Financial Mathematics

Matrix types

Diagonal matrices

Definition:

A matrix is said to be a **diagonal matrix** if it's a square matrix such that all of its entries off the diagonal are zero.

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

Triangular matrices

Definition:

A matrix is an **upper triangluar matrix** if it's a square matrix such that all of its entries strictly below the diagonal are zero.

$$\begin{bmatrix} a & b & c & d & e \\ 0 & f & g & h & i \\ 0 & 0 & j & k & l \\ 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & o \end{bmatrix}$$

Exercise: Write out the only reasonable definitions of **strictly upper triangular**, **lower triangular**, **strictly lower triangular matrices**.

Symmetric matrices

Definition:

A matrix M is a **symmetric matrix**

if $M^t = M$.

$$\begin{bmatrix} a & b & c & d & e \\ b & f & g & h & i \\ c & g & j & k & l \\ d & h & k & m & n \\ e & i & l & n & o \end{bmatrix}$$

etc.

Anti-symmetric matrices

Definition:

A matrix M is an **anti-symmetric matrix** if $M^t = -M$.

$$\begin{bmatrix} 0 & b & c & d & e \\ -b & 0 & g & h & i \\ -c & -g & 0 & k & l \\ -d & -h & -k & 0 & n \\ -e & -i & -l & -n & 0 \end{bmatrix}$$

Nilpotent matrices

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 \subseteq \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition:

A (square) matrix is **nilpotent** if some power of it is zero.

Nilpotent matrices

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 \subseteq \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition:

A (square) matrix is **nilpotent** if some power of it is zero.

Nilpotent matrices

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}^4 \subseteq \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

Definition:

A (square) matrix is **nilpotent** if some power of it is zero.

e.g.: any strictly upper triangluar,
any conjugate of a str. upper triangluar

Scalar matrices

$$\underbrace{\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}}_{9I}$$

Definition: A matrix is **scalar** if it's a scalar multiple of the identity.

e.g.: $9I$

scalar → 9, identity → I

Standard nilpotent matrices

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\substack{\| \\ \ddot{N}}}$$

Definition: A matrix is **standard nilpotent** if it's square and has 1s just above the diagonal and 0s everywhere else.

e.g.: N is the 4×4 standard nilpotent.

Jordan blocks

$$\underbrace{\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}}_{\substack{\parallel \\ \because \\ B}} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: A matrix is a **Jordan block** if it's the sum of a scalar matrix and a standard nilpotent matrix.

e.g.: B, the 1×1 matrix $[4]$

$$e^x := \exp(x) := \lim_{n\to\infty} [1 + (x/n)]^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Definition: $\forall$square matrices M,

$$\boxed{e^M} := \boxed{\exp(M)} := \lim_{n\to\infty} [I + (M/n)]^n$$

$$= I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$$

Fact: If $AB = BA$,

then $e^{A+B} = e^A e^B$

Multiplication of diagonal matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} = ??$$

shape of answer??

$$= \begin{bmatrix} 1 \cdot 5 & 0 & 0 & 0 \\ 0 & 2 \cdot 6 & 0 & 0 \\ 0 & 0 & 3 \cdot 7 & 0 \\ 0 & 0 & 0 & 4 \cdot 8 \end{bmatrix}$$

Exponentiation of diagonal matrices

$$\exp\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{bmatrix} = ??$$

shape of answer??

Exponentiation of diagonal matrices

Is the answer diagonal??

(3, 3)-entry of answer??

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\frac{1}{2!}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^2 = \boxed{\frac{1}{2!}}\begin{bmatrix} 1^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & \boxed{3^2} & 0 \\ 0 & 0 & 0 & 4^2 \end{bmatrix}$$

$$\frac{1}{3!}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^3 = \boxed{\frac{1}{3!}}\begin{bmatrix} 1^3 & 0 & 0 & 0 \\ 0 & 2^3 & 0 & 0 \\ 0 & 0 & \boxed{3^3} & 0 \\ 0 & 0 & 0 & 4^3 \end{bmatrix}$$

Exponentiation of diagonal matrices

Is the answer diagonal??	Yes	1	3
(3, 3)-entry of answer??	e^3		
			$3^2/(2!)$
			$3^3/(3!)$

Exponentiation of diagonal matrices

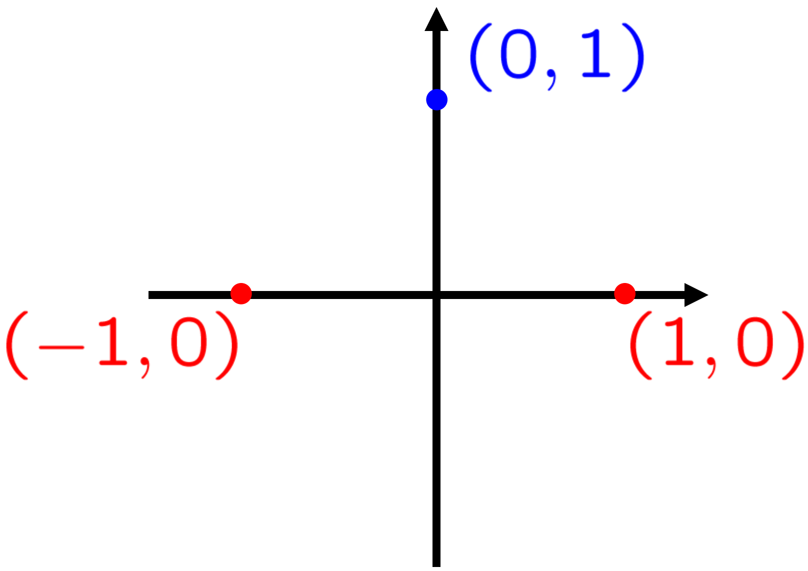
$$\exp \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^3 & 0 \\ 0 & 0 & 0 & e^4 \end{bmatrix}$$

Key idea: Diagonal matrices are much easier to study than generic matrices.

Question:

What is the 2×2 real matrix M such that L_M is reflection in the y-axis?



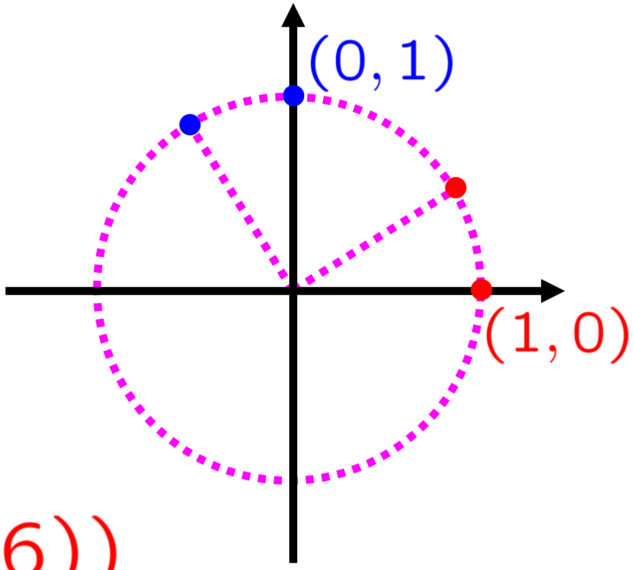
$$L_M(1,0) = (-1,0)$$

$$L_M(0,1) = (0,1)$$

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question:

What is the 2×2 real matrix M such that L_M is counterclockwise rotation by $\pi/6$ radians?



$$L_M(1,0) = (\cos(\pi/6), \sin(\pi/6))$$
$$L_M(0,1) = (-\sin(\pi/6), \cos(\pi/6))$$

$$M = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

Definition:

A matrix $M \in \mathbb{R}^{n\times n}$ is **orthogonal**

if $L_M : \mathbb{R}^n \to \mathbb{R}^n$ is distance-preserving,

i.e., if $\forall v, w \in \mathbb{R}^n$,

$$\text{dist}(L_M(v), L_M(w)) = \text{dist}(v, w).$$

e.g.:
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

Fact:

Every orthogonal 2×2 or 3×3 matrix is
either the matrix of a reflection
or the matrix of a rotation.

Fact: A matrix $M \in \mathbb{R}^{n \times n}$ is orthogonal

iff L_M is length-preserving,

i.e., iff $\forall v \in \mathbb{R}^n$, $|L_M(v)| = |v|$.

distance-preserving

$v :\to v - w$

Proof:

only if: $\forall v \in \mathbb{R}^n$,

$$|L_M(v)| = |[L_M(v)] - 0|$$

$L_M(0)$

$$= |v - 0| = |v|.$$

if: $\forall v, w \in \mathbb{R}^n$,

$$|[L_M(v)] - [L_M(w)]| = |L_M(v - w)|$$

$$= |v - w|.$$

QED

Fact: Any orthogonal matrix is dot-product-preserving

i.e.: if M is orthogonal and

length-preserving

if v and w are vectors,

then $(L_M(v)) \cdot (L_M(w)) = v \cdot w$.

Proof: Let $x := L_M(v)$ and $y := L_M(w)$.

Want: $x \cdot y = v \cdot w$

$|x| = |v|$ and $|y| = |w|$ and $|x + y| = |v + w|$.

The polarization formula:

$$v \cdot w = \frac{|v + w|^2 - |v|^2 - |w|^2}{2}$$

$$\|$$

$$x \cdot y = \frac{|x + y|^2 - |x|^2 - |y|^2}{2}$$

QED

Definition:

A matrix $M \in \mathbb{R}^{n \times n}$ is **orthogonal**
if $L_M : \mathbb{R}^n \to \mathbb{R}^n$ is distance-preserving,

Fact: A (real) square matrix M is orthogonal
if and only if
$M^t M = I$. ← the identity matrix

Proof: $M^t M = I$

if and only if

$\forall v, w, \quad [L_{M^t M}(v)] \cdot w = [L_I(v)] \cdot w$

if and only if

$\forall v, w, \quad [L_M(v)] \cdot [L_M(w)] = v \cdot w$

if and only if

M is orthogonal.

QED