

# Financial Mathematics

## Combinatorics

“Where people count!”

Question: How many ways to order “ab” ?

Answer: ab ba  $2 = 2 \times 1 = 2!$

better: (a,b), (b,a)

Question: How many ways to order “ac” ?

Answer: ac ca  $2 = 2 \times 1 = 2!$

Question: How many ways to order “bc” ?

Answer: bc cb  $2 = 2 \times 1 = 2!$

Question: How many ways to order “abc” ?

Answer: “a” followed by all orderings of “bc”  
“b” followed by all orderings of “ac”  
“c” followed by all orderings of “ab”

$$6 = 3 \times 2 \times 1 = 3!$$

Question: How many ways to order "abc" ?

Answer: "a" followed by all orderings of "bc"  
"b" followed by all orderings of "ac"  
"c" followed by all orderings of "ab"

$$6 = 3 \times 2 \times 1 = 3!$$

Question: How many ways to order "abc" ?

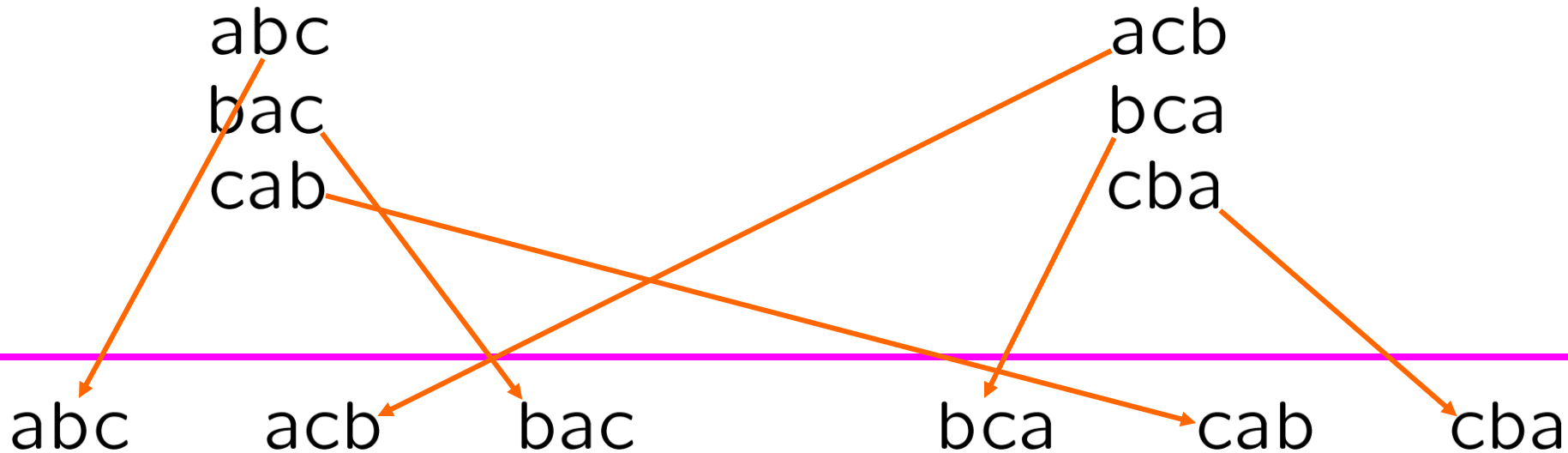
Answer: "a" followed by all orderings of "bc"  
"b" followed by all orderings of "ac"  
"c" followed by all orderings of "ab"

$$6 = 3 \times 2 \times 1 = 3!$$

Question: How many ways to order "abc" ?

Answer: "a" followed by all orderings of "bc"  
"b" followed by all orderings of "ac"  
"c" followed by all orderings of "ab"

$$6 = 3 \times 2 \times 1 = 3!$$

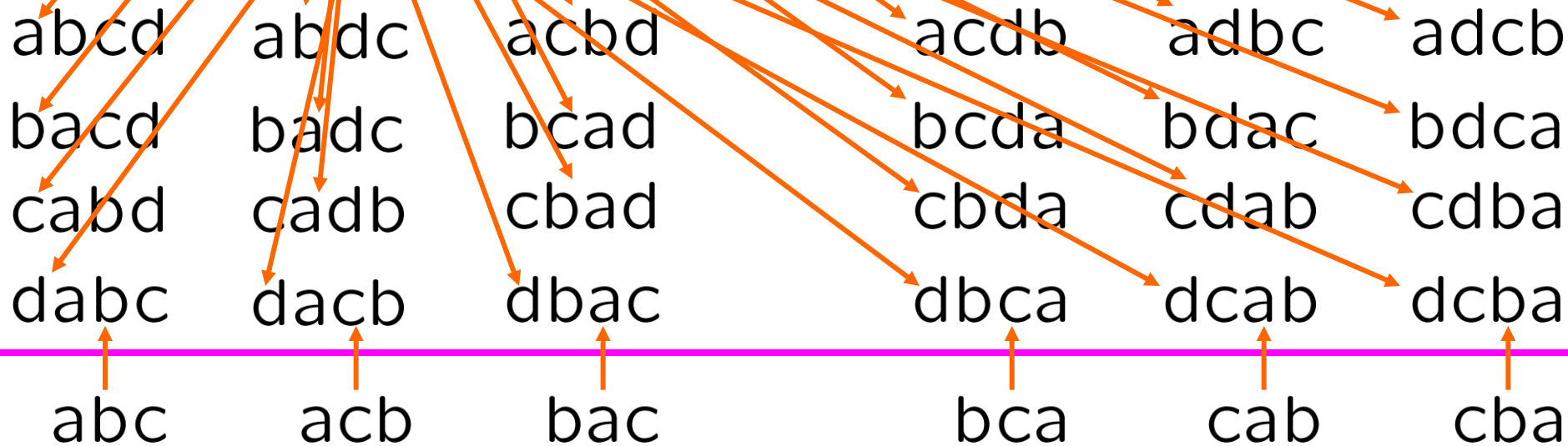


are the ways to order "abc".

Question: How many ways to order "abcd" ?

Answer: "a" followed by all orderings of "bcd"  
"b" followed by all orderings of "acd"  
"c" followed by all orderings of "abd"  
"d" followed by all orderings of "abc"

$$24 = 4 \times 3 \times 2 \times 1 = 4!$$



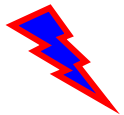
are the ways to order "abc".

Question: How many ways to order “abcde” ?

Answer: “a” followed by all orderings of “bcde”  
“b” followed by all orderings of “acde”  
“c” followed by all orderings of “abde”  
“d” followed by all orderings of “abce”  
“e” followed by all orderings of “abcd”

$$120 = 5 \times 4 \times 3 \times 2 \times 1 = 5!$$

BIG... DISPLAYED LATER



Next subtopic: Choosing subsets

Question: How many ways of choosing two objects from among five?

Motivation for such a question...

STOCK  
SCENARIOS:

UUUUU five upticks  
UUUUD four upticks, then a downtick  
UUUDU three upticks, then a downtick, then an uptick  
UUUDD three upticks, then two downticks  
⋮  
⋮  
⋮  
DDDDD five downticks

Question: How many ways of choosing two objects from among five?

Motivation for such a question...



Next subtopic:  
How to answer such a question

STOCK SCENARIOS:

UUUUU  
UUUUD  
UUUDU  
UUUDD  
.  
.  
.  
DDDDD

Question:  
How many scenarios have two upticks and three downticks?

Choose two letters from abcde

Change them to U

Change the others to D



Question: How many ways of choosing two objects from among five?

Question: How many ways of choosing two letters from "abcde" ?

Answer: ab ac ad ae  
same as ba bc bd be  
better: {a,b} cd ce  
not enough room de

Question: How many ways of choosing complement three letters from "abcde" ?

Answer: cde bde bce bcd

$10 = \binom{5}{2}$  Read: "5 choose 2"

Question: How many ways of choosing two objects from among five?

Question: How many ways of choosing two letters from "abcde" ?

Answer: ab                      ac                      ad                      ae  
   bc                      bd                      be  
   cd                      ce  
   de

Question: How many ways of choosing complement three letters from "abcde" ?

Answer: cde                      bde                      bce                      bcd  
   ade                      ace                      acd

$10 = \binom{5}{2}$  Read: "5 choose 2"

Question: How many ways of choosing two objects from among five?

Question: How many ways of choosing two letters from "abcde" ?

Answer: ab                  ac                  ad                  ae  
                                  bc                  bd                  be  
  cd                  ce  
  de

Question: How many ways of choosing complement three letters from "abcde" ?

Answer: cde                  bde                  bce                  bcd  
                                  ade                  ace                  acd  
  abe                  abd  
  abc

$10 = \binom{5}{2}$  Read: "5 choose 2"

Question: How many ways of choosing two objects from among five?

Question: How many ways of choosing two letters from "abcde" ?

Answer: ab                      ac                      ad                      ae  
   bc                      bd                      be  
   cd                      ce  
   de

Question: How many ways of choosing three letters from "abcde" ?

Answer: cde                      bde                      bce                      bcd  
   ade                      ace                      acd  
   abe                      abd  
   abc

$$10 = \binom{5}{2} = \binom{5}{3}$$

Question: How many ways to partition "abcde" into two letters, three letters?

Answer:

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

Question: How many ways of reordering ab?

Answer: ab ba  
2!

Question: How many ways of reordering cde?

Answer: cde ced dce dec ecd edc  
3!

Question: How many ways of reordering ab,cde?

A: No letter is allowed to move across the comma.

ab, orderings of cde

ba, orderings of cde

**Question:** How many ways to partition "abcde" into two letters, three letters?

**Answer:**

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

**Question:** How many ways of reordering ab?

**Answer:** ab ba  
2!

**Question:** How many ways of reordering cde?

**Answer:** cde ced dce dec ecd edc  
3!

**Question:** How many ways of reordering ab,cde?

**A:** ab,cde ab,ced ab,dce ab,dec ab,ecd ab,edc  
ba,cde ba,ced ba,dce ba,dec ba,ecd ba,edc

(2!)(3!)

ab, orderings of cde

ba, orderings of cde

Question: How many ways to partition "abcde" into two letters, three letters?

Answer:

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

Question: How many ways of reordering abcde?

Answer: DISPLAYED LATER

$$5! = 120$$

e.g.:

badec	bceda	ecbad
ba,dec	bc,eda	ec,bad

Note: Each leads to a reordering of one of the partitions above...

Question: How many ways to partition "abcde" into two letters, three letters?

Answer:

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$\binom{5}{2} = \binom{5}{3} = 10$

Question: How many ways of reordering abcde?

Answer: 5!

5!

Listing algorithm:

For each of the  $\binom{5}{2}$  partitions above,



Question: How many ways to partition "abcde" into two letters, three letters?

Answer:

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$\binom{5}{2} = \binom{5}{3} = 10$

Question: How many ways of reordering abcde?

Answer:  $5! = \binom{5}{2} (2!)(3!)$  Let's see this...

Listing algorithm:

For each of the  $\binom{5}{2}$  partitions above,

write out its  $(2!)(3!)$  reorderings and then take out all commas.

Question: How many ways to partition "abcde" into two letters, three letters?

Answer:

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

ab,cde ab,ced ab,ecd  
 ab,dce ab,dec ab,edc  
 ba,cde ba,ced ba,ecd  
 ba,dce ba,dec ba,edc

(2!)(3!)

NEXT STEP IS SLOW...

reorderings  
 respecting  
 the comma

Listing algorithm:

For each of the  $\binom{5}{2}$  partitions above,

write out its (2!)(3!) reorderings  
 and then take out all commas.

**Question:** How many ways to partition "abcde" into two letters, three letters?

**Answer:**

ab, cde	ac, bde	ad, bce	ae, bcd
	bc, ade	bd, ace	be, acd
		cd, abe	ce, abd
			de, abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

ab, cde	ab, ced	ab, ecd	ac, bde	ac, bed	ac, ebd	ad, bce	ad, bec	ad, ebc	ae, bcd	ae, bdc	ae, dbc
ab, dce	ab, dec	ab, edc	ac, dbe	ac, deb	ac, edb	ad, cbe	ad, ceb	ad, ecb	ae, cbd	ae, cdb	ae, dcb
ba, cde	ba, ced	ba, ecd	ca, bde	ca, bed	ca, ebd	da, bce	da, bec	da, ebc	ea, bcd	ea, bdc	ea, dbc
ba, dce	ba, dec	ba, edc	ca, dbe	ca, deb	ca, edb	da, cbe	da, ceb	da, ecb	ea, cbd	ea, cdb	ea, dcb
$(2!)(3!)$			bc, ade	bc, aed	bc, ead	bd, ace	bd, aec	bd, eac	be, acd	be, adc	be, dac
reorderings respecting the comma			bc, dae	bc, dea	bc, ead	bd, cae	bd, cea	bd, eca	be, cad	be, cda	be, dca
			cb, ade	cb, aed	cb, ead	db, ace	db, aec	db, eac	eb, acd	eb, adc	eb, dac
			cb, dae	cb, dea	cb, ead	db, cae	db, cea	db, eca	eb, cad	eb, cda	eb, dca
						cd, abe	cd, aeb	cd, eab	ce, abd	ce, adb	ce, dab
						cd, bae	cd, bea	cd, eba	ce, bad	ce, bda	ce, dba
						dc, abe	dc, aeb	dc, eab	ec, abd	ec, adb	ec, dab
						dc, bae	dc, bea	dc, eba	ec, bad	ec, bda	ec, dba
									de, abc	de, acb	de, cab
									de, bac	de, bca	de, cba
									dc, abc	dc, acb	ed, cab
									dc, bac	dc, bca	ed, cba

$$\binom{5}{2} (2!)(3!) = \binom{5}{3} (2!)(3!)$$

NEXT STEP IS SLOW...

Take out all of the commas

**Question:** How many ways to partition "abcde" into two letters, three letters?

**Answer:**

ab,cde	ac,bde	ad,bce	ae,bcd
	bc,ade	bd,ace	be,acd
		cd,abe	ce,abd
			de,abc

$$\binom{5}{2} = \binom{5}{3} = 10$$

abcde	abced	abecd	acbde	acbed	acebd	adbce	adbec	adebc	aebcd	aebdc	aedbc
abdce	abdec	abedc	acdbe	acdeb	acedb	adcbe	adceb	adecb	aecbd	aecdb	aedcb
bacde	baced	baecd	cabde	cabed	caebd	dabce	dabec	daebc	eabcd	eabdc	eadbc
badce	badec	baedc	cadbe	cadeb	caedb	dacbe	daceb	daecb	eacbd	eacdb	eadcb

(2!)(3!)

reorderings respecting the comma

bcade	bcaed	bcead	bdace	bdaec	bdeac	beacd	beadc	bedac
bcdae	bcdea	bceda	bdcae	bdcea	bdeca	becad	becda	bedca
cbade	cbaed	cbead	dbace	dbaec	dbeac	ebacd	ebadc	ebdac
cbdae	cbdea	cbeda	dbcae	dbcea	dbeqa	ebcad	ebdea	ebdca
			cdabe	cdaeb	cdeab	ceabd	ceadb	cedab
			cdbae	cdbea	cdeba	cebada	cebda	cedba
			dcabe	dcaeb	dceab	ecabd	ecadb	ecdab
			dcbae	dcbea	dceba	ecbad	ecbda	ecdab
						deabc	deacb	decab
						debac	debca	decba
						dcabc	dcacb	edcab
						dcbac	dcbca	edcba

$$5! = \binom{5}{2} (2!)(3!) = \binom{5}{3} (2!)(3!)$$

Take out all of the commas

$$5! = \binom{5}{2} \frac{5!}{(2!)(3!)} = \binom{5}{2}$$

$$5! = \binom{5}{2} (2!)(3!)$$

$$\binom{5}{2} = \frac{5!}{(2!)(3!)}$$

$$\frac{5!}{(2!)(3!)} = \binom{5}{2}$$

$$5! = \binom{5}{2} (2!)(3!)$$

binomial  
coefficients

$$\binom{n}{k} = \frac{n!}{(k!)((n-k)!)}$$

$$\binom{5}{2} = \frac{5!}{(2!)(3!)}$$

$$\frac{5!}{(2!)(3!)} = \binom{5}{2}$$

$$4! = \binom{4}{2} (2!)(2!)$$

$$\binom{n}{k} = \frac{n!}{(k!)((n-k)!)}$$

$$\binom{100}{9} = \frac{100!}{(9!)(91!)}$$

$$= \frac{\overset{5}{(100)} \overset{11}{(99)} \overset{7}{(98)} \overset{4}{(97)} \overset{16}{(96)} \overset{47}{(95)} \overset{31}{(94)} \overset{31}{(93)} \overset{92}{(92)}}{\overset{9}{(9)} \overset{8}{(8)} \overset{7}{(7)} \overset{6}{(6)} \overset{5}{(5)} \overset{4}{(4)} \overset{3}{(3)} \overset{2}{(2)} \overset{1}{(1)}}$$

$$= (5)(11)(7)(97)(4)(95)(47)(31)(92)$$

$$= (666,981,700)(31)(92) = (666,981,700)(2,852)$$

$$= 1,902,231,808,400$$

**SKILL**  
 Compute  
 binomial  
 coefficients



$$\binom{n}{k} = \frac{n!}{(k!)((n-k)!)}$$

$$\binom{n}{1} = \frac{n!}{(1!)((n-1)!)} = n$$

$$n! = n \underbrace{(n-1)(n-2) \cdots 1}_{(n-1)!}$$

polynomial  
in  $n$  of  
degree 1

$$\frac{n!}{(n-1)!} = n$$

$$\binom{n}{k} = \frac{n!}{(k!)((n-k)!)}$$

$$\binom{n}{2} = \frac{n!}{(2!)((n-2)!)} = \frac{n(n-1)}{2!}$$

$$n! = n(n-1)\underbrace{(n-2)\cdots 1}_{(n-2)!}$$

polynomial  
in  $n$  of  
degree 2

$$\frac{n!}{(n-2)!} = n(n-1)$$

$$\binom{n}{k} \binom{n}{k} \frac{n!}{(k!)((n-k)!)}$$

$$\binom{n}{3} = \frac{n!}{(3!)((n-3)!)}$$

$$\binom{n}{k} = \frac{n!}{(k!)((n-k)!)}$$



$$\binom{n}{3} = \frac{n!}{(3!)((n-3)!)} = \frac{n(n-1)(n-2)}{3!}$$

$$n! = n(n-1)(n-2)\underbrace{(n-3)\cdots 1}_{(n-3)!}$$

$$\frac{n!}{(n-3)!} = n(n-1)(n-2)$$

polynomial  
in  $n$  of  
degree 3

**SKILL**  
For fixed  $d$ ,

write  $\binom{n}{d}$

as a degree  $d$   
polynomial in  $n$