Financial Mathematics

Conditional convergence of series

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = \infty$$

$$-1 - 1 - 1 - \cdots = -\infty$$

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$\underbrace{1 + (1/2) + (1/4) + (1/8) + \cdots}_{\displaystyle \| \atop \displaystyle \sum_{n=0}^{\infty} 2^{-n}} = 2$$

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} +$$

$$\tfrac{1}{8} + \tfrac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} +$$

$$\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} +$$

$$\frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}$$

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

The harmonic series

Goal:

Lower bound on 31st partial sum

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} +$$

$$\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} +$$

$$\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + +$$

$$\frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}$$

$$\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} +$$

$$\frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}$$

group terms

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$\boxed{1} + \boxed{\frac{1}{2} + \frac{1}{3}} + \boxed{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} +$$

$$\boxed{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}} +$$

$$\boxed{\begin{array}{l}\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \\ \quad \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}\end{array}}$$

group terms

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

black $\vee$ blue

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) +$$

$$\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) +$$

$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

$$\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) +$$

$$\left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) +$$

$\frac{8}{16}$

$$\left(\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}\right)$$

$$\left(\frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32}\right)$$

$\frac{16}{32}$

group terms

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

black $\vee$ blue

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) +$$

$$\boxed{\frac{1}{2}} + \boxed{\frac{2}{4}} + \boxed{\frac{4}{8}} +$$

$$\frac{1}{2} \quad \frac{2}{4} \quad \frac{4}{8}$$

$$\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) +$$

$$\boxed{\frac{8}{16}} +$$

$$\frac{8}{16}$$

$$\left(\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}\right)$$

$$\boxed{\frac{16}{32}}$$

$$\frac{16}{32}$$

ALL ARE $\frac{1}{2}$

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) +$$

$$\boxed{\frac{1}{2}} + \boxed{\frac{1}{2}} + \boxed{\frac{1}{2}} +$$

black $\vee$ blue

$$\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) +$$

$$\boxed{\frac{1}{2}} +$$

$$\left(\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}\right)$$

$$\boxed{\frac{1}{2}}$$

$$= \frac{5}{2}$$

ALL ARE $\frac{1}{2}$

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) +$$

black $\vee$ blue

$$\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) +$$

$$\left(\frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31}\right)$$

$$> \frac{5}{2}$$

Goal: ☺

Lower bound on 31st partial sum

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

there is a partial sum that is $> \frac{5}{2}$
(31st)

$> \frac{5}{2}$

The harmonic series

$$1 + (1/2) + (1/3) + (1/4) + \cdots = ??$$

there is a partial sum that is $> \frac{5}{2}$
(31st)

there is a partial sum that is $> \frac{6}{2}$
(63rd)

there is a partial sum that is $> \frac{7}{2}$
(127th)

etc.

The partial sums are unbounded.

$$1 + (1/2) + (1/3) + \cdots = \infty$$

The alternating harmonic series

$$1 - (1/2) + (1/3) - (1/4) + \cdots = ??2$$

$\forall x \in (-1, 1]$,

$$\ln(1 + x) =$$

$$1 - (x/2) + (x^2/3) - (x^3/4) + \cdots$$

$$\ln(1 + 1) =$$

$$1 - (1/2) + (1^2/3) - (1^3/4) + \cdots$$

$$1 - (1/2) + (1/3) - (1/4) + \cdots = \ln 2$$

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots = \ln 2$$

$$1 - (1/2) + (1/3) - (1/4) + \cdots = \ln 2$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots=\frac{2}{2}\ln 2$$

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots=\frac{1}{2}\ln 2$$

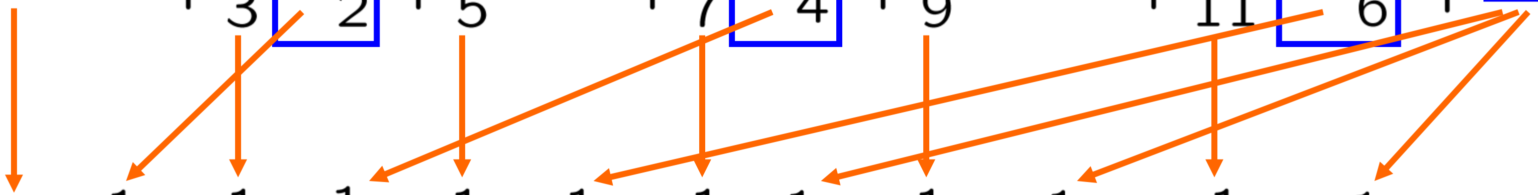
$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\ln 2$$

ADD

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$\frac{1}{2} \quad -\frac{1}{4} \quad +\frac{1}{6} \quad -\frac{1}{8} \quad +\frac{1}{10} \quad -\frac{1}{12}+\cdots = \frac{1}{2}\ln 2$$

$$1 \quad +\frac{1}{3}-\frac{1}{2}+\frac{1}{5} \quad +\frac{1}{7}-\frac{1}{4}+\frac{1}{9} \quad +\frac{1}{11}-\frac{1}{6}+\cdots = \frac{3}{2}\ln 2$$



$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

REARRANGE

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots=\frac{2}{2}\ln 2$$

$$\frac{1}{2}\qquad-\frac{1}{4}\qquad+\frac{1}{6}\qquad-\frac{1}{8}\qquad+\frac{1}{10}\qquad-\frac{1}{12}+\cdots=\frac{1}{2}\ln 2$$

$$1\qquad+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\qquad+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}\qquad+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\ln 2$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots=\ln 2$$

equal

The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

converges to $\ln 2$, but it has a rearrangement

$$1\qquad+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\qquad+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}\qquad+\frac{1}{11}-\frac{1}{6}+\cdots$$

that converges to $\frac{3}{2}\ln 2$.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots=\frac{2}{2}\ln 2$$

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots=\frac{1}{2}\ln 2$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\ln 2$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots=\ln 2$$

The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

converges to $\ln 2$, but it has a rearrangement

$$\underbrace{1+\frac{1}{3}}_{\text{two pos.}}\underbrace{-\frac{1}{2}}_{\text{one neg.}}\underbrace{+\frac{1}{5}+\frac{1}{7}}_{\text{two pos.}}\underbrace{-\frac{1}{4}}_{\text{one neg.}}\underbrace{+\frac{1}{9}+\frac{1}{11}}_{\text{two pos.}}\underbrace{-\frac{1}{6}}_{\text{one neg.}}+\cdots \quad \text{etc.}$$

that converges to $\frac{3}{2}\ln 2$.

Find sum of negative terms. Find sum of positive terms.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots = \ln 2$$

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \cdots = -\infty$$

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \cdots = +\infty$$

First this!

The partial sums are unbounded.

$$1 + (1/2) + (1/3) + \cdots = \infty$$

The partial sums are unbounded.

$$(1/2) + (1/4) + (1/6) + \cdots = \infty$$

The partial sums are unbounded.

$$-(1/2) - (1/4) - (1/6) - \cdots = -\infty$$

Find sum of positive terms.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2}\quad-\frac{1}{4}\quad-\frac{1}{6}\quad-\frac{1}{8}\quad-\frac{1}{10}\quad-\frac{1}{12}\quad-\cdots = -\infty$$

Partial sums are unbounded.

$$1\quad+\frac{1}{3}\quad+\frac{1}{5}\quad+\frac{1}{7}\quad+\frac{1}{9}\quad+\frac{1}{11}\quad+\cdots = +\infty$$

$$\vee\quad\vee\quad\vee\quad\vee\quad\vee\quad\vee$$

$$\frac{1}{2}\quad+\frac{1}{4}\quad+\frac{1}{6}\quad+\frac{1}{8}\quad+\frac{1}{10}\quad+\frac{1}{12}\quad+\cdots = +\infty$$

Partial sums are unbounded.

The partial sums are unbounded.

$$1+(1/2)+(1/3)+\cdots = \infty$$

The partial sums are unbounded.

$$(1/2)+(1/4)+(1/6)+\cdots = \infty$$

The partial sums are unbounded.

$$-(1/2)-(1/4)-(1/6)-\cdots = -\infty$$

Find sum of positive terms.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad -\cdots = -\infty$$

$$1 \quad +\frac{1}{3} \quad +\frac{1}{5} \quad +\frac{1}{7} \quad +\frac{1}{9} \quad +\frac{1}{11} \quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2}\quad -\frac{1}{4}\quad -\frac{1}{6}\quad -\frac{1}{8}\quad -\frac{1}{10}\quad -\frac{1}{12}\quad -\cdots = -\infty$$

$$1\quad +\frac{1}{3}\quad +\frac{1}{5}\quad +\frac{1}{7}\quad +\frac{1}{9}\quad +\frac{1}{11}\quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

sum to $+\infty$, while the negative terms sum to $-\infty$.

There is a rearrangement that sums to 1000. Proof:

less than

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1} \leq 1000$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1} > 1000 \qquad \left(\frac{1}{2M+1} < 1/2\right)$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2} < 1000$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2} + \frac{1}{2M+3}+\frac{1}{2M+5}+\cdots+\frac{1}{2N+1} > 1000 \qquad \left(\frac{1}{2N+1} < 1/4\right)$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad -\cdots = -\infty$$

$$1 \quad +\frac{1}{3} \quad +\frac{1}{5} \quad +\frac{1}{7} \quad +\frac{1}{9} \quad +\frac{1}{11} \quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

sum to $+\infty$, while the negative terms sum to $-\infty$.

There is a rearrangement that sums to 1000. Proof:

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1} \leq 1000$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1} > 1000$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2} < 1000$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2}$$
$$+\frac{1}{2M+3}+\frac{1}{2M+5}+\cdots+\boxed{\frac{1}{2N+1}}-\frac{1}{4} \quad \text{~~> 1000~~}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad -\cdots = -\infty$$

$$1 \quad +\frac{1}{3} \quad +\frac{1}{5} \quad +\frac{1}{7} \quad +\frac{1}{9} \quad +\frac{1}{11} \quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

sum to $+\infty$, while the negative terms sum to $-\infty$.

There is a rearrangement that sums to 1000. Proof:

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2}$$
$$+\frac{1}{2M+3}+\frac{1}{2M+5}+\cdots+\frac{1}{2N+1}-\frac{1}{4}$$
$$+\cdots$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2}$$
$$+\frac{1}{2M+3}+\frac{1}{2M+5}+\cdots+\frac{1}{2N+1}-\frac{1}{4}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad -\cdots = -\infty$$

$$1 \quad +\frac{1}{3} \quad +\frac{1}{5} \quad +\frac{1}{7} \quad +\frac{1}{9} \quad +\frac{1}{11} \quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

sum to $+\infty$, while the negative terms sum to $-\infty$.

There is a rearrangement that sums to 1000. Proof:

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2M-1}+\frac{1}{2M+1}-\frac{1}{2}$$
$$+\frac{1}{2M+3}+\frac{1}{2M+5}+\cdots+\frac{1}{2N+1}-\frac{1}{4}$$
$$+\cdots-\frac{1}{6}+\cdots-\frac{1}{8}+\cdots = 1000 \quad \text{QED}$$

partial sums → 1000

There is a rearrangement that sums to any desired real number.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

$$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad -\cdots = -\infty$$

$$1 \quad +\frac{1}{3} \quad +\frac{1}{5} \quad +\frac{1}{7} \quad +\frac{1}{9} \quad +\frac{1}{11} \quad +\cdots = +\infty$$

The positive terms in

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots$$

sum to $+\infty$, while the negative terms sum to $-\infty$.

There is a rearrangement that sums to 1000.

There is a rearrangement that sums to any desired real number.

There is a rearrangement that sums to $+\infty$.

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$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots = \ln 2$$

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There is a rearrangement that sums to $-\infty$.
There is a rearrangement that has no sum.

This happens for any series whose positive terms sum to $+\infty$, and whose negative terms sum to $-\infty$.

Def'n: A series $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + \cdots$ is **nonnegative** if: $\forall$integers $j \geq 1$, $a_j \geq 0$.

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Def'n: A **generalized partial sum** of a series is a sum of finitely many terms of the series.

---

e.g.: One generalized partial sum of the series

$$1 + 3 + \underline{5} + 7 + \underline{9} + 11 + 13 + 15 + 17 + \underline{19} + \cdots$$

is $\underline{5} + \underline{9} + \underline{19} = 33$.

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e.g.: One generalized partial sum of the series $1+3+5+7+9+11+13+15+17+19+\cdots$ is $5+9+19=33$.

When $a_1 + a_2 + \cdots$ is a nonnegative series, $a_1 + a_2 + \cdots$ has a sum, and this sum is the supremum of the set of all generalized partial sums of $a_1 + a_2 + a_3 + \cdots$.

pf omitted

(This set is unaffected by rearrangement.)

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$$x_+ := \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

$$x_- := (-x)_+ = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Reproducing equation: $x = x_+ - x_-$

Absolute value equation: $|x| = x_+ + x_-$

Assume that $a_1, a_2, a_3, \ldots \to 0$

Fact: If *both* $\sum (a_j)_+ = \infty$ *and* $\sum (a_j)_- = \infty$, then every extended real number is the sum of some rearrangement of $\sum a_j$.

Fact: If *either* $\sum (a_j)_+ \neq \infty$ *or* $\sum (a_j)_- \neq \infty$, then every rearrangement of $\sum a_j$ has sum equal to $[\sum (a_j)_+] - [\sum (a_j)_-]$.

$$x_+ := \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases} \qquad x_- := (-x)_+ = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Reproducing equation: $x = x_+ - x_-$

Absolute value equation: $|x| = x_+ + x_-$

Summary

There is an approach to summing countable collections of real numbers
(even allowing for repeats),

BUT sometimes we don't get a sum,

AND even when we do,

to get best results (rearrangment-invariant), it's important either that
the positive terms have finite sum,
or that
the negative terms have finite sum.

(No ambiguity occurs if all terms are ≥ 0.)

(No ambiguity occurs if all terms are ≤ 0.)

Similar issues crop up in integration theory...

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

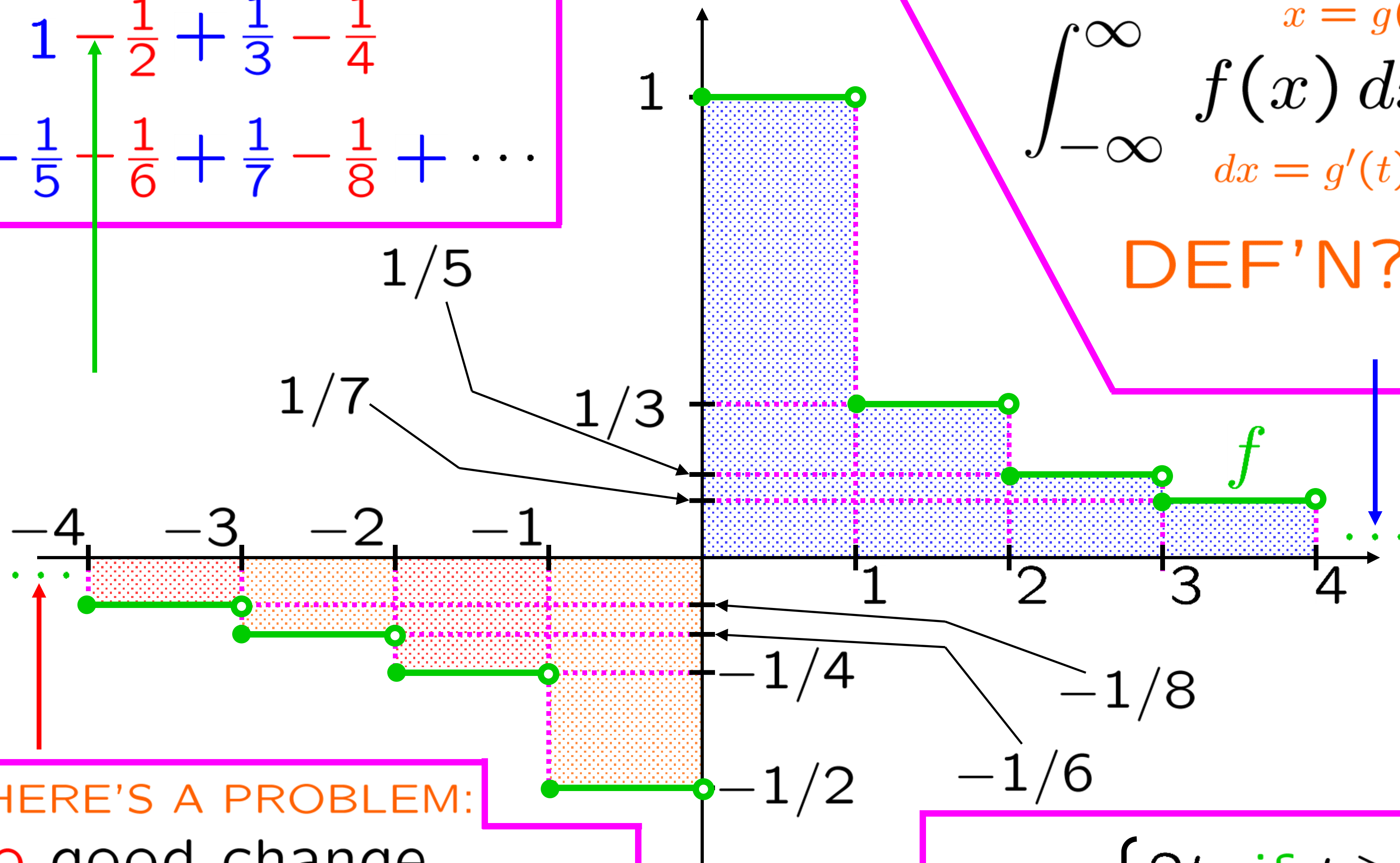
$+ \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$x = g(t)$

$dx = g'(t)\,dt$

DEF'N?



THERE'S A PROBLEM:

No good change of variables formula!

$$\lim_{K\to\infty} \int_{-K}^{K} f(x)\,dx = \ln 2$$

MAYBE?

$$g(t) = \begin{cases} 2t, & \text{if } t \geq 0 \\ t, & \text{if } t \leq 0 \end{cases}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

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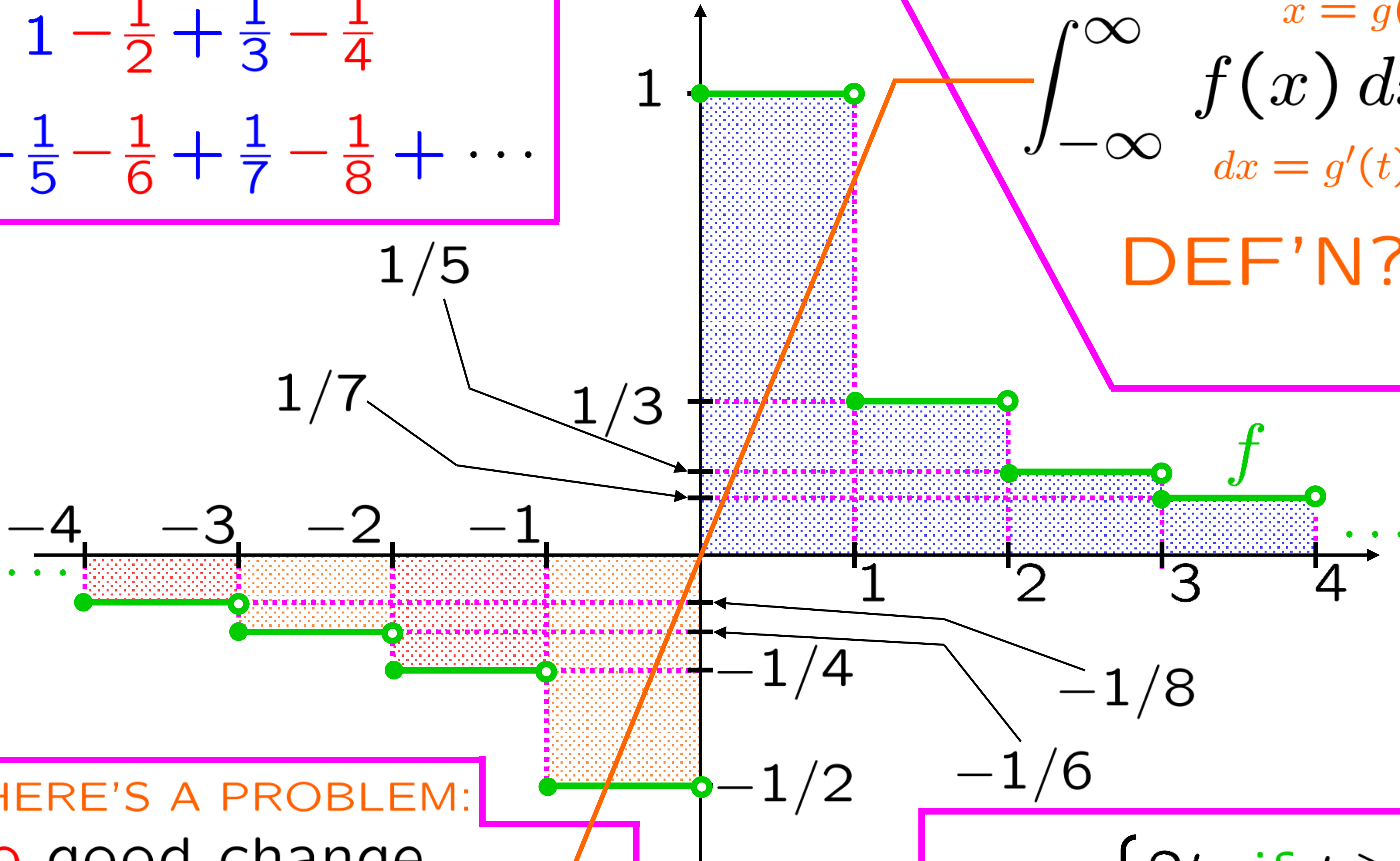

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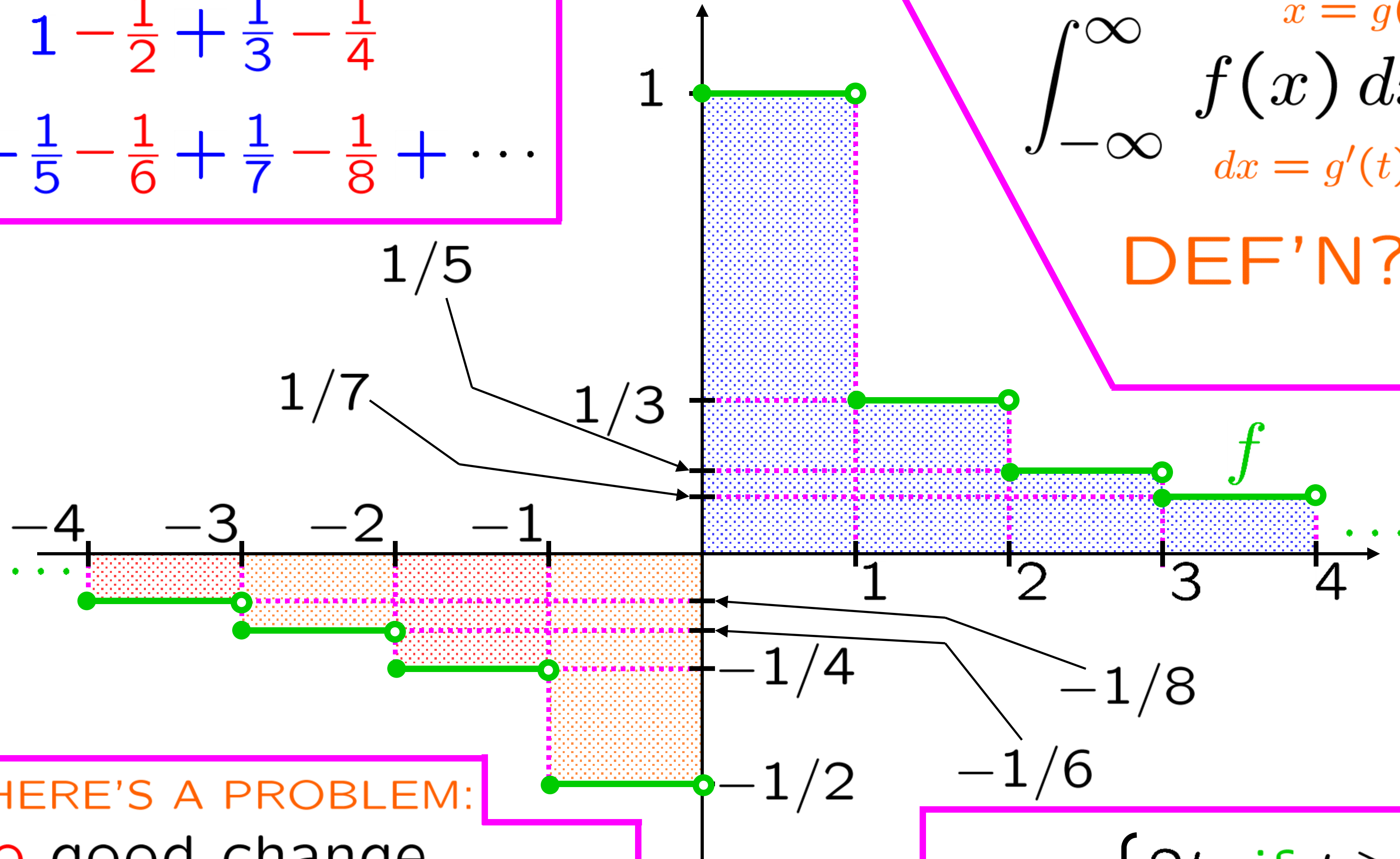

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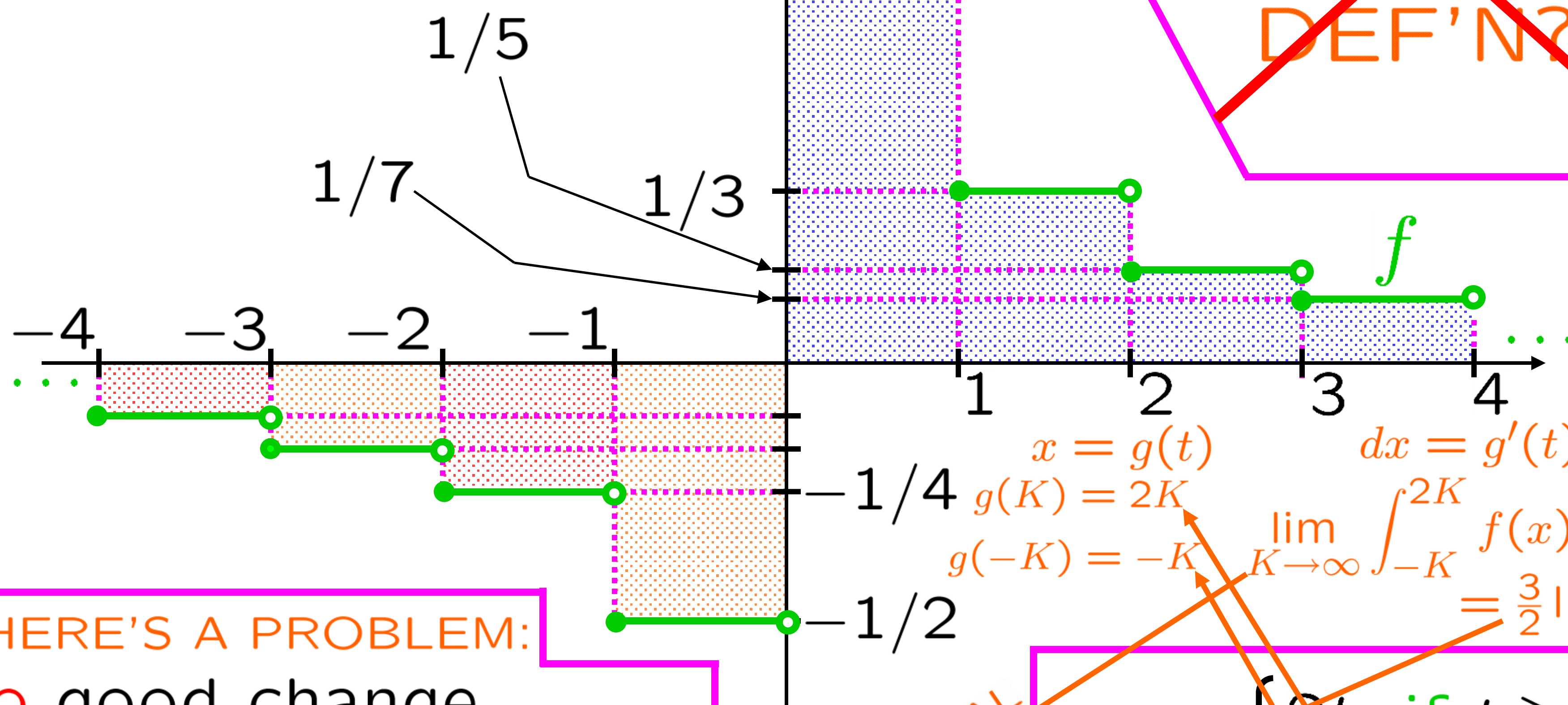
$\int_{-\infty}^{\infty} f(x)\, dx$

DEF'N?



$x = g(t)$ $\quad dx = g'(t)\, dt$

$g(K) = 2K$

$g(-K) = -K$

$\lim_{K\to\infty} \int_{-K}^{2K} f(x)\, dx$
$= \frac{3}{2} \ln 2$

THERE'S A PROBLEM:

No good change of variables formula!

EQUAL

$\lim_{K\to\infty} \int_{-K}^{K} [f(g(t))][g'(t)]\, dt = \frac{3}{2} \ln 2$ MAYBE?

change from t back to x...

K

$-K$

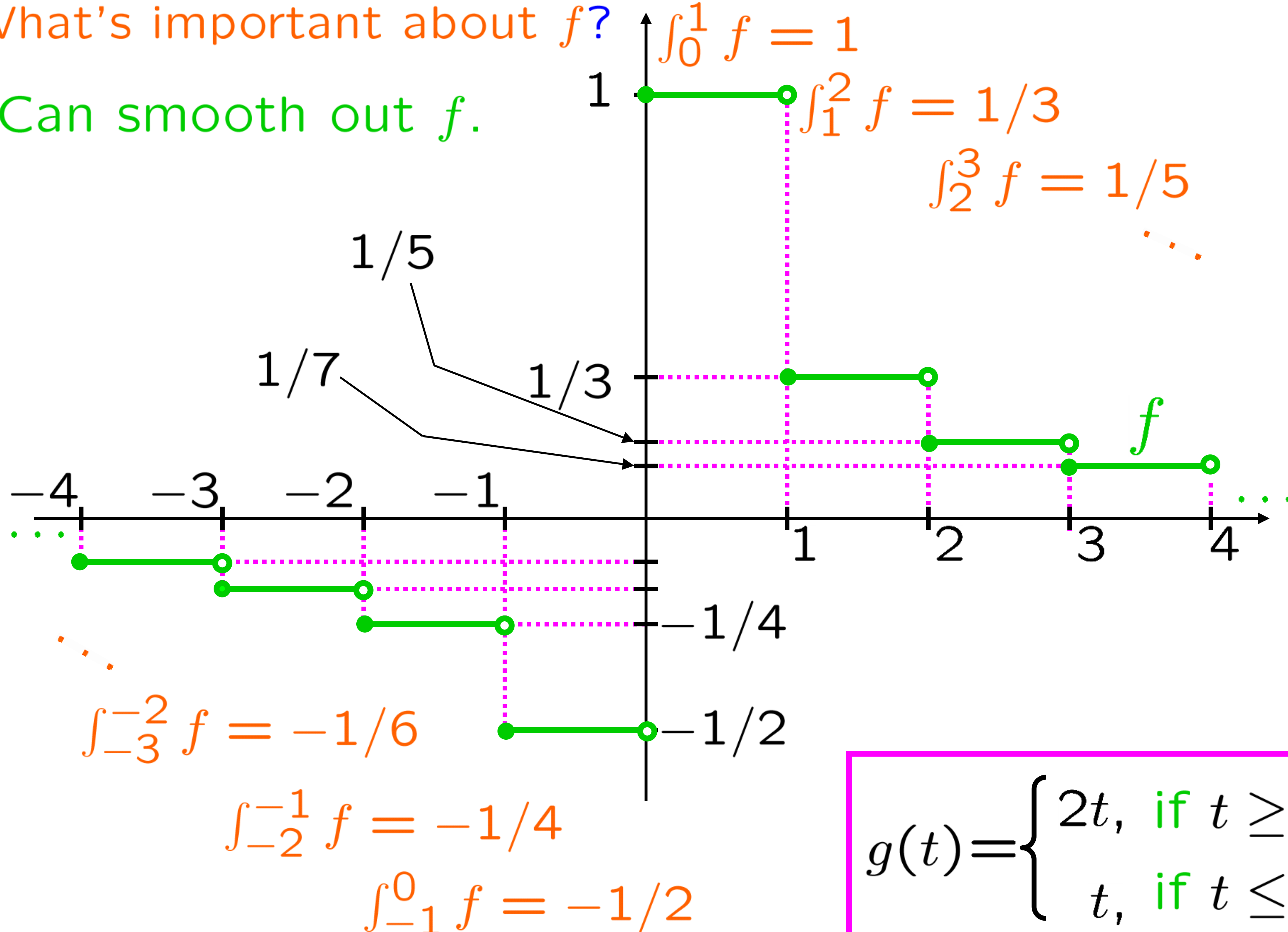
$g(t) = \begin{cases} 2t, & \text{if } t \geq 0 \\ t, & \text{if } t \leq 0 \end{cases}$

K

$-K$

What's important about f?

Can smooth out f.



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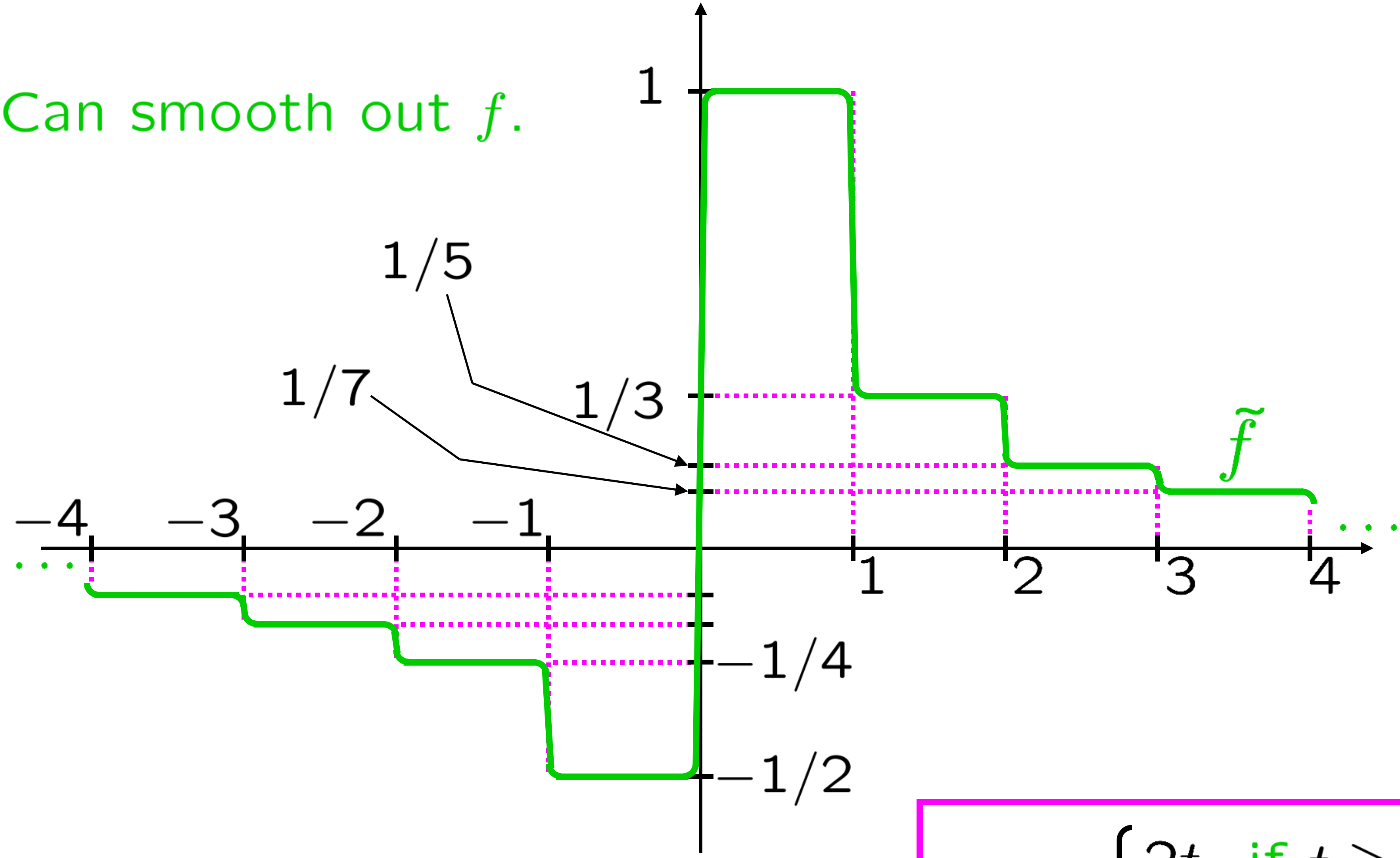

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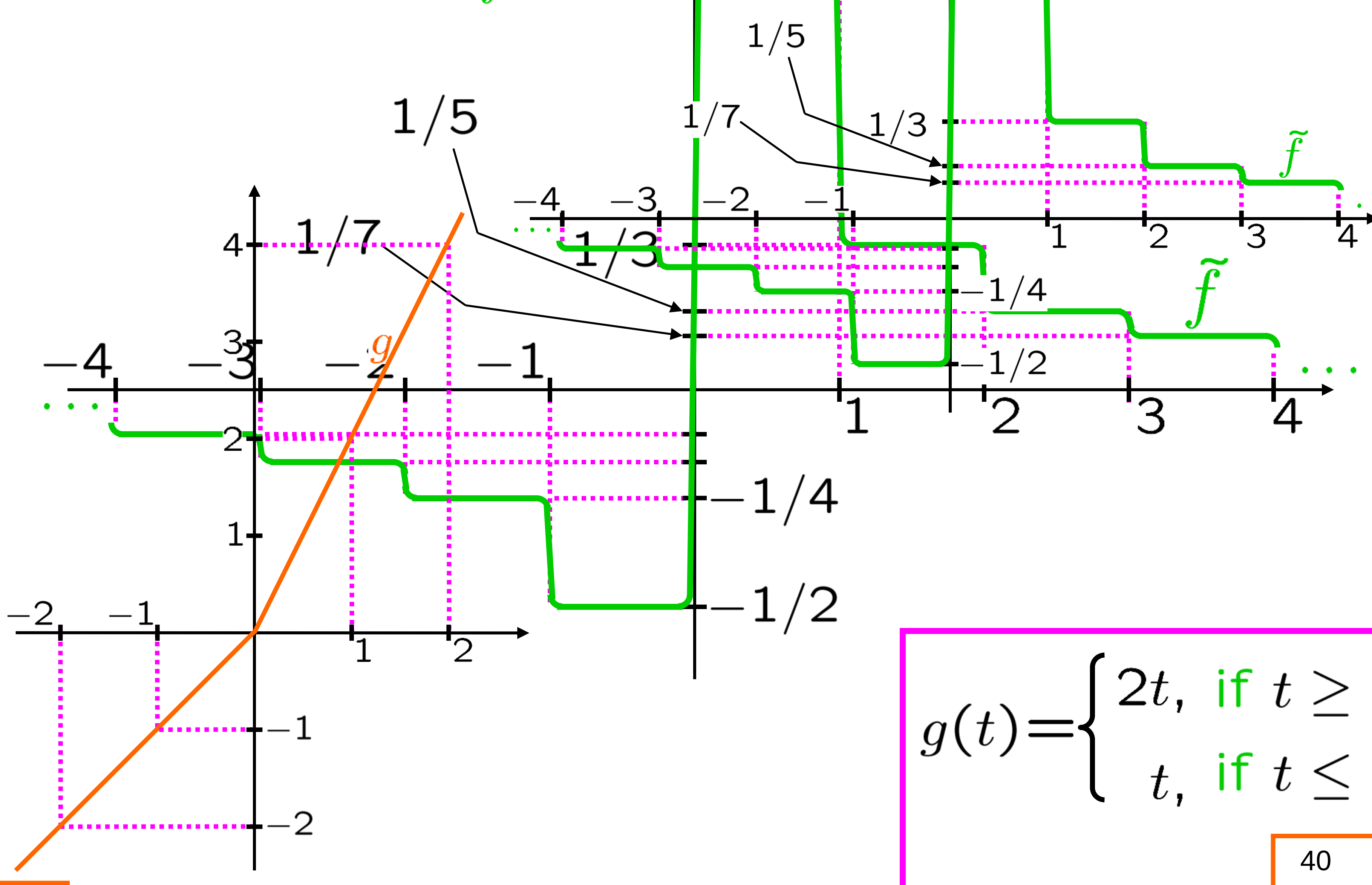

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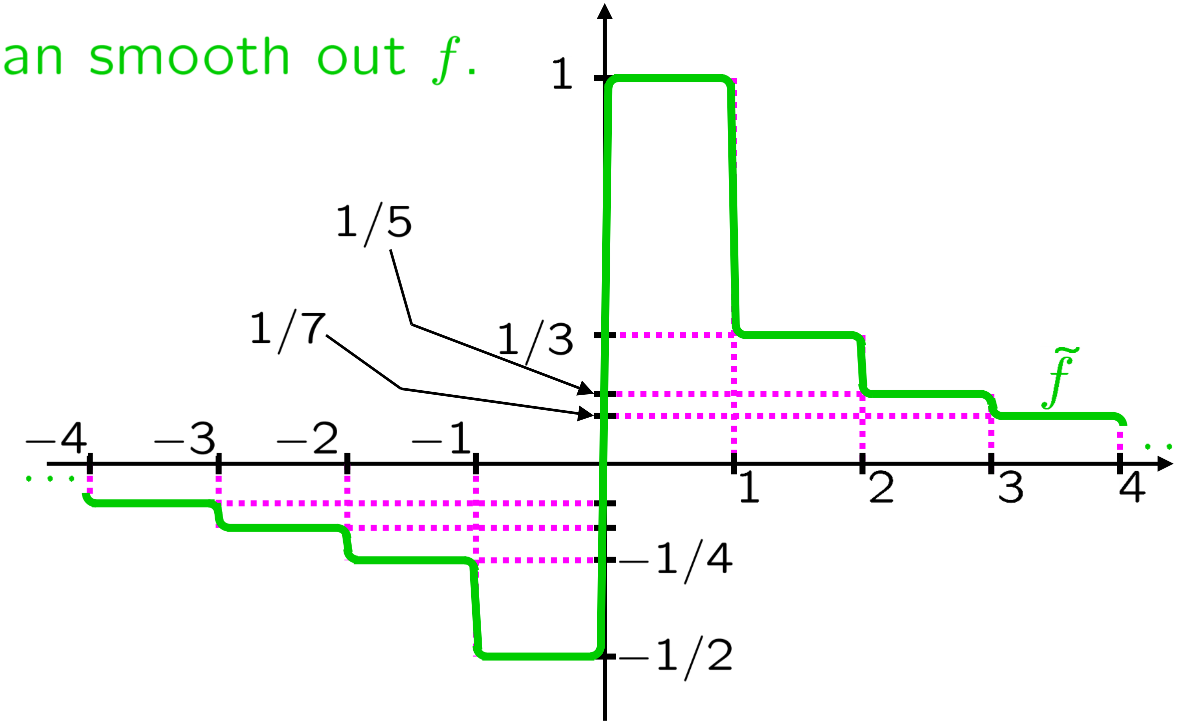
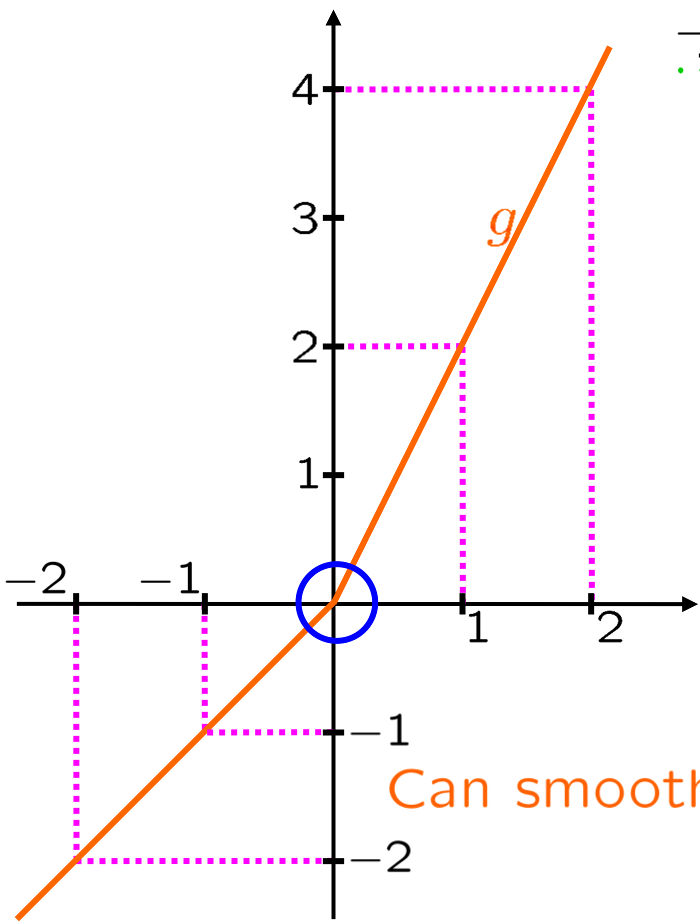
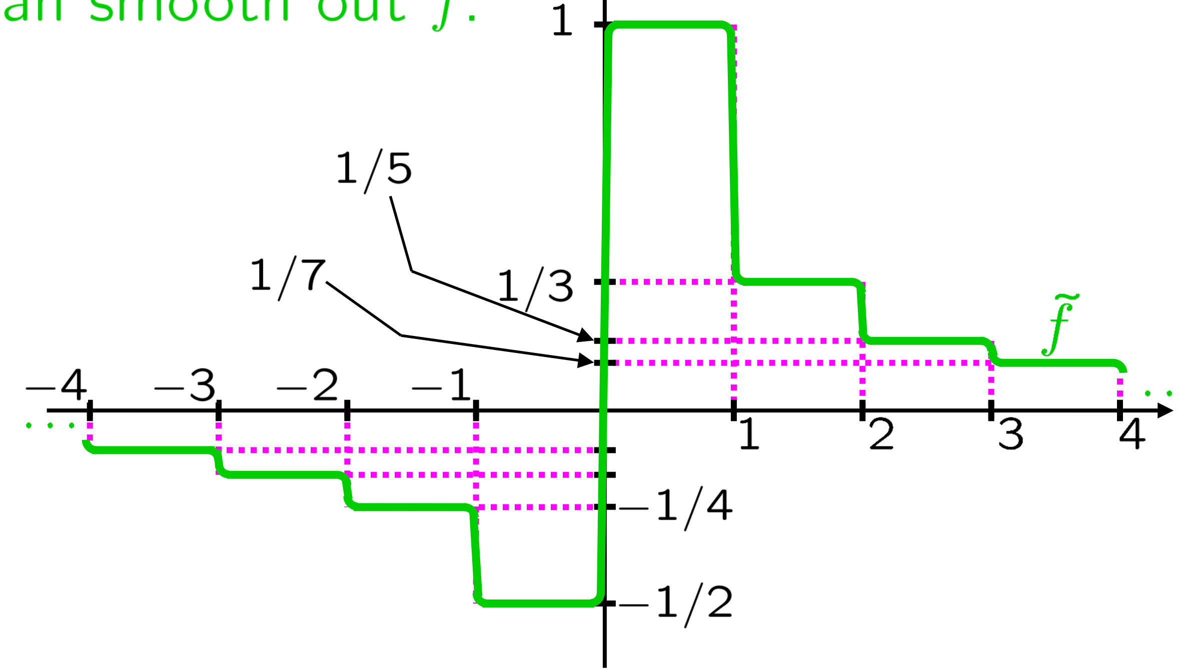


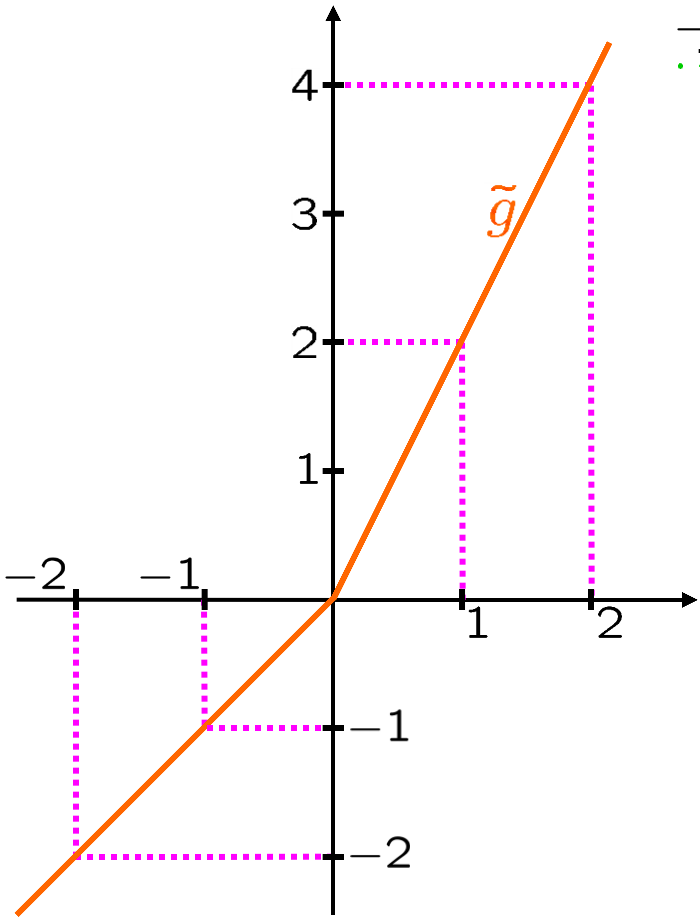


Can smooth out g.

$$g(t) = \begin{cases} 2t, & \text{if } t \geq 0 \\ t, & \text{if } t \leq 0 \end{cases}$$

Can smooth out f.

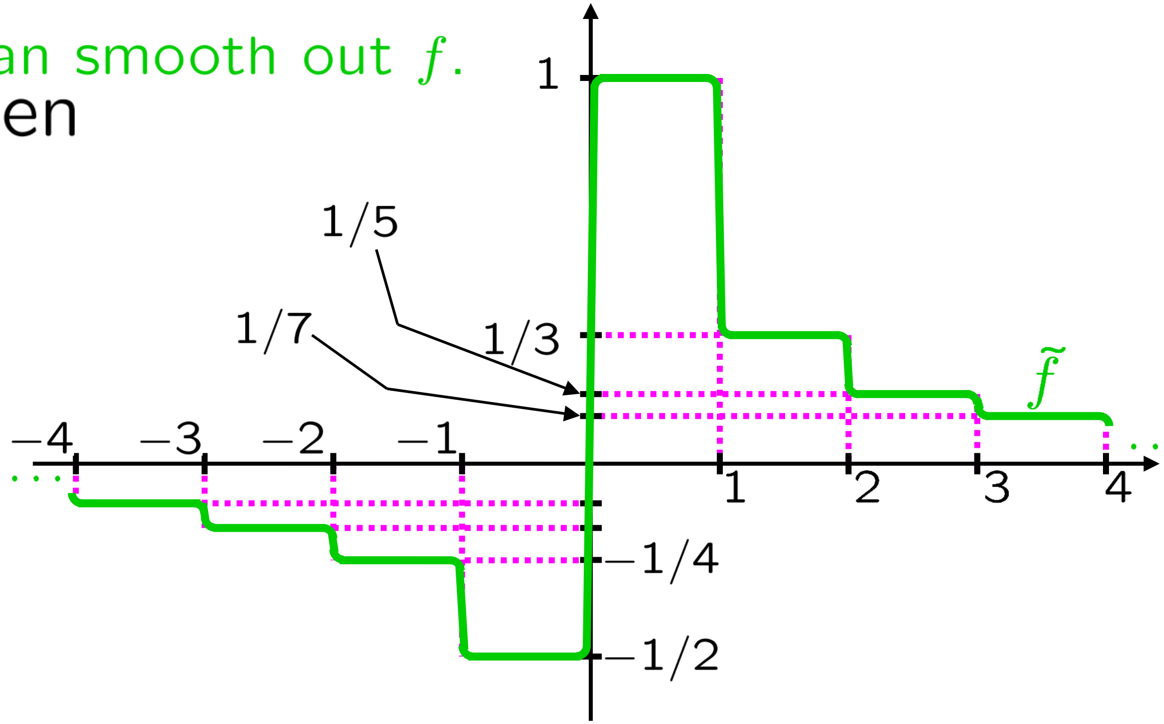


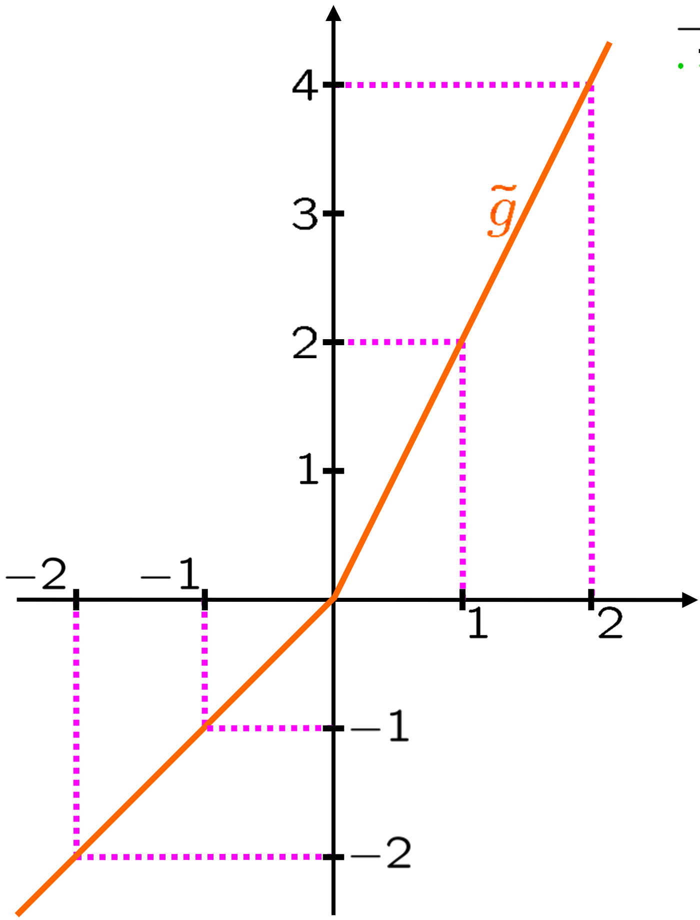


$$g(t) = \begin{cases} 2t, & \text{if } t \geq 0 \\ t, & \text{if } t \leq 0 \end{cases}$$

Problem persists even for integration on a bounded interval, like $(-1, 1)$ . . .

Can smooth out f.





Problem persists:

$$\lim_{K\to\infty} \int_{-K}^{K} \tilde{f}(x)\,dx$$

$$\not\parallel$$

$$\lim_{K\to\infty} \int_{-K}^{K} [\tilde{f}(\tilde{g}(t))][\tilde{g}'(t)]\,dt$$

Problem persists even for integration on a bounded interval, like $(-1,1)$ . . .

That is: $\exists$smooth $F : (-1,1) \to \mathbb{R}$

$\exists$smooth increasing $G : (-1,1/2) \to (-1,1)$

s.t. $$\lim_{K\uparrow 1} \int_{-K}^{K} F(x)\,dx \neq \lim_{K\uparrow 1} \int_{-K}^{K} [F(G(t))][G'(t)]\,dt$$

How to do this:

Make $0 < a_1 < a_2 < a_3 < \cdots$, with $a_n \to 1$.

Make $\int_0^{a_1} F = 1$, $\int_{a_1}^{a_2} F = 1/3$, $\int_{a_2}^{a_3} F = 1/5$, . . .

$\int_{-a_1}^{0} F = -1/2$, $\int_{-a_2}^{-a_1} F = -1/4$, $\int_{-a_3}^{-a_2} F = -1/6$, ...

Make $G(x) = x$, for $x \in (-1, -\varepsilon)$,

$G(x) = 2x$, for $x \in (\varepsilon, 1/2)$.

etc., etc., etc.