Financial Mathematics

Counting monomials

Definition:

A **monomial** is a word in a certain alphabet, in which the letters commute.

e.g.: $abbab$ is a monomial in a and b,
but it is the same as $bbaab$;
canonical form is $aabbb$,
more commonly written a^2b^3.

e.g.: $x^3y^7z^{1{,}000{,}000}$ is a monomial in x, y, z.
Its **degree** is

$$3+7+1{,}000{,}000=1{,}000{,}010$$

Question:
How many monomials are there degree $\leq d$ in n variables?

e.g.: How many monomials are there degree ≤ 3 in 2 variables?

Let's use x and y for the variables.

$$\underbrace{1,}_{1} + \underbrace{x,\ y,}_{2} + \underbrace{x^2,\ xy,\ y^2,}_{3} + \underbrace{x^3,\ x^2y,\ xy^2,\ y^3}_{4} = 10$$

$$\binom{3+2}{2} = \binom{3+2}{3} = \frac{5!}{(2!)(3!)} = 10$$

Coincidence? I think not.

e.g.: How many monomials are there degree $\leq \boxed{3}$ in 4 variables?

Exercise: List them, and count.
Check that there are 35.

$$\binom{3+4}{3} = \binom{3+4}{4} = \frac{7!}{(3!)(4!)}$$

$$= \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})(\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1})} = 7 \cdot 5 = 35$$

Coincidence? I think not.

e.g.: How many monomials are there degree $\leq \boxed{8}$ in 4 variables?

Solution: List them?

$$\binom{8+4}{4} = \frac{\cancel{12}\cdot 11\cdot 10\cdot 9}{\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1} = 11\cdot 5\cdot 9 = 495$$

NO WAY!!

Can we count them without listing them?

e.g.: How many monomials are there degree ≤ 8 in 4 variables?

$$\binom{8+4}{4}$$

$$\stackrel{??}{=}$$

\# monomials of degree ≤ 8 in w, x, y, z

$$\binom{8+4}{4}$$

Why do these two counts come out the same?

We set up a one-to-one correspondence between

choices of four from $a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l$ $= \binom{8+4}{4}$

??=

and

monomials of degree ≤ 8 in w, x, y, z

Why do these two counts come out the same?

We set up a one-to-one correspondence between

choices of four from

$a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l$

and

monomials of degree ≤ 8 in w, x, y, z ■

e.g.: $\{c, g, h, j\}$ ↕ w^2x^3z

$\underbrace{a\ b}_{2}\ \boxed{c}\ \underbrace{d\ e\ f}_{3}\ \boxed{g}\ \underset{0}{\uparrow}\ \boxed{h}\ \underbrace{i}_{1}\ \boxed{j}\ \underbrace{k\ l}_{\text{don't count this}}$

yields

$$w^2x^3y^0z^1 = w^2x^3z$$

Exercise: What corresponds to w^3x^2yz?

Why do these two counts come out the same?

We set up a one-to-one correspondence

between

choices of four from

$a \; b \; c \; d \; e \; f \; g \; h \; i \; j \; k \; l$

e.g.: $\{d, ??i, k\}$

and

monomials of degree ≤ 8 in w, x, y, z

w^3x^2yz

$w^3x^2y^1z^1$

yields

$\underbrace{a \; b \; c}_{3} \; \boxed{d} \; \underbrace{e \; f}_{2} \; \boxed{g} \; \underbrace{h}_{1} \; \boxed{i} \; \underbrace{j}_{1} \; \boxed{k} \; \underbrace{l}$

don't count this

Exercise: What corresponds to w^3x^2yz?

Question: How many monomials are there degree $\boxed{\leq}$ d in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$ $\quad d :\to 6$, $\; n :\to 3$

Question: How many monomials are there degree $\boxed{=}$ d in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

$$\#\{\text{monomials of degree} = 6 \text{ in } \{x, y, z, t\}\}$$

Claim: $\|$ Pf. . .

$$\#\{\text{monomials of degree} \leq 6 \text{ in } \{x, y, z\}\}$$

$$\|$$

$$\binom{6+3}{6} = \binom{6+3}{3} = 84$$

exercise

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $$\binom{d+n}{d} = \binom{d+n}{n}$$

Question: How many monomials are there degree $= d$ in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

{monomials of degree $= 6$ in $\{x, y, z, t\}$}

$\{x, y, z\}\}$

We will set up a 1-1 coorespondence between this and this. . .

{monomials of degree ≤ 6 in $\{x, y, z\}$}

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

$$\{\text{monomials of degree} = 6 \text{ in } \{x, y, z, t\}\}$$

$$\rotatebox{90}{$\in$}$$

x^2y^2zt ← eliminate all ts

$\updownarrow$

x^2y^2z

$$\rotatebox{90}{$\in$}$$

$$\{\text{monomials of degree} \leq 6 \text{ in } \{x, y, z\}\}$$

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

$$\{\text{monomials of degree} = 6 \text{ in } \{x, y, z, t\}\}$$

$$\begin{array}{cccc} \ni & \ni & \ni & \ni \\ x^2y^2zt & xy^2t^3 & xy^2zt^2 & x^2yz^3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ x^2y^2z & xy^2 & xy^2z & x^2yz^3 \\ \in & \in & \in & \in \end{array}$$

$$\{\text{monomials of degree} \leq 6 \text{ in } \{x, y, z\}\}$$

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

{monomials of degree $= 6$ in $\{x, y, z, t\}$}

$\in$

xyz^2t^2

$\updownarrow$

$xyz^2 \leftarrow$ add in ts until degree $= 6$

$\in$

{monomials of degree ≤ 6 in $\{x, y, z\}$}

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

e.g.: How many monomials are there degree $= 6$ in 4 variables?

$\{$monomials of degree $= 6$ in $\{x, y, z, t\}\}$

$\ni$ $\qquad$ $\ni$ $\qquad$ $\ni$ $\qquad$ $\ni$

$xyz^2t^2 \qquad x^3t^3 \qquad x^2y^2z^2 \qquad xy^3zt$

$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$

$xyz^2 \qquad x^3 \qquad x^2y^2z^2 \qquad xy^3z$

$\in$ $\qquad$ $\in$ $\qquad$ $\in$ $\qquad$ $\in$

$\{$monomials of degree ≤ 6 in $\{x, y, z\}\}$

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $$\binom{d+n}{d} = \binom{d+n}{n} \qquad n :\longrightarrow n-1$$

Question: How many monomials are there degree $= d$ in n variables?

Solution:

$$\#\{\text{monomials of degree} \leq d \text{ in } n-1 \text{ variables}\}$$
$$\|$$
$$\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$$

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

Solution: $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

\#{monomials of degree ≤ 2 in 5 variables}

$\binom{d+n-1}{d} + \binom{d+n-1}{n-1}$

\#{monomials of degree $= 3$ in 5 variables}

Question: How many monomials are there degree $\leq d$ in n variables?

Solution: $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree $= d$ in n variables?

Solution: $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

$\#\{\text{monomials of degree} \leq 2 \text{ in } 5 \text{ variables}\}$

$+$

$\#\{\text{monomials of degree} = 3 \text{ in } 5 \text{ variables}\}$

$\|$

$\#\{\text{monomials of degree} \leq 3 \text{ in } 5 \text{ variables}\}$

$$\binom{2+5}{2} \qquad \binom{3+5}{3}$$

Question: How many monomials are there degree $= d$ in n variables?

Solution: $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

$\#\{$monomials of degree ≤ 2 in 5 variables$\}$

$+$

$\#\{$monomials of degree $= 3$ in 5 variables$\}$

$\|$

$\#\{$monomials of degree ≤ 3 in 5 variables$\}$

$$\binom{2+5}{2} \qquad \binom{3+5-1}{3} \qquad \binom{3+5}{3}$$

Question: How many monomials are there degree $= d$ in n variables?

Solution: $$\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$$

$\#\{$monomials of degree ≤ 2 in 5 variables$\}$

$+$

$\#\{$monomials of degree $= 3$ in 5 variables$\}$

$\|$

$\#\{$monomials of degree ≤ 3 in 5 variables$\}$

$$\binom{2+5}{2} + \binom{3+5-1}{3} = \binom{3+5}{3}$$

Question: How many monomials are there degree $= d$ in n variables?

Solution: $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

$\#\{$monomials of degree ≤ 2 in 5 variables$\}$

$+$

$\#\{$monomials of degree $= 3$ in 5 variables$\}$

$\|$

$\#\{$monomials of degree ≤ 3 in 5 variables$\}$

$$\binom{2+5}{2} + \binom{3+5-1}{3} = \binom{3+5}{3}$$

$$\binom{7}{2} + \binom{7}{3} = \binom{8}{3}$$