Financial Mathematics

Matrix operations

Problem of general interest:

Given $A \in \mathbb{R}^{k \times n}$ and $B \in \mathbb{R}^{n \times q}$,

form $L_A : \mathbb{R}^n \to \mathbb{R}^k$ and $L_B : \mathbb{R}^q \to \mathbb{R}^n$

and then compose:

$$L_A \circ L_B : \mathbb{R}^q \to \mathbb{R}^k$$

L_B then L_A

and try to find $C \in \mathbb{R}^{k \times q}$

such that $L_C : \mathbb{R}^q \to \mathbb{R}^k$ is equal to $L_A \circ L_B$.

E.g.: $k = 2$, $n = 3$, $q = 4$,

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2 \times 4}$ s.t. $L_C = L_A \circ L_B$.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C $\boxed{\text{"is"}}$ $L_C(1,0,0,0)$.

$L_C(1,0,0,0)$ is horizontal,

with commas and parentheses.

The first column of C is vertical,

with no commas and no parentheses.

Same entries, though!

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

$$A = \begin{bmatrix} \boxed{4 \quad -2 \quad 1} \\ \boxed{0 \quad 5 \quad 9} \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$$L_A(\,L_B(1,0,0,0)\,) = L_A(\,(4,0,7)\,)$$

$$\| \qquad\qquad\qquad \|$$

$$L_C(1,0,0,0) \qquad (\,(4,0,7)\cdot(4,-2,1)\;,\;(4,0,7)\cdot(0,5,9)\,)$$

DOT PRODUCT
IS COMMUTATIVE

$$C = \begin{bmatrix} (4,0,7)\cdot(4,-2,1) & ??? & ??? & ??? \\ (4,0,7)\cdot(0,5,9) & ??? & ??? & ??? \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ \boxed{0 \quad 5 \quad 9} \end{bmatrix} \text{ and } B = \begin{bmatrix} \boxed{\begin{matrix}4\\0\\7\end{matrix}} & \begin{matrix}-2 & 1 & 2 \\ 5 & 9 & 1 \\ 3 & 8 & 2\end{matrix} \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ \boxed{(0,5,9)} \cdot \boxed{(4,0,7)} & ??? & ??? & ??? \end{bmatrix}$$

DOT PRODUCT
IS COMMUTATIVE

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ (0,5,9)\cdot(4,0,7) & ??? & ??? & ??? \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ \boxed{0 \quad 5 \quad 9} \end{bmatrix} \text{ and } B = \begin{bmatrix} \boxed{\begin{matrix}4\\0\\7\end{matrix}} & \begin{matrix}-2 & 1 & 2\\ 5 & 9 & 1 \\ 3 & 8 & 2\end{matrix} \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ \boxed{\boxed{(0,5,9)} \cdot \boxed{(4,0,7)}} & ??? & ??? & ??? \end{bmatrix}$$

Note: The $\boxed{(2,1) \text{ entry of } C}$

is equal to

the $\boxed{\text{second row of } A}$

dotted against

the $\boxed{\text{first column of } B}$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ (0,5,9)\cdot(4,0,7) & ??? & ??? & ??? \end{bmatrix}$$

Note: The (j,k) entry of C

is equal to

the jth row of A

dotted against

the kth column of B

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} 23 & ??? & ??? & ??? \\ 63 & ??? & ??? & ??? \end{bmatrix}$$

Note: The (j,k) entry of C

is equal to

the jth row of A

dotted against

the kth column of B

$$A = \begin{bmatrix} \boxed{4 \quad -2 \quad 1} \\ 0 \quad\ \ 5 \quad\ \ 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & \boxed{1} & 2 \\ 0 & 5 & \boxed{9} & 1 \\ 7 & 3 & \boxed{8} & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of C "is" $L_C(1,0,0,0)$.

$4 - 18 + 8 =$

$$C = \begin{bmatrix} 23 & -15 & \boxed{-6} & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

Note: The (j,k) entry of C ($j = 1$, $k = 3$)

is equal to

$(4, -2, 1)$ the jth row of A (1st)

dotted against

$(1, 9, 8)$ the kth column of B (3rd)

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2 \times 4}$ s.t. $L_C = L_A \circ L_B$.

$$C = \begin{bmatrix} 23 & -15 & -6 & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

$$C = \begin{bmatrix} 23 & -15 & -6 & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

Def'n: When $L_Z = L_X \circ L_Y$, we say that

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

$$\underline{AB = C} = \begin{bmatrix} 23 & -15 & -6 & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

Def'n: When $L_Z = L_X \circ L_Y$, we say that
Z is the **product of** X **by** Y,
and we write $Z = \boxed{XY}$.

The (j, k) entry of XY
is equal to
the jth row of X
dotted against
the kth column of Y

Matrix multiplication

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second.

If A is $p \times q$ and B is $q \times r$, then AB is $p \times r$.

Warning:

Matrix multiplication is not commutative.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\nparallel$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Matrix multiplication

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second.

If A is $p \times q$ and B is $q \times r$, then AB is $p \times r$.

Warning:

Matrix multiplication is not commutative.

However, it *is* associative:

$$\forall A \in \mathbb{R}^{p\times q},\ \forall B \in \mathbb{R}^{q\times r},\ \forall C \in \mathbb{R}^{r\times s},$$

$$(L_A \circ L_B) \circ L_C = L_A \circ (L_B \circ L_C),$$

$$\text{so } (AB)C = A(BC).$$

Matrix multiplication

$$\boxed{3 \times 4} \rightarrow \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 6 & 7 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 5 & 6 \\ -1 & -3 \\ 2 & 0 \end{bmatrix} \leftarrow \boxed{4 \times 2}$$

$$\|$$

$$\begin{bmatrix} 28 + 15 - 2 + 2 & 12 + 18 - 6 + 0 \\ 35 + 30 - 7 + 16 & 15 + 36 - 21 + 0 \\ 14 + 5 + 0 - 2 & 6 + 6 + 0 + 0 \end{bmatrix}$$

$$\|$$

$$\boxed{3 \times 2} \rightarrow \begin{bmatrix} 43 & 24 \\ 74 & 30 \\ 17 & 12 \end{bmatrix}$$

SKILL:

Given two matrices, find their product.

e.g.:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & -2 & 4 & 6 & -3 \\ 0 & 9 & 2 & -8 & 3 \end{bmatrix} \begin{bmatrix} 6 & 7 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

Discussion:

Shape of answer??

First row of answer??

Last two columns??

Matrices and linear transformations

$$\boxed{(a,b,c)^{\mathsf{CV}}} := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix}^{\mathsf{V}}} := (a,b,c)$$

$$\boxed{(a,b,c)^{\mathsf{RV}}} := \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{V}}} := (a,b,c)$$

$$\boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix}^{t}} := \begin{bmatrix} a & b & c \end{bmatrix} \qquad \boxed{\begin{bmatrix} a & b & c \end{bmatrix}^{t}} := \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Matrices and linear transformations

$$\boxed{(a,b,c)^{\mathsf{CV}}} := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix}^{\mathsf{V}}} := (a,b,c)$$

$$[L_M(v)]^{\mathsf{CV}} = M[v^{\mathsf{CV}}]$$

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$v = (-1,-2,-3,-4)$$

$$L_M(v) = (-30,-70)$$

$$M[v^{\mathsf{CV}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -30 \\ -70 \end{bmatrix}$$

Matrix addition

Matrices can only be added if they have the same shape, in which case, addition is done entry by entry.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 1+13 & 2+14 & 3+15 & 4+16 \\ 5+17 & 6+18 & 7+19 & 8+20 \\ 9+21 & 10+22 & 11+23 & 12+24 \end{bmatrix}$$

SKILL: Matrix addition

Direct sums

Def'n: If $A \in \mathbb{R}^{p\times q}$ and if $B \in \mathbb{R}^{r\times s}$,
then $A \oplus B \in \mathbb{R}^{(p+r)\times(q+s)}$ is defined by

$$\boxed{A \oplus B} = \begin{bmatrix} \overset{p\times q}{A} & \overset{p\times s}{0} \\ \underset{r\times q}{0} & \underset{r\times s}{B} \end{bmatrix}$$

e.g.:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 & 14 & 19 \\ 10 & 15 & 20 \\ 11 & 16 & 21 \\ 12 & 17 & 22 \\ 13 & 18 & 23 \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 14 & 19 \\ 0 & 0 & 0 & 0 & 10 & 15 & 20 \\ 0 & 0 & 0 & 0 & 11 & 16 & 21 \\ 0 & 0 & 0 & 0 & 12 & 17 & 22 \\ 0 & 0 & 0 & 0 & 13 & 18 & 23 \end{bmatrix}$$

SKILL: Matrix direct sum

Tensor products

Def'n: If $A = \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} \end{bmatrix} \in \mathbb{R}^{p\times q}$

and if $B \in \mathbb{R}^{r\times s}$

then $$A \otimes B := \begin{bmatrix} \overset{r\times s}{a_{11}B} & \cdots & \overset{r\times s}{a_{1q}B} \\ \vdots & \ddots & \vdots \\ \underset{r\times s}{a_{p1}B} & \cdots & \underset{r\times s}{a_{pq}B} \end{bmatrix}$$

$$\rotatebox{90}{$\in$}$$

$$(\mathbb{R}^{p\times q})^{r\times s}$$

Tensor products

e.g.: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ $\quad B = \begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & \boxed{2\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix}} & 3\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 4\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \\ 5\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 6\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 7\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 8\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \\ 9\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 10\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 11\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 12\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \end{bmatrix}$$

etc.

Tensor products

e.g.: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ $\quad B = \begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & \begin{bmatrix} 26 & 36 \\ 28 & 38 \\ 30 & 40 \\ 32 & 42 \\ 34 & 44 \end{bmatrix} & 3\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 4\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \\ 5\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 6\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 7\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 8\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \\ 9\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 10\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 11\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} & 12\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix} \end{bmatrix}$$

Tensor products

e.g.: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ $\quad B = \begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} \boxed{\begin{bmatrix} 13 & 18 \\ 14 & 19 \\ 15 & 20 \\ 16 & 21 \\ 17 & 22 \end{bmatrix}} & \begin{bmatrix} 26 & 36 \\ 28 & 38 \\ 30 & 40 \\ 32 & 42 \\ 34 & 44 \end{bmatrix} & \boxed{\begin{bmatrix} 39 & 54 \\ 42 & 57 \\ 45 & 60 \\ 48 & 63 \\ 51 & 66 \end{bmatrix}} & \boxed{\begin{bmatrix} 52 & 72 \\ 56 & 76 \\ 60 & 80 \\ 64 & 84 \\ 68 & 88 \end{bmatrix}} \\ \boxed{\begin{bmatrix} 65 & 90 \\ 70 & 95 \\ 75 & 100 \\ 80 & 105 \\ 85 & 110 \end{bmatrix}} & \boxed{\begin{bmatrix} 78 & 108 \\ 84 & 114 \\ 90 & 120 \\ 96 & 126 \\ 102 & 132 \end{bmatrix}} & \boxed{\begin{bmatrix} 91 & 126 \\ 98 & 133 \\ 105 & 140 \\ 112 & 147 \\ 119 & 154 \end{bmatrix}} & \boxed{\begin{bmatrix} 104 & 144 \\ 112 & 152 \\ 120 & 160 \\ 128 & 168 \\ 136 & 176 \end{bmatrix}} \\ \boxed{\begin{bmatrix} 117 & 162 \\ 126 & 171 \\ 135 & 180 \\ 144 & 189 \\ 153 & 198 \end{bmatrix}} & \boxed{\begin{bmatrix} 130 & 180 \\ 140 & 190 \\ 150 & 200 \\ 160 & 210 \\ 170 & 220 \end{bmatrix}} & \boxed{\begin{bmatrix} 143 & 198 \\ 154 & 209 \\ 165 & 220 \\ 176 & 231 \\ 187 & 242 \end{bmatrix}} & \boxed{\begin{bmatrix} 156 & 216 \\ 168 & 228 \\ 180 & 240 \\ 192 & 252 \\ 204 & 264 \end{bmatrix}} \end{bmatrix}$$

SKILL: Matrix tensor product

Standard basis of $\mathbb{R}^{2\times 2}$

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$

$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

Let $A_{11} = \begin{bmatrix} 4 & -1 \\ 4 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 & 6 \\ 0 & -4 \end{bmatrix},$

$A_{21} = \begin{bmatrix} 1 & 3 \\ -2 & -3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -5 & -6 \\ 8 & -2 \end{bmatrix}.$

Compute $(E_{11} \otimes A_{11}) + (E_{12} \otimes A_{12}) + (E_{21} \otimes A_{21}) + (E_{22} \otimes A_{22})$.

$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ | $E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ | $E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ | $E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

Let $E_{11} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad E_{12} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$

$$E_{21} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \quad E_{22} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Let $A_{11} = \begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix}, \quad A_{12} = \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix},$

$$A_{21} = \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix}, \quad A_{22} = \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}.$$

Compute $(E_{11} \otimes A_{11}) + (E_{12} \otimes A_{12}) + (E_{21} \otimes A_{21}) + (E_{22} \otimes A_{22})$.

$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	$E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$	$E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$
$A_{11}=\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix}$	$A_{12}=\begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}$	$A_{21}=\begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix}$	$A_{22}=\begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}$

$E_{11} \otimes A_{11} =$

Let $A_{11} = \begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix}, \quad A_{12} = \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix},$

$A_{21} = \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix}, \quad A_{22} = \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}.$

Compute $(E_{11} \otimes A_{11}) + (E_{12} \otimes A_{12}) + (E_{21} \otimes A_{21}) + (E_{22} \otimes A_{22})$.

$$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\qquad E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\qquad E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\qquad E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix}\qquad A_{12}=\begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\qquad A_{21}=\begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix}\qquad A_{22}=\begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}$$

$$E_{11}\otimes A_{11}=\begin{bmatrix}\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}\qquad E_{12}\otimes A_{12}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}$$

$$E_{21}\otimes A_{21}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}\qquad E_{22}\otimes A_{22}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}\end{bmatrix}$$

Compute $(E_{11}\otimes A_{11}) + (E_{12}\otimes A_{12}) + (E_{21}\otimes A_{21}) + (E_{22}\otimes A_{22})$.

$$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \quad E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} \quad A_{12}=\begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix} \quad A_{21}=\begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} \quad A_{22}=\begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}$$

$$E_{11}\otimes A_{11}=\begin{bmatrix}\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \quad E_{12}\otimes A_{12}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}$$

$$E_{21}\otimes A_{21}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \quad E_{22}\otimes A_{22}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}\end{bmatrix}$$

$$\begin{matrix}E_{11}\otimes A_{11}\\ +\\ E_{12}\otimes A_{12}\\ +\\ E_{21}\otimes A_{21}\\ +\\ E_{22}\otimes A_{22}\end{matrix} = (E_{11}\otimes A_{11}) + (E_{12}\otimes A_{12}) + (E_{21}\otimes A_{21}) + (E_{22}\otimes A_{22})$$

$$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \qquad E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \qquad E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \qquad E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} \qquad A_{12}=\begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix} \qquad A_{21}=\begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} \qquad A_{22}=\begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}$$

$$E_{11}\otimes A_{11}=\begin{bmatrix}\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \qquad E_{12}\otimes A_{12}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}$$

$$E_{21}\otimes A_{21}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \qquad E_{22}\otimes A_{22}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}\end{bmatrix}$$

$$\begin{matrix}E_{11}\otimes A_{11}\\ +\\ E_{12}\otimes A_{12}\\ +\\ E_{21}\otimes A_{21}\\ +\\ E_{22}\otimes A_{22}\end{matrix} = \begin{bmatrix}\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} & \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\\ \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} & \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$$

$$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \quad E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$A_{11}=\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} \quad A_{12}=\begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix} \quad A_{21}=\begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} \quad A_{22}=\begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}$$

$$E_{11}\otimes A_{11}=\begin{bmatrix}\begin{bmatrix}4 & -1\\ 4 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \quad E_{12}\otimes A_{12}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-3 & 6\\ 0 & -4\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}$$

$$E_{21}\otimes A_{21}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 3\\ -2 & -3\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{bmatrix} \quad E_{22}\otimes A_{22}=\begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}-5 & -6\\ 8 & -2\end{bmatrix}\end{bmatrix}$$

$$\begin{matrix}E_{11}\otimes A_{11}\\ +\\ E_{12}\otimes A_{12}\\ +\\ E_{21}\otimes A_{21}\\ +\\ E_{22}\otimes A_{22}\end{matrix} \qquad \begin{matrix}E_{11}\otimes A_{11}\\ +\\ E_{12}\otimes A_{12}\\ +\\ E_{21}\otimes A_{21}\\ +\\ E_{22}\otimes A_{22}\end{matrix} \quad = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \in (\mathbb{R}^{2\times 2})^{2\times 2}$$

$E_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$E_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	$E_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$	$E_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

The multiplication map $\mathcal{M}$

Define $\mathcal{M} : (\mathbb{R}^{2\times 2})^{2\times 2} \to \mathbb{R}^{2\times 2}$ by the rule:
$\forall X_{11}, X_{12}, X_{21}, X_{22} \in \mathbb{R}^{2\times 2}$,

$$\begin{aligned}&\mathcal{M}((E_{11} \otimes X_{11}) + (E_{12} \otimes X_{12}) + \\ &\qquad (E_{21} \otimes X_{21}) + (E_{22} \otimes X_{22})) \\ &= E_{11}X_{11} + E_{12}X_{12} + E_{21}X_{21} + E_{22}X_{22}\end{aligned}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}, \quad B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}.$

Compute $\mathcal{M}(A \otimes B)$.

$$\begin{aligned}&\mathcal{M}((E_{11}\otimes X_{11})+(E_{12}\otimes X_{12})+\\&(E_{21}\otimes X_{21})+(E_{22}\otimes X_{22}))\\&=E_{11}X_{11}+E_{12}X_{12}+E_{21}X_{21}+E_{22}X_{22}\end{aligned}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$\begin{aligned}&\mathcal{M}((E_{11}\otimes X_{11})+(E_{12}\otimes X_{12})+\\&(E_{21}\otimes X_{21})+(E_{22}\otimes X_{22}))\\&=E_{11}X_{11}+E_{12}X_{12}+E_{21}X_{21}+E_{22}X_{22}\end{aligned}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$.

Compute $\mathcal{M}(A \otimes B)$.

$$\mathcal{M}\left(\boxed{(E_{11} \boxed{\otimes} X_{11}) + (E_{12} \boxed{\otimes} X_{12}) + (E_{21} \boxed{\otimes} X_{21}) + (E_{22} \boxed{\otimes} X_{22})}\right)$$

REMOVE $\otimes$S

$$= \boxed{E_{11}X_{11} + E_{12}X_{12} + E_{21}X_{21} + E_{22}X_{22}}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$A = 4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}$$

$$A \otimes B = [(4E_{11}) \otimes B] + [(2E_{12}) \otimes B] + [(3E_{21}) \otimes B] + [(-6E_{22}) \otimes B]$$

$$= \boxed{[E_{11} \boxed{\otimes} (\boxed{4}B)] + [E_{12} \boxed{\otimes} (\boxed{2}B)] + [E_{21} \boxed{\otimes} (\boxed{3}B)] + [E_{22} \boxed{\otimes} (\boxed{-6}B)]}$$

REMOVE $\otimes$S

$$\mathcal{M}(A \otimes B) = \boxed{[E_{11}(4B)] + [E_{12}(2B)] + [E_{21}(3B)] + [E_{22}(-6B)]}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$A = 4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$\begin{aligned}\mathcal{M}(A \otimes B) &= [E_{11}(4B)] + [E_{12}(2B)] + \\ &\quad [E_{21}(3B)] + [E_{22}(-6B)] \\ &= [(\boxed{4}E_{11})B] + [(\boxed{2}E_{12})B] + \\ &\quad [(\boxed{3}E_{21})B] + [(\boxed{-6}E_{22})B]\end{aligned}$$

$$\begin{aligned}\mathcal{M}(A \otimes B) &= [E_{11}(4B)] + [E_{12}(2B)] + \\ &\quad [E_{21}(3B)] + [E_{22}(-6B)]\end{aligned}$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$A = \underline{4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}}$$

$$\begin{aligned}\mathcal{M}(A \otimes B) &= [E_{11}(4B)] + [E_{12}(2B)] + \\ &\quad [E_{21}(3B)] + [E_{22}(-6B)] \\ &= [(4E_{11})\boxed{B}] + [(2E_{12})\boxed{B}] + \\ &\quad [(3E_{21})\boxed{B}] + [(-6E_{22})\boxed{B}] \\ &= (\underline{4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}})\boxed{B} \\ &= A\boxed{B}\end{aligned}$$

FACTOR OUT B

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$A = 4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}$$

$\mathcal{M}(A \otimes B) =$

$$\begin{aligned} \mathcal{M}(A \otimes B) &= AB \\ &= \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix} \end{aligned}$$

$$AB$$

Let $A := \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$, $B := \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix}$. Compute $\mathcal{M}(A \otimes B)$.

$$A = 4E_{11} + 2E_{12} + 3E_{21} - 6E_{22}$$

$$\begin{aligned} \mathcal{M}(A \otimes B) &= AB \\ &= \begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 8 \\ 39 & 6 \end{bmatrix} \end{aligned}$$

The zero matrix

Definition:

The $n \times k$ **zero matrix** is the $n \times k$ matrix with all entries equal to 0.

e.g.: The 5×4 zero matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The zero matrix is an additive identity

Fact: If 0 is the $n \times k$ zero matrix,
and if M is an $n \times k$ matrix
then $0 + M = M + 0 = M$.

e.g.:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 7 & -6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 7 & -6 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 5 & 3 \\ 7 & -6 \\ 3 & 4 \end{bmatrix}$$

The diagonal of a matrix

Definition:

The **diagonal of a (square) matrix** consists of the entries on the straight line from the upper left to the lower right.

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{bmatrix}$$

For non-square matrices, we take the $(1,1)$-entry, the $(2,2)$-entry, *etc.*, until we hit an edge.

The diagonal of a matrix

For non-square matrices, we take the $(1,1)$-entry, the $(2,2)$-entry, *etc.*, until we hit an edge.

$$\begin{bmatrix} a & b & c & d & e & \alpha & \beta \\ f & g & h & i & j & \gamma & \delta \\ k & l & m & n & o & \epsilon & \zeta \\ p & q & r & s & t & \eta & \theta \\ u & v & w & x & y & \psi & \omega \end{bmatrix} \qquad \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \\ \alpha & \beta & \gamma & \delta & \epsilon \\ \zeta & \eta & \theta & \psi & \omega \end{bmatrix}$$

For non-square matrices, we take the $(1,1)$-entry, the $(2,2)$-entry, *etc.*, until we hit an edge.

The identity matrix

Definition:
The $n \times n$ **identity matrix** is the $n \times n$ matrix with 1s on the diagonal, and 0s off.

e.g.: The 5×5 identity matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix is a left identity

Fact: If I is the $n \times n$ identity,
and if M is an $n \times k$ matrix
then $IM = M$.

e.g.:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix}$$

The identity matrix is a right identity

Fact: If M is an $n \times k$ matrix
and if I is the $k \times k$ identity,
then $MI = M$.

e.g.:

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix}$$

The inverse of a matrix

identity matrix

Def'n: If $AB = BA = \boxed{I}$, then we say that B is the **inverse of** A,
and write $\boxed{A^{-1}}$ for B.

Discussion:
Why can't a matrix have two inverses?

Def'n: A matrix is said to be **invertible** if it has an inverse.

Discussion:
Why must an invertible matrix be square?

Matrix conjugation

Definition:

Let $M \in \mathbb{R}^{n \times n}$.

Let $C \in \mathbb{R}^{n \times n}$ be invertible.

The **left conjugate** **of** M **by** C is CMC^{-1}.

The **right conjugate** **of** M **by** C is $C^{-1}MC$.

Definition:

Let $M, M' \in \mathbb{R}^{n \times n}$.

We say M **and** M' **are** **conjugate** if

$\exists$an invertible $C \in \mathbb{R}^{n \times n}$

s.t. $M' = CMC^{-1}$.

SKILL: Matrix conjugation

Matrix transpose

The transpose of a matrix M is denoted $\boxed{M^t}$, and is obtained by "reflecting the entries through the diagonal".

e.g.:
$$\begin{bmatrix} 4 & 1 & 2 & 6 \\ -8 & 7 & 3 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 4 & -8 & 2 \\ 1 & 7 & 1 \\ 2 & 3 & 0 \\ 6 & 8 & -1 \end{bmatrix}$$

The (i,j) entry of M^t is the (j,i) entry of M.

Fact: Let $M \in \mathbb{R}^{n\times k}$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^n$.

Then $[L_M(v)] \cdot w = v \cdot [L_{M^t}(w)]$.

"As you pull M across the dot product, it gets transposed."

Fact: Let $M \in \mathbb{R}^{n \times k}$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^n$.
Then $[L_M(v)] \cdot w = v \cdot [L_{M^t}(w)]$.

$$M := \begin{bmatrix} 4 & 1 & 2 & 6 \\ -8 & 7 & 3 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$$

$v := (3, 4, 6, 1) \in \mathbb{R}^4$

$w := (2, 4, 9) \in \mathbb{R}^3$

Fact: Let $M \in \mathbb{R}^{n \times k}$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^n$.
Then $[L_M(v)] \cdot w = v \cdot [L_{M^t}(w)]$.

Want: $\boxed{[L_M(v)] \cdot w} = v \cdot [L_{M^t}(w)]$.

Fact: Let $M \in \mathbb{R}^{n\times k}$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^n$.

Then $[L_M(v)] \cdot w = v \cdot [L_{M^t}(w)]$.

$$M := \begin{array}{cc} \begin{array}{cccc} 3 & 4 & 6 & 1 \end{array} & \\ \begin{bmatrix} 4 & 1 & 2 & 6 \\ -8 & 7 & 3 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix} & \begin{array}{c} 2 \\ 4 \\ 9 \end{array} \end{array}$$

$v := (3, 4, 6, 1) \in \mathbb{R}^4$

$w := (2, 4, 9) \in \mathbb{R}^3$

Want: $\boxed{[L_M(v)] \cdot w} = v \cdot [L_{M^t}(w)]$.

$$M^t = \begin{array}{c} \begin{array}{ccc} \color{blue}{2} & \color{blue}{4} & \color{blue}{9} \end{array} \\ \begin{bmatrix} 4 & -8 & 2 \\ 1 & 7 & 1 \\ 2 & 3 & 0 \\ 6 & 8 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ \color{blue}{3} \\ \color{blue}{4} \\ \color{blue}{6} \\ \color{blue}{1} \end{array}$$

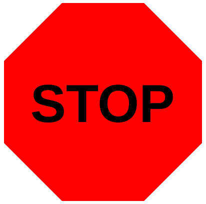
Multiply each entry by the blue number above and then multiply that by the blue number to the right.

Then add results.

$$M := \begin{array}{c} \begin{array}{cccc} \color{blue}{3} & \color{blue}{4} & \color{blue}{6} & \color{blue}{1} \end{array} \\ \begin{bmatrix} 4 & 1 & 2 & 6 \\ -8 & 7 & 3 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ \color{blue}{2} \\ \color{blue}{4} \\ \color{blue}{9} \end{array}$$

$v := (3, 4, 6, 1) \in \mathbb{R}^4$

$w := (2, 4, 9) \in \mathbb{R}^3$



$\cdots + (3)(\color{blue}{6})(\color{blue}{4}) + \cdots \qquad\qquad \cdots + (3)(\color{blue}{4})(\color{blue}{6}) + \cdots$

Want: $\boxed{[L_M(v)] \cdot w} \boxed{=} \boxed{v \cdot [L_{M^t}(w)]}$.