

# Financial Mathematics

## Counting monomials

## Definition:

A **monomial** is a word in a certain alphabet, in which the letters commute.

*e.g.*:  $abbab$  is a monomial in  $a$  and  $b$ ,  
**but** it is the same as  $bbaab$ ;  
canonical form is  $aabbb$ ,  
more commonly written  $a^2b^3$ .

*e.g.*:  $x^3y^7z^{1,000,000}$  is a monomial in  $x, y, z$ .  
Its **degree** is  
 $3+7+1,000,000=1,000,010$

Question:

How many monomials are there degree  $\leq d$  in  $n$  variables?

e.g.: How many monomials are there degree  $\leq 3$  in 2 variables?

Let's use  $x$  and  $y$  for the variables.

$$\underbrace{1}_{1} + \underbrace{x, y}_{2} + \underbrace{x^2, xy, y^2}_{3} + \underbrace{x^3, x^2y, xy^2, y^3}_{4} = 10$$

$$\binom{3+2}{2} = \binom{3+2}{3} = \frac{5!}{(2!)(3!)} = 10$$

Coincidence? I think not.

e.g.: How many monomials are there  
degree  $\leq 3$  in 4 variables?

Exercise: List them, and count.  
Check that there are 35.

$$\binom{3+4}{3} = \binom{3+4}{4} = \frac{7!}{(3!)(4!)} \\ = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})(\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1})} = 7 \cdot 5 = 35$$

Coincidence? I think not.

e.g.: How many monomials are there  
degree  $\leq 8$  in 4 variables?

Solution: List them?

$$\binom{8+4}{4} = \frac{\cancel{12} \cdot 11 \cdot 10 \cdot 9}{\cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1} = 11 \cdot 5 \cdot 9 = 495$$

NO WAY!!

Can we count them without listing them?

e.g.: How many monomials are there  
degree  $\leq 8$  in 4 variables?

$$\binom{8+4}{4}$$

??

$\binom{8+4}{4}$  # monomials of degree  $\leq 8$  in  $w, x, y, z$

# Why do these two counts come out the same?

We set up a one-to-one correspondence between

$$\# \text{ choices of four from } a \ b \ c \ d \ e \ f \ g \ h \ i \ j \ k \ l = \boxed{\binom{8+4}{4}}$$

and

$$\# \text{ monomials of degree } \leq 8 \text{ in } w, x, y, z$$

# Why do these two counts come out the same?

We set up a one-to-one correspondence between

choices of four from

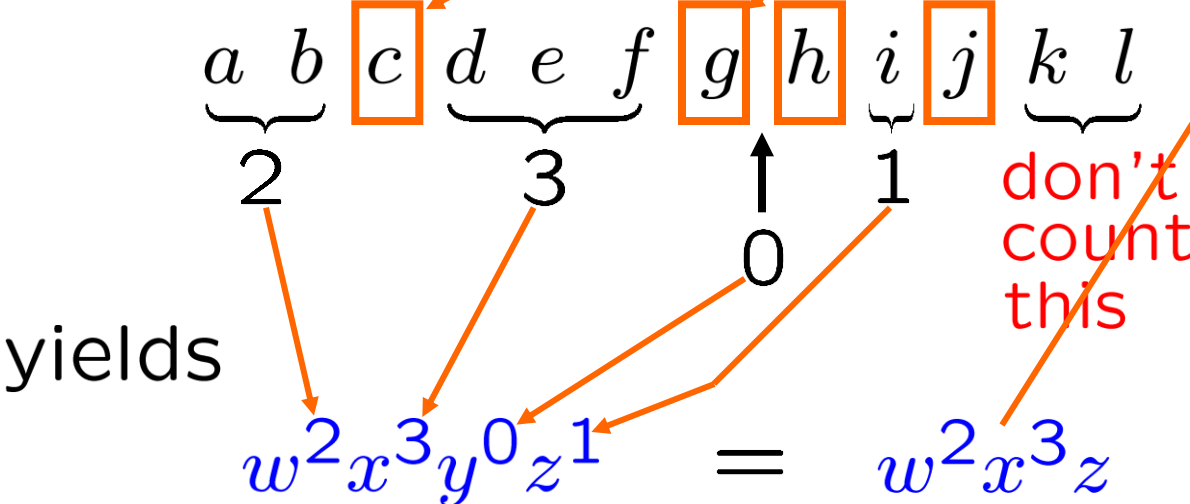
$a \ b \ c \ d \ e \ f \ g \ h \ i \ j \ k \ l$

e.g.:  $\{c, g, h, j\}$

and

monomials of degree  $\leq 8$  in  $w, x, y, z$  ■

$w^2 x^3 z$



Exercise: What corresponds to  $w^3 x^2 y z$ ?

8



# Why do these two counts come out the same?

We set up a one-to-one correspondence between

choices of four from

$a \ b \ c \ d \ e \ f \ g \ h \ i \ j \ k \ l$

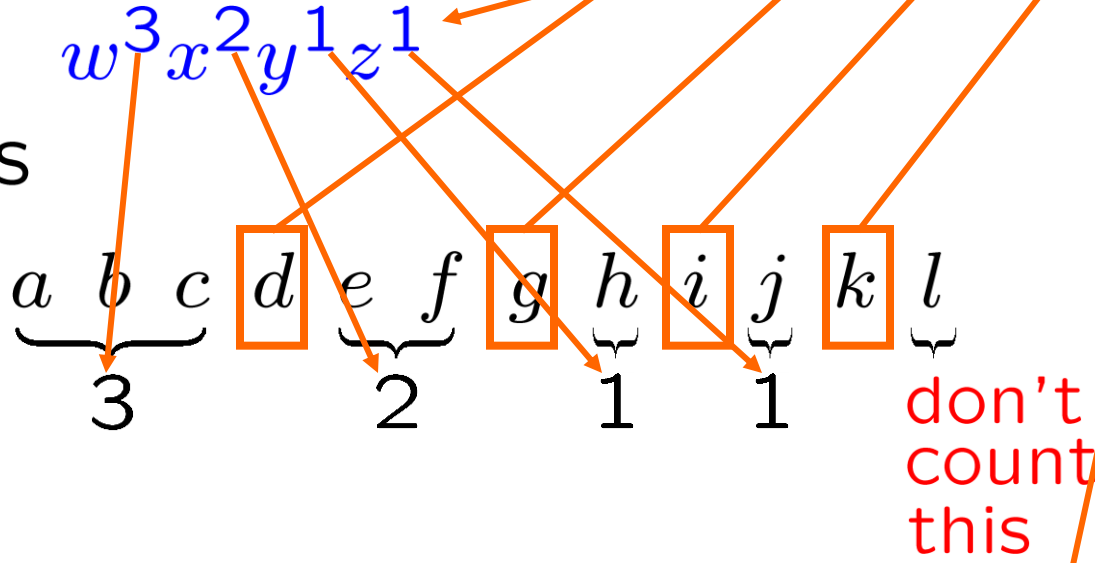
and

monomials of degree  $\leq 8$  in  $w, x, y, z$

e.g.:  $\{d, \dots, i, k\}$

$w^3 x^2 y z$

yields



Exercise: What corresponds to  $w^3 x^2 y z$ ?

Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution:  $\binom{d+n}{d} = \binom{d+n}{n}$   $d \rightarrow 6$   
 $n \rightarrow 3$

Question: How many monomials are there degree  $= d$  in  $n$  variables?

e.g.: How many monomials are there degree = 6 in 4 variables?

$\#\{\text{monomials of degree} = 6 \text{ in } \{x, y, z, t\}\}$

Claim:  $\parallel$  Pf...

$\#\{\text{monomials of degree} \leq 6 \text{ in } \{x, y, z\}\}$

$$\parallel$$
$$\binom{6+3}{6} = \binom{6+3}{3} \stackrel{\text{exercise}}{=} 84$$

Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution: 
$$\binom{d+n}{d} = \binom{d+n}{n}$$

Question: How many monomials are there degree  $= d$  in  $n$  variables?

*e.g.*: How many monomials are there degree  $= 6$  in 4 variables?

{monomials of degree  $= 6$  in  $\{x, y, z, t\}$ }

We will set up a 1-1 correspondence between  $\{x, y, z\}$  and this...

{monomials of degree  $\leq 6$  in  $\{x, y, z\}$ }

Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution:  $\binom{d+n}{d} = \binom{d+n}{n}$

Question: How many monomials are there degree  $= d$  in  $n$  variables?

e.g.: How many monomials are there degree  $= 6$  in 4 variables?

{monomials of degree  $= 6$  in  $\{x, y, z, t\}$ }

$x^2 y^2 z t$  ← eliminate all  $ts$

$x^2 y^2 z$

{monomials of degree  $\leq 6$  in  $\{x, y, z\}$ }

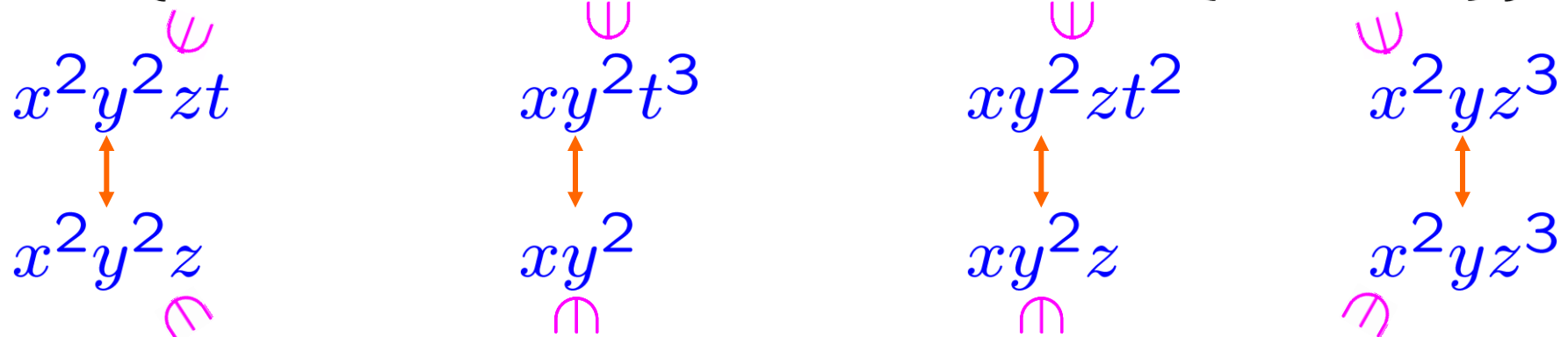
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{monomials of degree  $= 6$  in  $\{x, y, z, t\}$ }

$xyz^2t^2$

$\updownarrow$   
 $xyz^2$  ← add in  $t$ s until degree  $= 6$

{monomials of degree  $\leq 6$  in  $\{x, y, z\}$ }

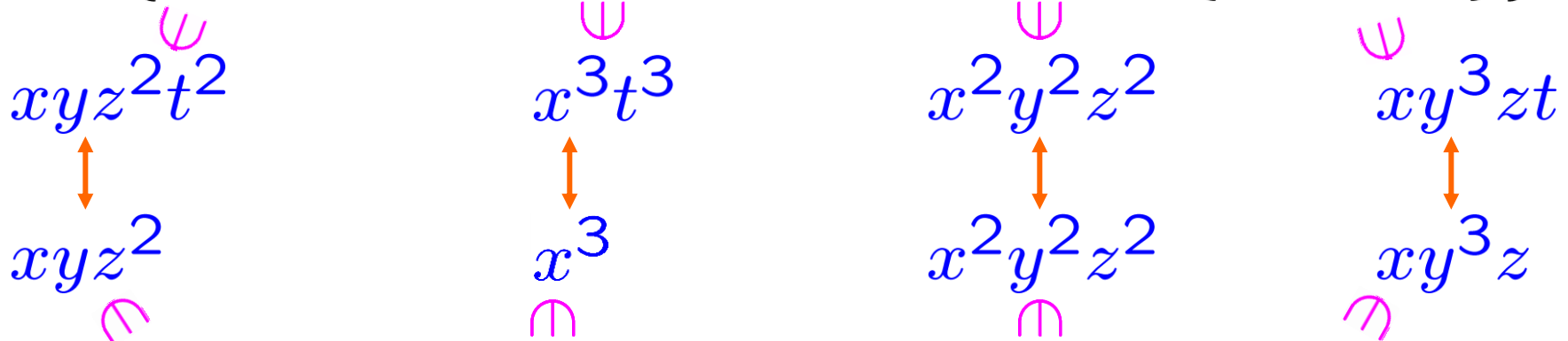
**Question:** How many monomials are there degree  $\leq d$  in  $n$  variables?

**Solution:** 
$$\binom{d+n}{d} = \binom{d+n}{n}$$

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*e.g.:* How many monomials are there degree  $= 6$  in 4 variables?

{monomials of degree  $= 6$  in  $\{x, y, z, t\}$ }



{monomials of degree  $\leq 6$  in  $\{x, y, z\}$ }

Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution:  $\binom{d+n}{d} = \binom{d+n}{n} \quad n \rightarrow n-1$

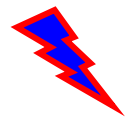
Question: How many monomials are there degree  $= d$  in  $n$  variables?

Solution:

$\#\{\text{monomials of degree } \leq d \text{ in } n-1 \text{ variables}\}$

$\parallel$

$$\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$$





Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution: 
$$\binom{d+n}{d} = \binom{d+n}{n}$$

Question: How many monomials are there degree  $= d$  in  $n$  variables?

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$$\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$$

# {monomials of degree  $\leq 2$  in 5 variables}

# {monomials of degree  $= 3$  in 5 variables}

$$\binom{d+n-1}{d} + \binom{d+n-1}{n-1}$$

Question: How many monomials are there degree  $\leq d$  in  $n$  variables?

Solution: 
$$\binom{d+n}{d} = \binom{d+n}{n}$$

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# {monomials of degree  $\leq 2$  in 5 variables}

# {monomials of degree  $= 3$  in 5 variables}

# {monomials of degree  $\leq 3$  in 5 variables}

$$\binom{2+5}{2}$$

$$\binom{3+5}{3}$$

Question: How many monomials are there  
degree =  $d$  in  $n$  variables?

Solution: 
$$\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$$

# {monomials of degree  $\leq 2$  in 5 variables}

# {monomials of degree = 3 in 5 variables}

# {monomials of degree  $\leq 3$  in 5 variables}

$$\binom{2+5}{2} \quad \binom{3+5-1}{3} \quad \binom{3+5}{3}$$

Question: How many monomials are there degree =  $d$  in  $n$  variables?

Solution:  $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

# {monomials of degree  $\leq 2$  in 5 variables}

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# {monomials of degree  $\leq 3$  in 5 variables}

$$\binom{2+5}{2} + \binom{3+5-1}{3} = \binom{3+5}{3}$$

Question: How many monomials are there degree =  $d$  in  $n$  variables?

Solution:  $\binom{d+n-1}{d} = \binom{d+n-1}{n-1}$

# {monomials of degree  $\leq 2$  in 5 variables}

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