Financial Mathematics

One variable integral calculus review

Def'n: Let I be an interval.

Let f be a function whose domain contains I.

A function F is called an **antiderivative of** f **on** I

if, $\forall x \in I$, we have: $F'(x) = f(x)$.

Def'n: Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

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e.g.: Find all antiderivatives of $f(x) = x^2$.

Guess: $F(x) = \frac{1}{3}x^3$

$F'(x) = x^2 = f(x)$

Guess: $F(x) = x^3$

$F'(x) = 3x^2 \neq f(x)$

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expressions. . .

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Guess: $F(x) = \frac{1}{3}x^3$ ☺

$F'(x) = x^2 = f(x)$

Guess: $F(x) = \frac{1}{3}x^3 + 6$ ☺

$F'(x) = x^2 = f(x)$

Other antiderivatives:

$\frac{1}{3}x^3 + 8$

$\frac{1}{3}x^3 + 3$

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Other antiderivatives: (of x^2 w.r.t. x)

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$\frac{1}{3}x^3 + 3$

d/dx

x^2

d/dx is not "1-1" and so is not invertible.

EQUALITY OF DERIVATIVES:

If $g'(x) = h'(x)$, for all x in an interval I, then $g - h$ is constant on I;

that is, $\exists c \in \mathbb{R}$ s.t., $\forall x \in (a, b)$, $g(x) = (h(x)) + c$.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

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Notation: The *set* of *all* antiderivatives of $f(x)$ w.r.t. x is denoted $\boxed{\int f(x)\,dx}$.

e.g.: $\int x^2\,dx \;= \{\frac{1}{3}x^3 + C \,|\, C \in \mathbb{R}\}$

Traditional to drop the set braces and everything after the vertical line (|)

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Note: $\{\frac{1}{3}x^3 + C \mid C \in \mathbb{R}\} = \{\frac{1}{3}x^3 - 6C \mid C \in \mathbb{R}\}$

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Looks strange, but it's correct, as sets.

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e.g.: $\int x^2\,dx = \frac{1}{3}x^3 + C$

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$x :\to v$

EQUALITY OF DERIVATIVES:

If $g'(v) = h'(v)$, for all v in an interval I,
then $g - h$ is constant on I;
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An expression $F(v)$ is called an **antiderivative of** $f(v)$ **w.r.t.** v if, $\forall v \in \mathbb{R}$, we have: $F'(v) = f(v)$.

Notation: The *set* of *all* antiderivatives of $f(v)$ w.r.t. v is denoted $\boxed{\int f(v)\,dv}$.

e.g.: $\int v^2\,dv = \frac{1}{3}v^3 + C$

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Note: $\frac{1}{3}v^3 + C = \frac{1}{3}v^3 - 6C$

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$$v :\to t$$

EQUALITY OF DERIVATIVES:

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

If $g'(t) = h'(t)$, for all t in an interval I,
then $g - h$ is constant on I;
that is, $\exists c \in \mathbb{R}$ s.t., $\forall t \in (a,b)$,
$g(t) = (h(t)) + c$.

Def'n: Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
An expression $F(t)$ is called an **antiderivative of** $f(t)$ **w.r.t.** t if, $\forall t \in \mathbb{R}$, we have: $F'(t) = f(t)$.

Notation: The *set* of *all* antiderivatives of $f(t)$ w.r.t. t is denoted $\boxed{\int f(t)\,dt}$.

Traditional to drop the set braces and everything after the vertical line (|)

e.g.: $\int t^2\,dt \;=\; \frac{1}{3}t^3 + C$

Note: $\frac{1}{3}t^3 + C = \frac{1}{3}t^3 - 6C$

Looks strange, but it's correct, as sets.

$t :\to s$

EQUALITY OF DERIVATIVES:

If $g'(s) = h'(s)$, for all s in an interval I,
then $g - h$ is constant on I;
that is, $\exists c \in \mathbb{R}$ s.t., $\forall s \in (a, b)$,
$g(s) = (h(s)) + c$.

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Def'n: Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
An expression $F(s)$ is called an **antiderivative of** $f(s)$ **w.r.t.** s if, $\forall s \in \mathbb{R}$, we have: $F'(s) = f(s)$.

Notation: The *set* of *all* antiderivatives of $f(s)$ w.r.t. s
is denoted $\boxed{\int f(s)\, ds}$.

e.g.: $\int s^2\, ds \; = \; \frac{1}{3}s^3 + C$

Traditional to drop the set braces and everything after the vertical line (|)

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expressions:→functions

EQUALITY OF DERIVATIVES:

If $g' = h'$ on an interval I,

then $g - h$ is constant on I;

that is, $\exists c \in \mathbb{R}$ s.t., on I,

$g = h + c$.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Def'n: Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

A function F is called an **antiderivative of** f

if: $F' = f$ on $\mathbb{R}$.

Notation: The *set* of *all* antiderivatives of f

is denoted $\boxed{\int f}$.

e.g.: $\int (\bullet)^2 = \frac{1}{3}(\bullet)^3 + C$

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Note: $\frac{1}{3}(\bullet)^3 + C = \frac{1}{3}(\bullet)^3 - 6C$

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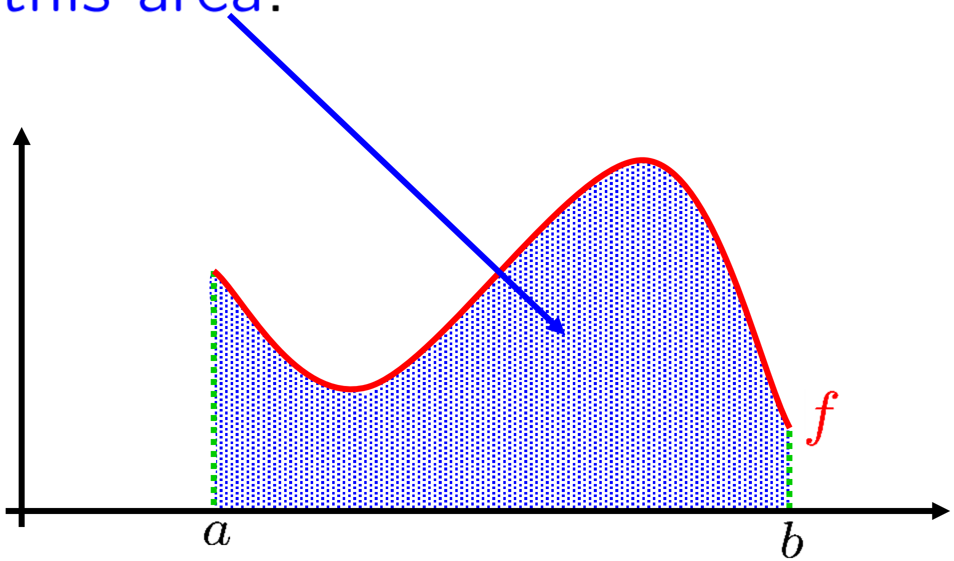
And now, for something completely different. . .or is it?

Next subtopic: Area

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let f be a function. Assume that f is continuous on $[a, b]$. $\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

Goal: Find this area.



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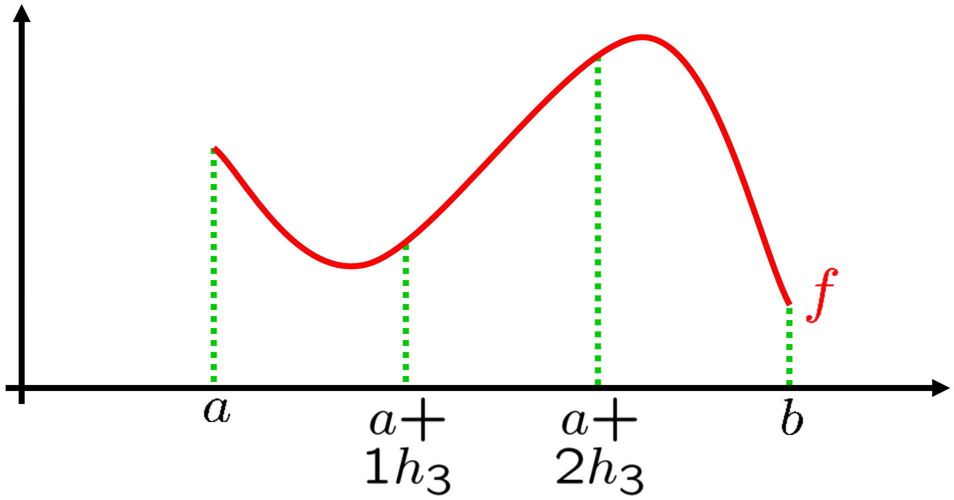
$$h_3 = \frac{b-a}{3}$$



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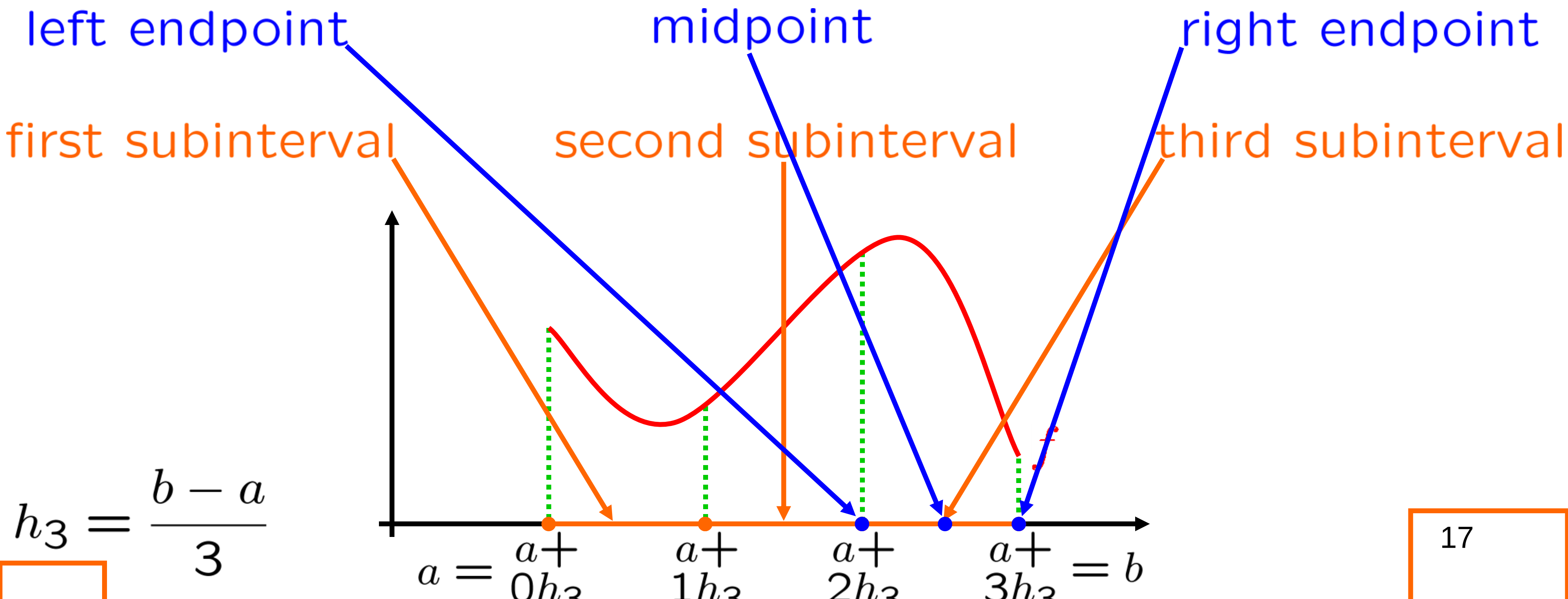
partition of $[a, b]$
into three subintervals
all of length h_3



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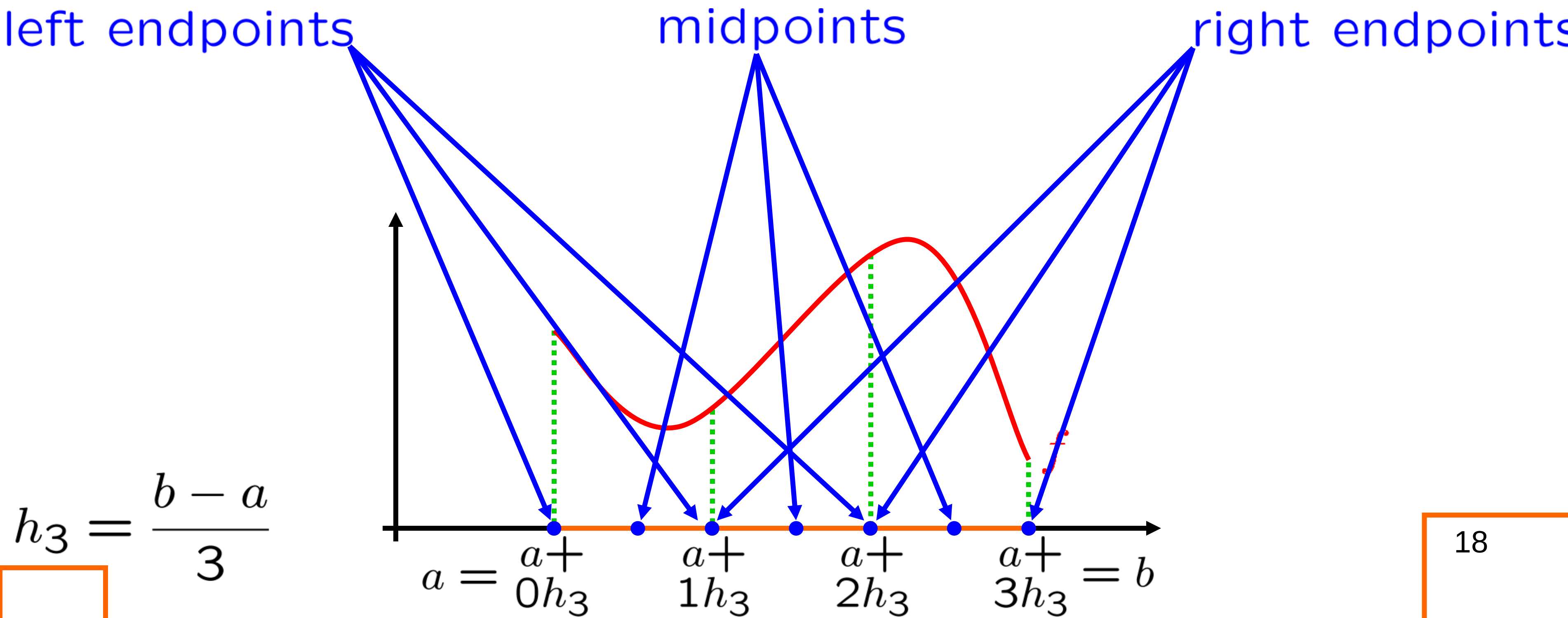

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Next: 10th partition of $[a, b]$...

3rd partition of $[a, b]$

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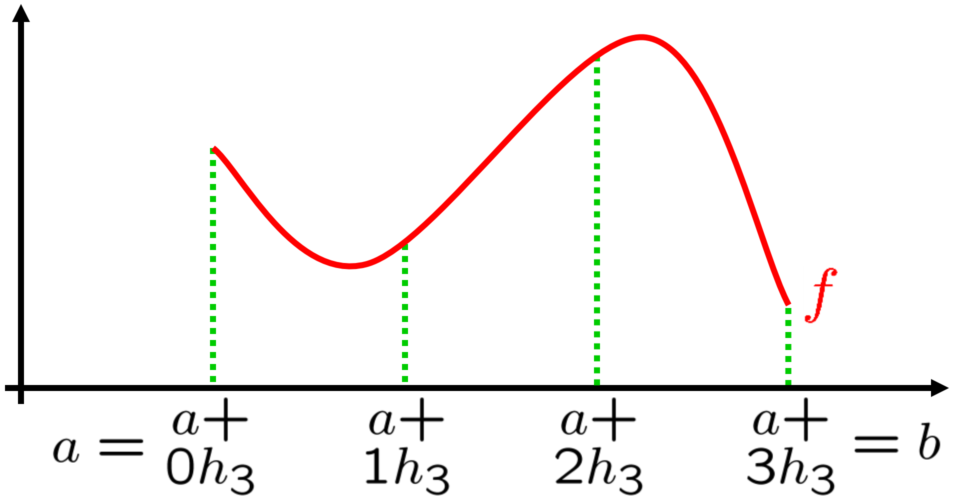

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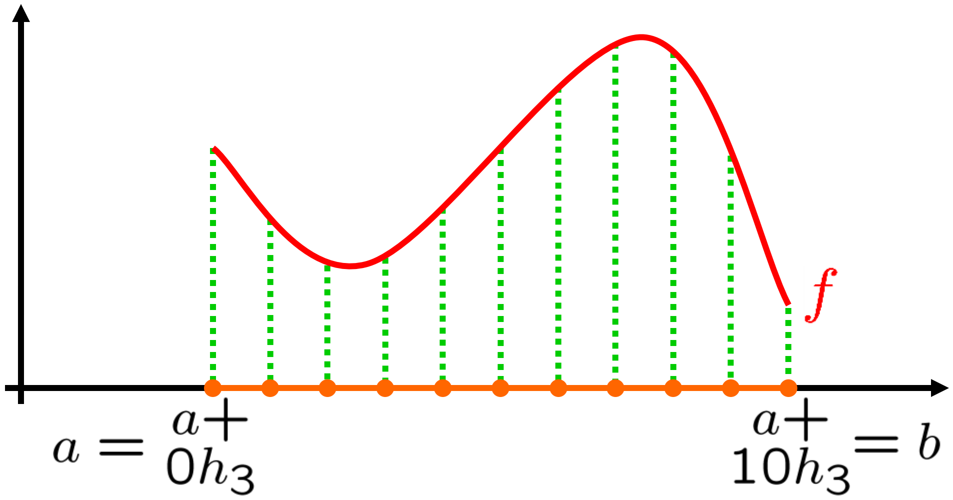
width of the subintervals in the nth partition

WARNING: h is for "horizontal", not "height"

10th partition of $[a, b]$

width of the subintervals in the 10th partition

$$h_{10} = \frac{b-a}{10}$$



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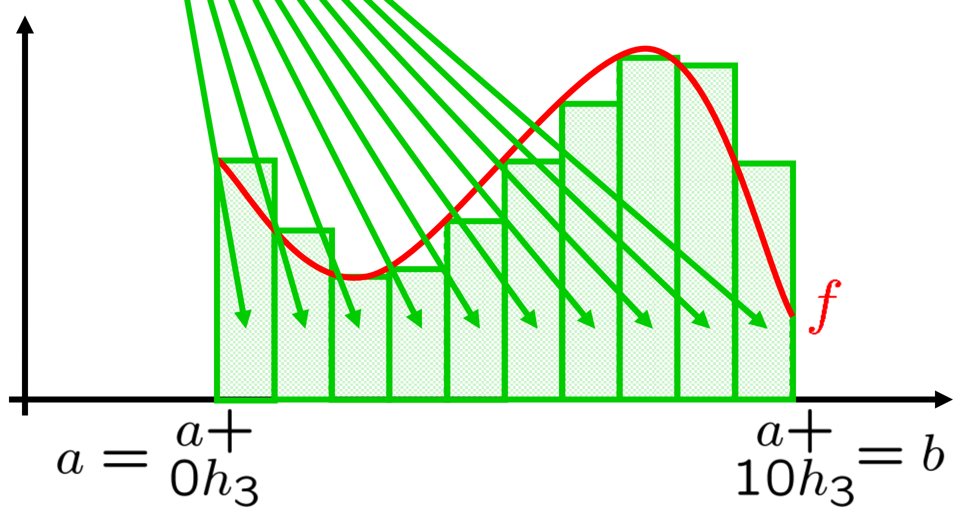
Back to the 3rd partition. . .

These rectangles have width h_{10}, not height.

10th partition of $[a, b]$

width of the subintervals in the 10th partition

$$h_{10} = \frac{b - a}{10}$$



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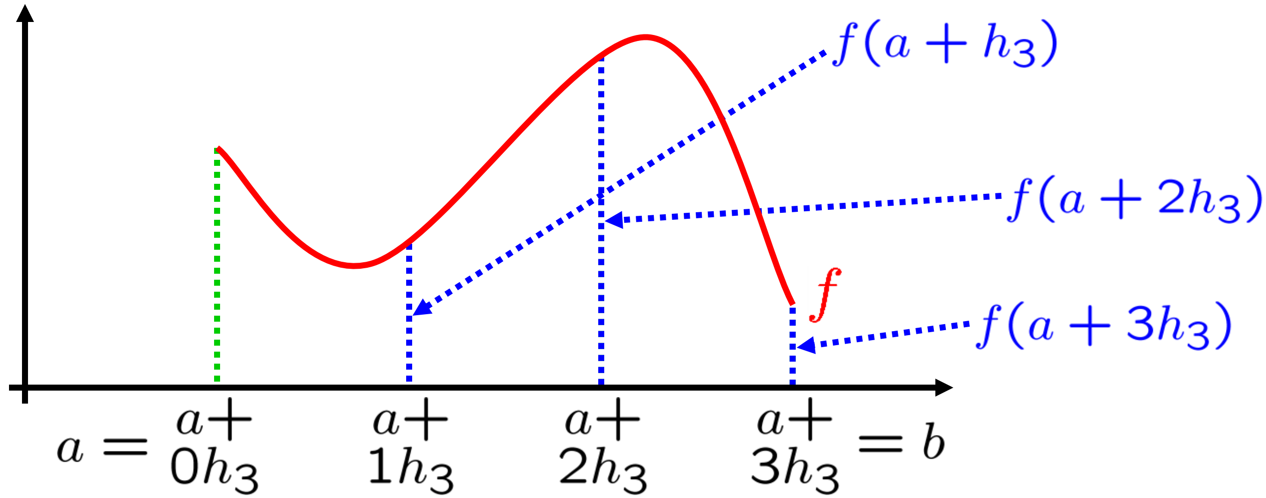


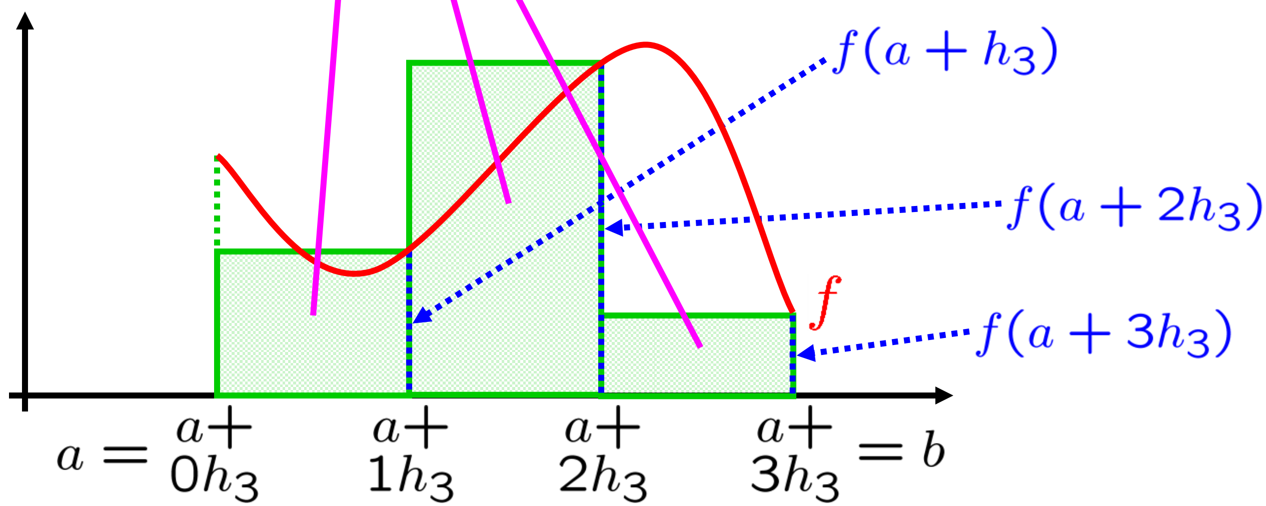
Values of f at the right endpts

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
Let f be a function. Assume that f is continuous on $[a, b]$.
$\forall$integers $n \geq 1$, let $h_n := (b-a)/n$,

let $\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$,

Right 3rd Riemann Sum from a to b of f

$$R_3 S_a^b f = \text{total shaded area} = \left\{ \begin{matrix} [h_3][f(a+h_3)] \\ + \\ [h_3][f(a+2h_3)] \\ + \\ [h_3][f(a+3h_3)] \end{matrix} \right\} = \sum_{j=1}^{3} [h_3][f(a+jh_3)]$$



Values of f at the right endpts

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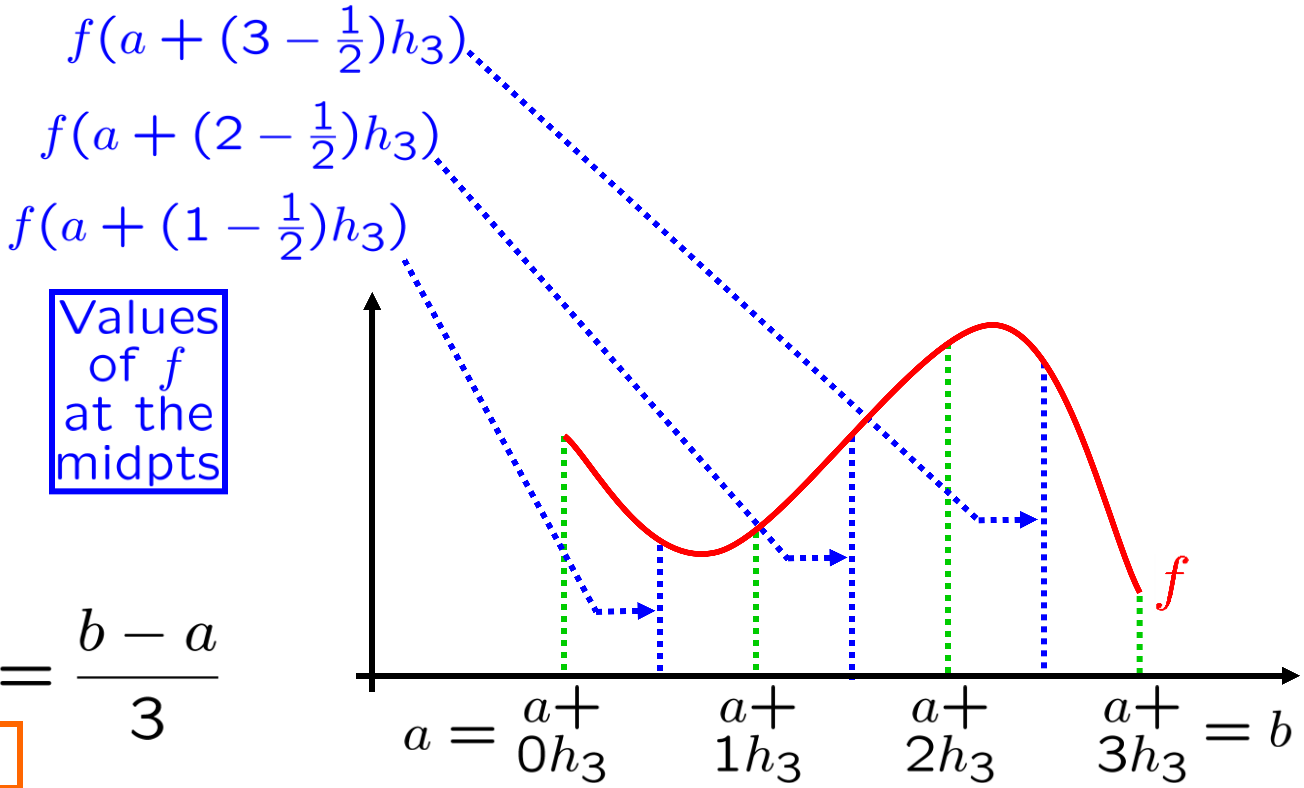

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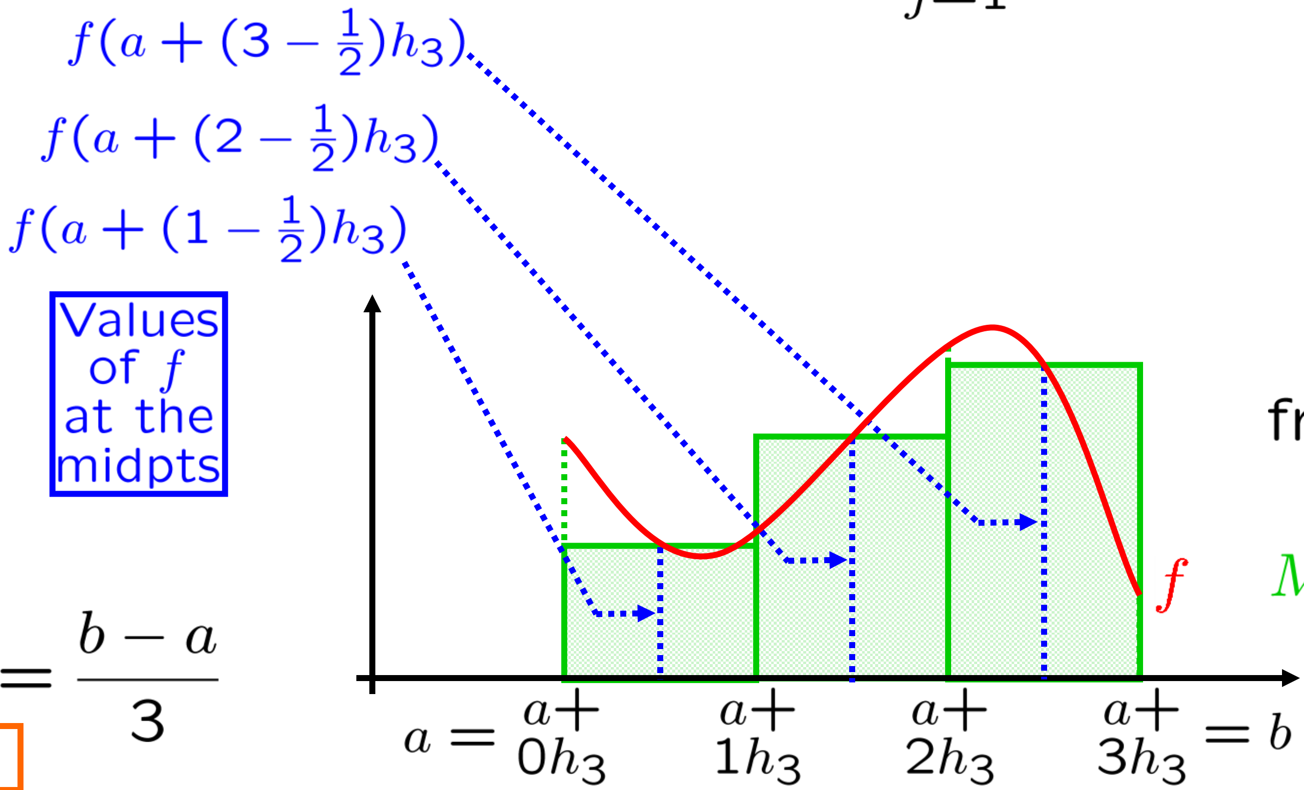

$h_3 = \dfrac{b - a}{3}$

*M*idpoint 3rd Riemann *S*um from a to b of f

$M_3 S_a^b f =$ total shaded area

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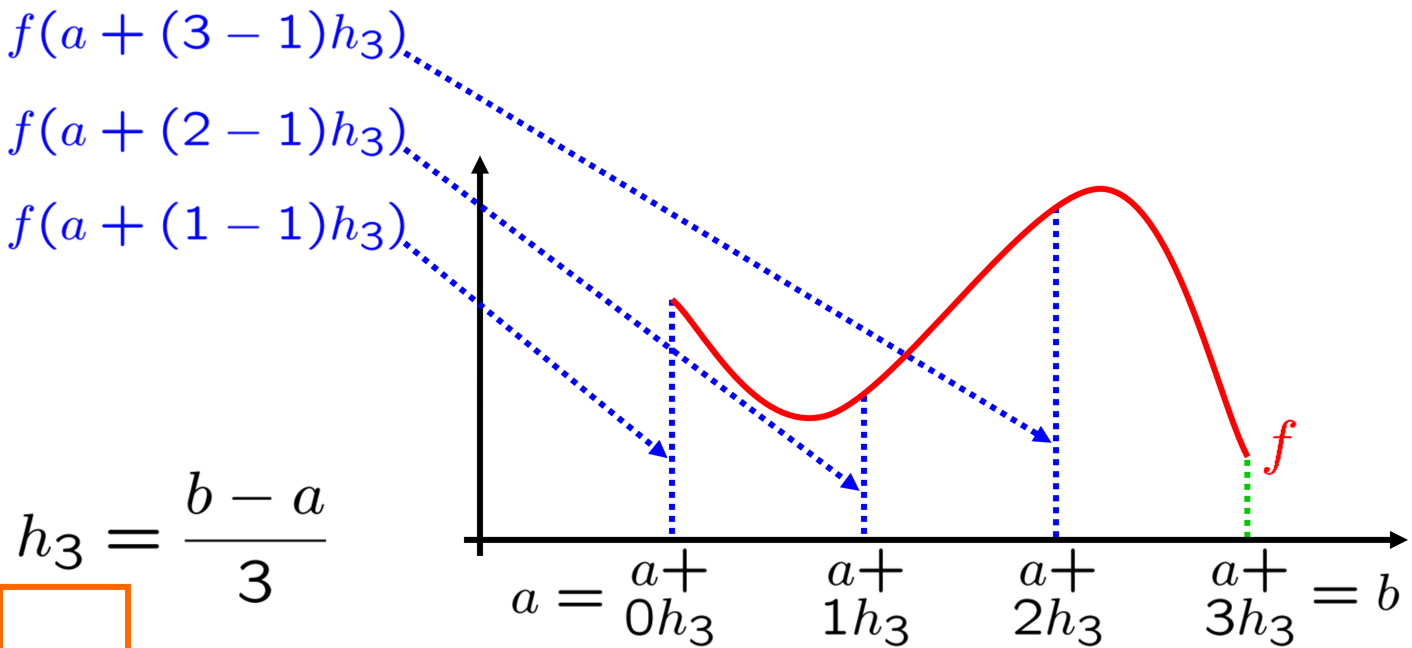
Values of f at the left endpts

$$\text{let } \boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$



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let $\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \frac{1}{2})h_n)]$

and let $\boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - 1)h_n)]$.

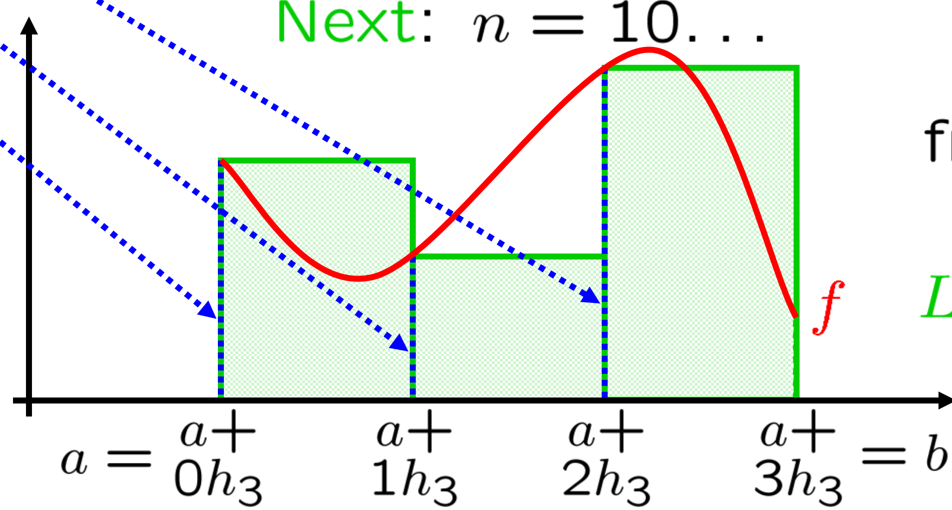
Values of f at the left endpts

$f(a + (3 - 1)h_3)$

$f(a + (2 - 1)h_3)$

$f(a + (1 - 1)h_3)$

Next: $n = 10\ldots$



Left 3rd Riemann Sum from a to b of f

$L_3 S_a^b f =$ total shaded area

$$h_3 = \frac{b - a}{3}$$

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let f be a function. Assume that f is continuous on $[a, b]$. $\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

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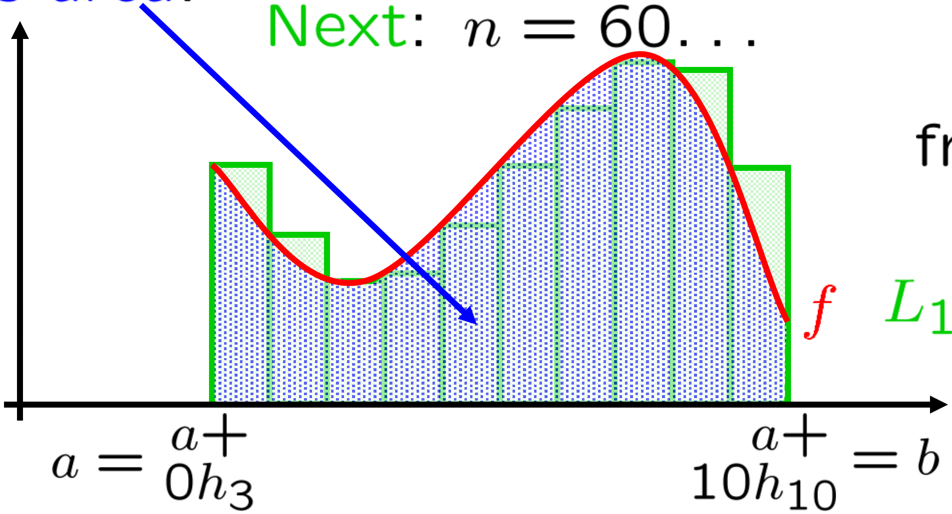
Goal: Find this area.

Next: $n = 60\ldots$



$h_{10} = \dfrac{b - a}{10}$

Left 10th Riemann Sum from a to b of f

$L_{10} S_a^b f$ = total shaded area

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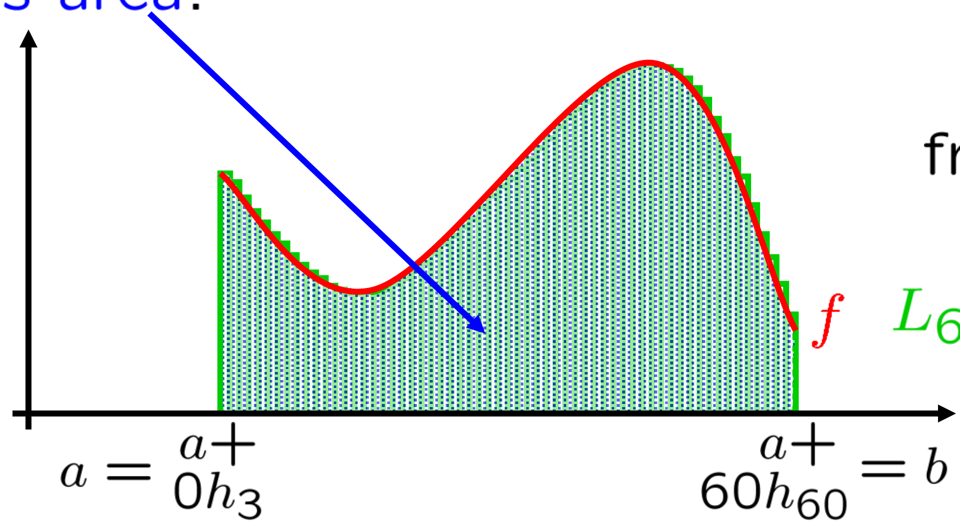
$$\text{and let } \boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - 1)h_n)].$$

Goal: Find this area.

Take $\lim_{n \to \infty} \ldots$



Left 60th Riemann Sum from a to b of f

$L_{60} S_a^b f$ = total shaded area

$$h_{60} = \frac{b - a}{60}$$

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$. Let f be a function. Assume that f is continuous on $[a, b]$. $\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

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Theorem:

$$\lim_{n \to \infty} L_n S_a^b f = \lim_{n \to \infty} M_n S_a^b f = \lim_{n \to \infty} R_n S_a^b f$$

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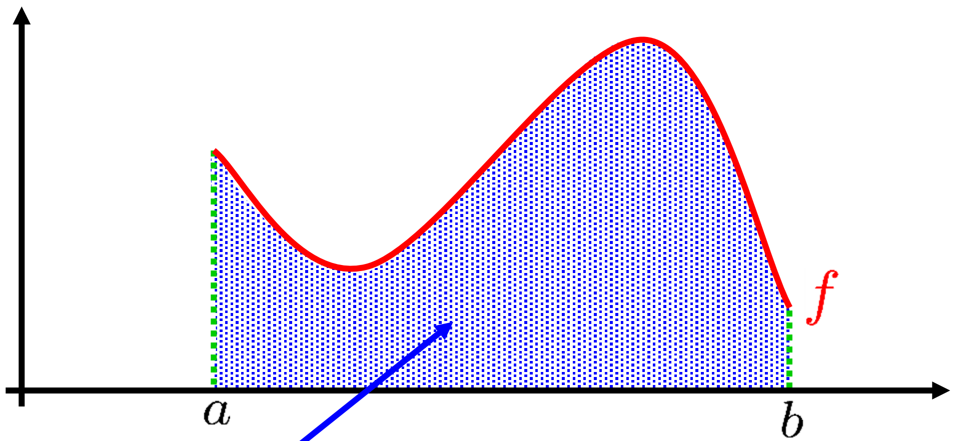
DEFINITION OF A DEFINITE INTEGRAL:

$$\boxed{\int_a^b f(x)\,dx} := \lim_{n \to \infty} L_n S_a^b f = \lim_{n \to \infty} M_n S_a^b f = \lim_{n \to \infty} R_n S_a^b f$$

$$\int_a^b f(x)\,dx = \int_a^b f(v)\,dv = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds = \int_a^b f$$



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Other limits yield the area...

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Let f be a function. Assume that f is continuous on $[a, b]$.
$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

let $p_n^{(1)} \in [a, a + h_n]$, $p_n^{(2)} \in [a + h_n, a + 2h_n]$,
$p_n^{(3)} \in [a + 2h_n, a + 3h_n], \ldots, p_n^{(n)} \in [a + (n - 1)h_n, b]$

and let $\mathrm{RS}_n := \sum_{j=1}^{n} [h_n][f(p_n^{(j)})]$.

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathrm{RS}_n.$$

REMARK: This kind of sum is a **Riemann sum** of f.

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Subintervals can have varying lengths. . .

THEOREM: Let $a, b \in \mathbb{R}$ satisfy $a < b$.

Let f be a function. Assume that f is continuous on $[a, b]$.

$\forall$integers $n \geq 1$, let $k_n \geq 1$ be an integer,

"nth partition"

let $a = x_n^{(0)} < \cdots < x_n^{(k_n)} = b$,

"points in subintervals in the nth partition"

let $p_n^{(1)} \in [x_n^{(0)}, x_n^{(1)}], \ldots, p_n^{(k_n)} \in [x_n^{(k_n-1)}, x_n^{(k_n)}]$,

"mesh of the nth partition"

let $\mu_n := \max\{ x_n^{(1)} - x_n^{(0)}, \ldots, x_n^{(k_n)} - x_n^{(k_n-1)} \}$

and let $\mathrm{RS}_n := \sum_{j=1}^{k_n} [x_n^{(j)} - x_n^{(j-1)}][f(p_n^{(j)})]$.

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Assume $\lim_{n\to\infty} \mu_n = 0$. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathrm{RS}_n$.

STOP