Graph $y = x^2 \boxed{+ 4x} + 5$.

the "linear" term

Replace x by $x + 5$.

expand and collect terms

$$y = (x+5)^2 + 4(x+5) + 5$$
$$= (x^2 + 10x + 25) + (4x + 20) + 5$$
$$= x^2 + 14x + 50$$

Question: Any easier to graph

$y = x^2 \boxed{+ 14x} + 50$?

the "linear" term

Question: Does this help in the original problem of graphing $y = x^2 + 4x + 5$?

Goal: Eliminate the linear term.

Problem: Graph $y = x^2 + 4x + 5$.

Replace x by $x + ?$. $? = -2$

expand and collect terms

$$y = (x + ?)^2 + 4(x + ?) + 5$$

$$= (x^2 + 2?x + ?^2) + (4x + 4?) + 5$$

$$= x^2 + (2? + 4)x + (?^2 + 4? + 5)$$

make zero

Replace x by $x - 2$.

$$y = x^2 + (4 - 8 + 5)$$

$$= x^2 + 1$$ NO "linear" term

Question: Does this help in the original problem of graphing $y = x^2 + 4x + 5$?

Problem: Graph $y = x^2 + 4x + 5$.

Replace x by $x - 2$.

$$y = (x-2)^2 + 4(x-2) + 5$$
$$= x^2 + 1$$ NO "linear" term

Question: Does this help in the

Replace x original problem of

$y = ($graphing $y = x^2 + 4x + 5$?

Note: $=(0,5)$ is on the graph ofn

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Problem: Graph $y = x^2 + 4x + 5$.

Replace x by $x-2$.

$$y = (x-2)^2 + 4(x-2) + 5$$
$$= x^2 + 1$$ NO "linear" term

Question: Does this help in the original problem of graphing $y = x^2 + 4x + 5$?

Note: $(0, 5)$ is on the graph of $y = x^2 + 4x + 5$.

shift two units to right

so $(0+2, 5)$ is on the graph of $y = (x-2)^2 + 4(x-2) + 5$

Problem: Graph $y = x^2 + 4x + 5.$

If we take all the points on

$$y = x^2 + 4x + 54(x-2) + 5$$

$$= x^2 + 1$$ NO "linear" term

$$y = (x-2)^2 + 4(x-2) + 5$$

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Note: $(0, 5)$ is on the graph of

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shift two units to right

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Problem: Graph $y = x^2 + 4x + 5$.

If we take all the points on

$$y = x^2 + 4x + 5$$

and shift them two units to the right, we'll get the graph of

$$y = (x-2)^2 + 4(x-2) + 5$$
$$= x^2 + 1$$ NO "linear" term

Note: $(0, 5)$ is on the graph of

$$y = x^2 + 4x + 5.$$

shift two units to right

so $(0+2, 5)$ is on the graph of

$$y = (x-2)^2 + 4(x-2) + 5$$

Problem: Graph $y = x^2 + 4x + 5$.

If we take all the points on

$$y = x^2 + 4x + 5$$

and shift them two units to the right, we'll get the graph of

$$\begin{aligned} y &= (x-2)^2 + 4(x-2) + 5 \\ &= x^2 + 1 \quad \text{NO "linear" term} \end{aligned}$$

Replacing x by $x-2$

causes the graph to move

two units to the *RIGHT*.

Problem: Graph $y = x^2 + 4x + 5$.

If we take all the points on

$$y = x^2 + 4x + 5$$

and shift them two units to the right, we'll get the graph of

$$y = (x-2)^2 + 4(x-2) + 5$$
$$= x^2 + 1 \quad \text{NO "linear" term}$$

now work backward:

To graph $y = x^2 + 4x + 5$,

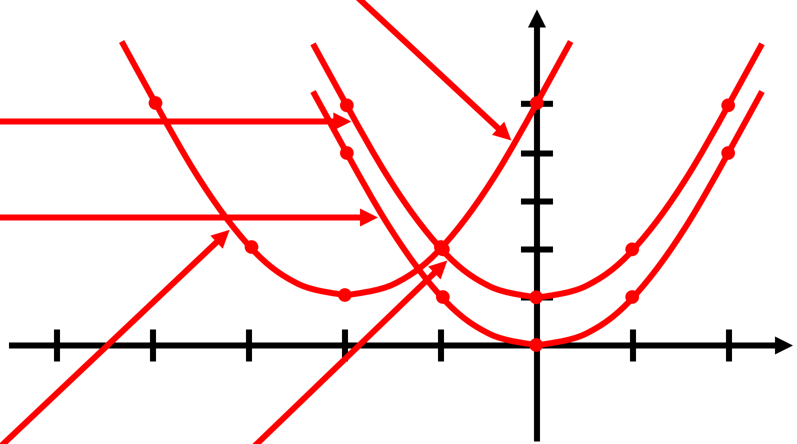
first graph $y = x^2 + 1$, NO "linear" term

then shift all the points two units to the left.

Problem: Graph $y = x^2 + 4x + 5.$

$y - 1 = x^2$

To graph $y = x^2 + 1$,
first graph $y = x^2$,
then shift all the points
one unit upward.



To graph $y = x^2 + 4x + 5$,
first graph $y = x^2 + 1$, NO "linear" term
then shift all the points
two units to the left.

Problem: Find the translation that eliminates the linear term in

$$y = 4x^2 + 6x - 2,$$

i.e., choose ? such that

$$y = 4(x + ?)^2 + 6(x + ?) - 2$$

$2?x$

has no linear term.

Solution:

$$\text{Linear term} = 4(2?x) + 6x$$

$$0 = (8? + 6)x$$

$$? = -6/8 = -3/4 \blacksquare$$

SKILL: Find the translation that eliminates the linear term.

Problem: Eliminate the linear term in

$$y = 4x^2 + 6x - 2,$$

i.e., choose ? such that

$$y = 4(x + ?)^2 + 6(x + ?) - 2$$

has no linear term.

eliminates the $y = 4x^2 + 6x - 2$

i.e.,

has no linear term, and collect terms in $y = 4(x + ?)^2 + 6(x + ?) - 2.$

$$? = -3/4$$

$$? \qquad = -3/4$$

Problem: Eliminate the linear term in

$$y = 4x^2 + 6x - 2,$$

i.e., choose ? such that

$$y = 4(x + ?)^2 + 6(x + ?) - 2$$

has no linear term, then expand and collect terms in $y = 4(x + ?)^2 + 6(x + ?) - 2$.

Solution: $? = -3/4$

$$\begin{aligned} y &= 4(x - (3/4))^2 + 6(x - (3/4)) - 2 \\ &= 4x^2 + 4(-3/4)^2 - 6(3/4) - 2 \\ &= 4x^2 + (9/4) - (9/2) - 2 \\ &= 4x^2 - (17/4) \end{aligned}$$

skip linear terms

SKILL: Eliminate the linear term.

Problem: Find the translation that eliminates the linear term in

$$y = ax^2 + bx + c,$$

i.e., choose ? such that

$$y = a(x + ?)^2 + b(x + ?) + c$$

has no linear term.

Solution:

$$\text{Linear term} = a(2?x) + bx$$

$$0 = (2a? + b)x$$

$$? = -b/(2a) \blacksquare$$

Problem: Eliminate the linear term in

$$y = ax^2 + bx + c,$$

i.e., choose ? such that

$$y = a(x + ?)^2 + b(x + ?) + c$$

has no linear term, then expand and collect terms in $y = a(x + ?)^2 + b(x + ?) + c$.

$$? = -b/(2a)$$

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$$y = ax^2 + bx + c,$$

i.e., choose ? such that

$$y = a(x + ?)^2 + b(x + ?) + c$$

has no linear term, then expand and collect terms in $y = a(x + ?)^2 + b(x + ?) + c$.

Solution: $? = -b/(2a)$

$$y = a\left(x - \frac{b}{2a}\right)^2 + b\left(x - \frac{b}{2a}\right) + c$$

$$? = -b/(2a)$$

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$$y = ax^2 + bx + c,$$

i.e., choose ? such that

$$y = a(x + ?)^2 + b(x + ?) + c$$

has no linear term, then expand and collect terms in $y = a(x + ?)^2 + b(x + ?) + c$.

Solution: $? = -b/(2a)$

$$y = a\left(x - \frac{b}{2a}\right)^2 + b\left(x - \frac{b}{2a}\right) + c = \underbrace{ax^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c}$$

$$= ax^2 - \frac{b^2 - 4ac}{4a} \quad \blacksquare$$

skip linear terms

$? = -b/(2a)$

Problem: Find the translation that eliminates the linear term in

$$y = -(x^2/2) + 19x + 5,$$

i.e., choose ? such that

$x :\to x + 19$

$$y = -((x + ?)^2/2) + 19(x + ?) + 5$$

has no linear term.

Solution: $a = -1/2$, $b = 19$

$$? = -b/(2a) = +b/(+1) = b$$

$$? = 19 \blacksquare$$

NOTE: When the quadratic term is $-x^2/2$,

$x :\to x +$ (the linear coefficient).

$? = b$

Problem: Find the translation that eliminates the linear term in

$$y = -(x^2/2) + 19x + 5,$$

i.e., choose ? such that

$$y = -((x + ?)^2/2) + 19(x + ?) + 5$$

has no linear term.

Solution: $a = -1/2$, $b = 19$

$$? = -b/(2a) = +b/(+1) = b$$

$$? = 19 \blacksquare$$

NOTE: When the quadratic term is $-x^2/2$,

$x :\rightarrow x +$ (the linear coefficient).

$? = b$

Problem: Eliminate the linear term in

$$y = -(x^2/2) + 19x + 5,$$

i.e., choose ? such that

$$y = -((x + ?)^2/2) + 19(x + ?) + 5$$

has no linear term, then expand and collect terms in $y = -((x + ?)^2/2) + 19(x + ?) + 5$.

Solution: $? = 19$

$$y = -((x + 19)^2/2) + 19(x + 19) + 5$$
$$= -(x^2/2) - (19^2/2) + (19^2) + 5$$
$$= -(x^2/2) + (371/2) \blacksquare$$

skip linear terms

$? = b$

Problem: Eliminate the linear term in

$$y = -(x^2/2) + bx + c,$$

i.e., choose ? such that

$$y = -(1/2)(x + ?)^2 + b(x + ?) + c$$

has no linear term, then expand and collect terms in $y = -(1/2)(x + ?)^2 + b(x + ?) + c$.

Solution: $? = b$

$$y = -\frac{1}{2}(x+b)^2 + b\,(x+b) + c = -\frac{x^2}{2} - \frac{b^2}{2} + b^2 + c$$

$= \frac{b^2}{2}$

$$= -(x^2/2) + (b^2/2) + c \quad \blacksquare$$

NOTE: When quad. term is $-x^2/2$, replace linear term by $(1/2)(\text{linear coeff})^2$.

$? = b$

Problem: Eliminate the linear term in

$$y = -(x^2/2) + 8x - 3,$$

i.e., choose ? such that

$$y = -((x + ?)^2/2) + 8(x + ?) - 3$$

has no linear term, then expand and collect terms in $y = -((x + ?)^2/2) + 8(x + ?) - 3$.

Solution: $? = 8$

$$x :\to x + 8$$

half the square of the linear coeff.

$$y = -(x^2/2) + (8^2/2) - 3$$

$$= -(x^2/2) + 29 \blacksquare$$

STOP

NOTE: When quad. term is $-x^2/2$, replace linear term by $(1/2)(\text{linear coeff})^2$.