Financial Mathematics

Basics of vector spaces

LINEAR ALGEBRA

The Dot Product

If $v := (a_1, \ldots, a_n)$ and $w := (b_1, \ldots, b_n)$ are $n$tuples then the dot product of v and w, written $v \cdot w$, is defined by:

$$v \cdot w := a_1 b_1 + \cdots + a_n b_n.$$

E.g.: $(2,3,4) \cdot (5,6,7) = 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7$
$= 10 + 18 + 28 = 56$

Note: $n$tuples are often called vectors.

SKILL: Vector dot product

$$(a_1, \cdots, a_n) \cdot (b_1, \cdots, b_n) := a_1 b_1 + \cdots + a_n b_n.$$

Game: I pick and tell you an integer $n > 0$.

I pick a secret vector $u \in \mathbb{R}^n$.

Your goal is to find u.

You pick and tell me finite sequence

$$v_1, \ldots, v_p \in \mathbb{R}^n$$

and I tell you $u \cdot v_1, \ldots, u \cdot v_p$.

How can you figure out u?

I'm thinking of a secret vector $u = (a, b, c, d) \in \mathbb{R}^4$.

Then $u \cdot (w, x, y, z)$ is equal to

$$aw + bx + cy + dz$$

Then $u \cdot (1, 0, 0, 0) = a$, so you can find the first entry of u. The other three entries can be found by asking for

$$u \cdot (0, 1, 0, 0),\ u \cdot (0, 0, 1, 0),\ u \cdot (0, 0, 0, 1).$$

Definition:

The **linear operations** are
vector addition
and scalar multiplication.

SKILLS:

Vector addition is done entry-by-entry:

$$(1,2,3)+(4,5,6) = (1+4,2+5,3+6)$$
$$= (5,7,9)$$

Scalar multiplication is done as follows:

$$2\cdot(6,8,7) = (2\cdot 6,2\cdot 8,2\cdot 7)$$
$$= (12,16,14)$$

Definition:

A vector v is a
linear combination of vectors $w_1, \ldots, w_k$
if it can be obtained from them
by linear operations,
i.e., if there are scalars $c_1, \cdots, c_k$ such that

$$v = c_1 w_1 + \cdots + c_k w_k$$

. . . . . .

coefficients

- -

"v is a linear combination of $w_1, \ldots, w_k$
with coefficients $c_1, \ldots, c_k$."

SKILL: Linear combinations

Def'n: A non$\emptyset$ subset $S \subseteq \mathbb{R}^n$ is a **subspace** (a.k.a. **vector subspace**, **linear subspace**)
if it's closed under the linear operations,
i.e., if both of the following hold:

- for all $v, w \in S$, we have: $v + w \in S$,
- for all $c \in \mathbb{R}$, for all $v \in S$, we have: $cv \in S$,

i.e., if any linear combination of elements of S is again an element of S,

i.e., if, for all integers $k > 0$,
for all scalars $c_1, \ldots, c_k$,
for all $v_1, \ldots, v_k \in S$,
we have: $c_1 v_1 + \cdots + c_k v_k \in S$.

---

e.g.: Any line through the origin in $\mathbb{R}^2$
Any line through the origin in $\mathbb{R}^3$
Any plane through the origin in $\mathbb{R}^3$

Question:

Subspaces (except $\{0\}$) are infinite,
so how can I discuss one with you?

I can't list all the elements.

So how do I communicate to you
a particular subspace of interest to me?

Definition: Let $A \subseteq \mathbb{R}^n$.

The **span** (a.k.a. **linear span**) of A
denoted $\boxed{\langle A \rangle}$,
is the set of all
linear combinations of elements of A

i.e., $\boxed{\langle A \rangle} := \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{R}, v_1, \ldots, v_k \in A\}$,

i.e., $\boxed{\langle A \rangle}$ is what one obtains after
"closing A under linear operations",

Definition: Let $A \subseteq \mathbb{R}^n$.

The **span** (a.k.a. **linear span**) of A denoted $\langle A \rangle$, is the set of all linear combinations of elements of A

i.e., $\langle A \rangle := \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{R}, v_1, \ldots, v_k \in A\}$,

Definition: Let $A \subseteq \mathbb{R}^n$.

The **span** (a.k.a. **linear span**) of A denoted $\langle A \rangle$,

i.e., $\langle A \rangle$ the smallest of all the subspaces of $\mathbb{R}^n$ containing A.

i.e., $\langle A \rangle := \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{R}, v_1, \ldots, v_k \in A\}$,

i.e., $\langle A \rangle$ is what one obtains after "closing A under linear operations",

Definition: Let $A \subseteq \mathbb{R}^n$.

The **span** (a.k.a. **linear span**) of A denoted $\langle A \rangle$, is the set of all linear combinations of elements of A

i.e., $\langle A \rangle := \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{R}, v_1, \ldots, v_k \in A\}$,

i.e., $\langle A \rangle$ is what one obtains after "closing A under linear operations",

i.e., $\langle A \rangle$ the smallest of all the subspaces of $\mathbb{R}^n$ containing A.

Notation: Say A is finite: $A = \{v_1, \ldots, v_k\}$.

Then $\langle A \rangle = \langle \{v_1, \ldots, v_k\} \rangle$ is usually written $\langle v_1, \ldots, v_k \rangle$, dropping the braces.

Example (of a subspace):

The subspace of $\mathbb{R}^4$ spanned by

$$(1, 3, 4, 2), \quad (2, 1, 2, -1), \quad (4, 7, 10, 3)$$

Question:

Subspaces (except $\{0\}$) are infinite,
so how can I discuss one with you?

I can't list all the elements.

So how do I communicate to you
a particular subspace of interest to me?

Answer:

I can give you a (finite) spanning set.

Notation: Say A is finite: $A = \{v_1, \ldots, v_k\}$.

Then $\langle A \rangle = \langle \{v_1, \ldots, v_k\} \rangle$ is usually written
$\langle v_1, \ldots, v_k \rangle$, dropping the braces.

Example (of a subspace):

The subspace of $\mathbb{R}^4$ spanned by

$$(1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3)$$

i.e.:

$$\left\langle (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \right\rangle$$

Note:

linear combination

Suppose v_k is a l.c. of $v_1, \ldots, v_{k-1}$.

Then $\langle v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_{k-1} \rangle$,

so v_k was redundant.

e.g.:

$$2 \cdot (1,3,4,2) + (2,1,2,-1) = (4,7,10,3),$$

so $(4,7,10,3)$ is a l.c. of $(1,3,4,2)$ and $(2,1,2,-1)$.

Example (of a subspace):

The subspace of $\mathbb{R}^4$ spanned by

$$(1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3)$$

i.e.:

$$\langle\, (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \,\rangle$$

---

e.g.:

$$2 \cdot (1,3,4,2) + (2,1,2,-1) = (4,7,10,3),$$

so $(4,7,10,3)$ is a l.c. of $(1,3,4,2)$ and $(2,1,2,-1)$.

---

e.g.:

$$\langle\, (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \,\rangle$$

$$\|$$

$$\langle\, (1,3,4,2), \quad (2,1,2,-1) \,\rangle$$

$$2 \cdot (1,3,4,2) + (2,1,2,-1) = (4,7,10,3),$$

so $(4,7,10,3)$ is a l.c. of $(1,3,4,2)$ and $(2,1,2,-1)$.

linear combination

Note:

Suppose v_k is a l.c. of $v_1, \ldots, v_{k-1}$.

Then $\langle v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_{k-1} \rangle$,

so v_k was redundant.

Definition:

(l.d.)

We say $v_1, \ldots, v_k$ are **linearly dependent**

if $\exists j$ s.t. $v_j \in \langle v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k \rangle$.

$$\langle (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \rangle$$

$$\|$$

$$\langle (1,3,4,2), \quad (2,1,2,-1) \rangle$$

Note:

Suppose v_k is a l.c. of $v_1, \ldots, v_{k-1}$.

linear combination

Then $\langle v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_{k-1} \rangle$,

so v_k was redundant.

Definition:

We say $v_1, \ldots, v_k$ are **linearly dependent** (l.d.)

if $\exists j$ s.t. $v_j \in \langle v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k \rangle$.

Definition:

We say $v_1, \ldots, v_k$ are **linearly independent** (l.i.)

if they are *not* l.d.

linearly dependent

Fact: Let F be a finite subset of $\mathbb{R}^n$.
Suppose F is l.i.
Say $v \in \mathbb{R}^n \backslash \langle F \rangle$, *i.e.*, $v \in \mathbb{R}^n$ and $v \notin \langle F \rangle$.
Then $F \cup \{v\}$ is l.i.

Proof:
Let $f_1, \ldots, f_k$ be the distinct elements of F.
$v \notin \langle F \rangle$, so v is not a l.c. of $f_1, \ldots, f_k$.
Say some vector in $F \cup \{v\}$ is a l.c. of the rest.
Want: Contradiction.
Say, *e.g.*, f_1 is a l.c. of $f_2, \ldots, f_k, v$.
Say, *e.g.*, $f_1 = a_1 v + a_2 f_2 + \cdots + a_k f_k$.
$f_1, \ldots, f_k$ are l.i., so $a_1 \neq 0$.
$(1/a_1) f_1 = v + (a_2/a_1) f_2 + \cdots + (a_k/a_1) f_k$
$(1/a_1) f_1 - (a_2/a_1) f_2 - \cdots - (a_k/a_1) f_k = v$
$v \in \langle F \rangle$. Contradiction. QED

MULTIPLY EQUATION BY $1/a_1$

Fact: The vectors $v_1, \ldots, v_k$ are l.i. iff the only l.c. of $v_1, \ldots, v_k$ that is $= 0$ is the one with all coefficients $= 0$.

the "trivial" l.c.

Proof of "only if" ($\Rightarrow$):

Say $v_1, \ldots, v_k$ are l.i. and

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Want: $c_1 = \cdots = c_k = 0$.

Say, *e.g.*, $c_1 \neq 0$. Want: Contradiction.

Divide the following by c_1:

$$c_1 \, v_1 = - \, c_2 \, v_2 - \cdots - c_k \, v_k$$

$$v_1 = -(c_2/c_1) v_2 - \cdots - (c_k/c_1) v_k$$

Then v_1 is a l.c. of $v_2, \ldots, v_k$.

Contradiction.

Fact: The vectors $v_1, \ldots, v_k$ are l.i. iff
the only l.c. of $v_1, \ldots, v_k$ that is $= 0$
is the one with all coefficients $= 0$.

the "trivial" l.c.

Proof of "if" ($\Leftarrow$):

Say the only l.c. of $v_1, \ldots, v_k$ that is $= 0$
is the one with all coefficients $= 0$.

Want: $v_1, \ldots, v_k$ are l.i.

Say, *e.g.*, v_1 is a l.c. of $v_2, \ldots, v_k$.

Want: Contradiction.

$$v_1 = c_2 v_2 + \cdots + c_k v_k$$

$$v_1 - c_2 v_2 - \cdots - c_k v_k = 0$$

Contradiction. QED

Definition:

Let S be a subspace of $\mathbb{R}^n$.

A **spanning set for** S is a subset $A \subseteq S$ such that $\langle A \rangle = S$.

A **basis of** S is a l.i. spanning set for S.

l.i. = linearly independent

$$S := \langle (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \rangle$$

$$\| $$

$$\langle (1,3,4,2), \quad (2,1,2,-1) \rangle$$

Question:

Is $\{(1,3,4,2), \quad (2,1,2,-1)\}$ a basis of S?

Definition:

Let S be a subspace of $\mathbb{R}^n$.
A **spanning set for** S is a subset
$$A \subseteq S \text{ such that } \langle A \rangle = S.$$
A **basis of** S is a l.i. spanning set for S.

l.i. = linearly independent

---

e.g.: The **standard basis of** $\mathbb{R}^n$ is

$$(1, 0, \ldots, 0), \quad (0, 1, 0, \ldots, 0), \quad \ldots, \quad (0, \ldots, 0, 1, 0), \quad (0, \ldots, 0, 1).$$

Questions: Why a spanning set?
Why linearly independent?

STOP