Financial Mathematics

The Fundamental Theorem of Calculus

Notation: The *set* of *all* antiderivatives of $f(x)$ w.r.t. x is denoted $\int f(x)\,dx$.

And now, for something completely different. . .or is it?
Next subtopic: Area

Connecting antidifferentiation to area:
The Fundamental Theorem of Calculus

The idea: The derivative of position is velocity,
So, position is an antiderivative of velocity.

antiderivative of velocity

We'll connect change in position
to the area under the graph of velocity. . .

Motion along a line:

3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$

midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Split in 3.

Want: position $p(t)$ at time t.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$v(6) = 36$, v(8)=64, v(10)=100

$[p(11)] - [p(9)] \approx [2][100]$

$[p(9)] - [p(7)] \approx [2][64]$

$[p(7)] - [p(5)] \approx [2][36]$

Between time 5 and time 7,
velocity ≈ 36

"Estimate velocity using the midpoint time."

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ADD
$$\begin{aligned}
[p(11)] - [p(9)] &\approx [2][100] \\
[p(9)] - [p(7)] &\approx [2]\ [64] \\
[p(7)] - [p(5)] &\approx [2]\ [36]
\end{aligned}$$

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$$\text{ADD}\left\{\begin{array}{lr} [p(11)] - \cancel{[p(9)]} & \approx [2][100] \\ \cancel{[p(9)]} - \cancel{[p(7)]} & \approx [2]\ [64] \\ \cancel{[p(7)]} - [p(5)] & \approx [2]\ [36] \end{array}\right.$$

$$[p(11)] \qquad - [p(5)] \approx [2][200]$$

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100	100
64	64
36	36

$[p(11)] - [p(5)] \approx [2][200] - [p(5)] \approx [2][200]$

Related Q: Compute $M_3 S_5^{11} v$.

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$$[p(11)]-[p(5)] \approx [2][200]$$

Related Q: Compute $M_3 S_5^{11} v$.

$$h_3 = \frac{11-5}{3} = 2$$

$a = 5$ $\quad a + h_3 = 7$ $\quad a + 2h_3 = 9$ $\quad a + 3h_3 = 11$

Midpoints: 6 8 10

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$$M_3 S_5^{11} v = [2][v(6)] + [2][v(8)] + [2][v(10)]$$

$$= [2][36] + [2][64] + [2][100]$$

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$$\begin{array}{r} 100 \\ 64 \\ \underline{36} \end{array}$$

$$[p(11)] - [p(5)] \approx [2][200] = M_3 S_5^{11} v$$

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$$[p(11)] - [p(5)] \approx [2][200] = M_3 S_5^{11} v$$

We'll connect change in position to
the area under the graph of velocity...

error $\to 0$, as $n \to \infty$

$$[p(11)] - [p(5)] \approx M_n S_5^{11} v$$

$$[p(11)] - [p(5)] = \lim_{n \to \infty} M_n S_5^{11} v$$

HARD TO CALCULATE, BUT...

$$= \int_5^{11} v(t)\, dt = \int_5^{11} t^2\, dt$$

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e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$p'(t) = v(t)$, *i.e.*, $p(t)$ is an antiderivative of t^2 w.r.t. t.

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

$$p(t) = (t^3/3) + C,$$

for some C

$$[(11^3/3) - \cancel{C}] - [(5^3/3) - \cancel{C}] = [11^3/3] - [5^3/3] \blacksquare$$

$$\|$$

$$[p(11)] - [p(5)] = \lim_{n\to\infty} M_n S_5^{11} v$$

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$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3] = [t^3/3]_{t:\to 5}^{t:\to 11}$$

$$= [11^3/3] - [5^3/3]$$

$$\int_5^{11} t^2\,dt$$

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$$= [(t^3/3) + C]_{t:\to 5}^{t:\to 11}$$

Key idea: To compute a definite integral,
find an antiderivative,
then evaluate at limits of integration,
then subtract.

THE FUNDAMENTAL THEOREM OF CALCULUS, DEFINITE INTEGRALS

Let v be any function, contin. on $[a,b]$.

Let V be an antiderivative of v on $[a,b]$.

$$\text{Then} \quad \int_a^b v(t)\,dt = [V(t)]_{t:\to a}^{t:\to b} = [V(b)] - [V(a)].$$

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

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Then $\displaystyle\int_a^b v(t)\,dt = [V(t)]_{t:\to a}^{t:\to b} = [V(b)] - [V(a)]$.

There's another version of this th'm, in which we integrate to a variable, then differentiate w.r.t. it.

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3]$$

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$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3]$$

e.g.: $\displaystyle\int_5^x t^2\,dt = [x^3/3] - [5^3/3]$

$$\frac{d}{dx}\int_5^x t^2\,dt = \frac{d}{dx}\Big([x^3/3] - [5^3/3]\Big) = x^2 = [t^2]_{t:\to x}$$

Key idea: To compute a definite integral,
find an antiderivative,
then evaluate at limits of integration,
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THE FUNDAMENTAL THEOREM OF CALCULUS, DEFINITE INTEGRALS

Let v be any function, contin. on $[a,b]$. $v :\longrightarrow f$, $V :\longrightarrow F$,

Let V be an antiderivative of v on $[a,b]$. $t :\longrightarrow x$

Then $$\int_a^b v(t)\,dt = [V(t)]_{t:\longrightarrow a}^{t:\longrightarrow b} = [V(b)] - [V(a)].$$

THE FUNDAMENTAL THEOREM OF CALCULUS, ANTIDERIVATIVES

If v is continuous on $[a,b]$, $v :\longrightarrow f$

then $$\frac{d}{dx}\int_a^x v(t)\,dt = [v(t)]_{t:\longrightarrow x} = v(x), \text{ for } x \in (a,b).$$

$$\frac{d}{dx}\int_5^x t^2\,dt = \frac{d}{dx}\Big([x^3/3] - [5^3/3]\Big) = x^2 = [t^2]_{t:\longrightarrow x}$$

THE FUNDAMENTAL THEOREM OF CALCULUS, DEFINITE INTEGRALS

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

THE FUNDAMENTAL THEOREM OF CALCULUS, ANTIDERIVATIVES

If f is continuous on $[a,b]$,

then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

Don't change t to x.

WARNING: $\displaystyle\int_a^x f$ is acceptable,

but $\displaystyle\int_a^x f(x)\,dx$ is not.

Don't use the same variable here and here.

THE FUNDAMENTAL THEOREM OF CALCULUS, DEFINITE INTEGRALS

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

An easier way to show $\int_0^1 x^2\,dx = \frac{1}{3}$

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_{x:\to 0}^{x:\to 1} = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \blacksquare$$

STOP