Financial Mathematics

Row and column operations and linear algebra

Question:

Is there an onto $\{1,2,3\} \to \{1,2,3,4\}$?

Answer: No.

Question: Is there an onto $\mathbb{R}^3 \to \mathbb{R}^4$?

Answer: Yes. Can even be made to be continuous.

"space-filling curve" $\times \mathbb{R}^2$

Question: Is there a linear onto $\mathbb{R}^3 \to \mathbb{R}^4$?

Answer: No.

Proof:

$L : \mathbb{R}^3 \to \mathbb{R}^4$ linear, onto.

Want: Contradiction.

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$L : \mathbb{R}^3 \to \mathbb{R}^4$ linear, onto.

Want: Contradiction.

$$C = \begin{bmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \\ 0 & 0 & 0 \end{bmatrix}$$

Proof:

$L : \mathbb{R}^3 \to \mathbb{R}^4$ linear, onto.

Want: Contradiction.

Choose $M \in \mathbb{R}^{4\times 3}$ such that $L = L_M$.

Choose $E \in \mathbb{R}^{4\times 4}$, $C \in \mathbb{R}^{4\times 3}$, $E' \in \mathbb{R}^{3\times 3}$

such that $M = ECE'$,

such that C is fully canonical,

such that E is a product of elementary 4×4 matrices,

and such that E' is a product of elementary 3×3 matrices.

$$L_C = L_E^{-1} \circ L_M \circ L_{E'}^{-1}$$

$L_M : \mathbb{R}^3 \to \mathbb{R}^4$ onto,

so $L_C : \mathbb{R}^3 \to \mathbb{R}^4$ onto.

$C \in \mathbb{R}^{4\times 3}$, so $L_C(\mathbb{R}^3) \subseteq \mathbb{R}^3 \times \{0\} \subsetneq \mathbb{R}^4$,

so $L_C : \mathbb{R}^3 \to \mathbb{R}^4$ is not onto.

Contradiction. QED

Question: Is there a linear onto $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$?

Answer: No.

Theorem:

If there is an onto linear $\mathbb{R}^n \longrightarrow \mathbb{R}^k$, then $n \geq k$.

covers implies
larger in dimension

Question:

Is there a one-to-one $\{1,2,3,4\} \to \{1,2,3\}$?

Answer: No.

Question: Is there a one-to-one $\mathbb{R}^4 \to \mathbb{R}^3$?

Answer: Yes, but it can't be made to be continuous.

"invariance of domain"

Question: Is there a linear 1-1 $\mathbb{R}^4 \to \mathbb{R}^3$?

one-to-one

Answer: No.

Proof: Similar: If there were one, then there'd be one coming from a fully canonical 3×4 matrix.

Discussion: Study all fully canonical 3×4 matrices, and show that none of them yields a 1-1 linear function.

Theorem:

If there is a one-to-one linear $\mathbb{R}^n \to \mathbb{R}^k$, then $n \le k$.

fits inside implies smaller in dimension

Theorem:

If there is an onto linear $\mathbb{R}^n \to \mathbb{R}^k$, then $n \ge k$.

covers implies larger in dimension

1-1, onto function

linear bijection

Theorem:

If there is an isomorphism $\mathbb{R}^n \to \mathbb{R}^k$, then $n = k$.

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Corollary:

Let V be a subspace of some Euclidean space.

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If there is an isomorphism $\mathbb{R}^n \to \mathbb{R}^k$, then $n = k$.

Corollary:

Let V be a subspace of some Euclidean space. Then any two bases of V have the same size.

Proof:

Let $(v_1, \ldots, v_n) \in V^n$ be an ordered basis of V.

Let $(v'_1, \ldots, v'_k) \in V^k$ be another. Want: $n = k$

Define isoms $F : \mathbb{R}^n \to V$ and $F' : \mathbb{R}^k \to V$ by

$$F(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n$$
$$F'(a_1, \ldots, a_k) = a_1 v'_1 + \cdots + a_k v'_k.$$

Then $(F')^{-1} \circ F : \mathbb{R}^n \to \mathbb{R}^k$ is an isomorphism.

Then $n = k$. QED

Theorem:

If there is an isomorphism $\mathbb{R}^n \to \mathbb{R}^k$, then $n = k$.

Corollary:

Any set of $k+1$ vectors in $\mathbb{R}^k$ is l.d.

Proof: Let $B := \{v_0, \ldots, v_k\} \subset \mathbb{R}^k$ be a set of $k+1$ vectors in $\mathbb{R}^k$.

Say B is l.i. Want: Contradiction

Let $S := \langle B \rangle$ be the span of B.

Then B is a basis of S.

Define $F : \mathbb{R}^{k+1} \to S$ by

$$F(a_0, \ldots, a_k) = a_0 v_0 + \cdots + a_k v_k.$$

Then $F : \mathbb{R}^{k+1} \to S$ is an isomorphism,

so $F : \mathbb{R}^{k+1} \to \mathbb{R}^k$ is 1-1.

Then $k+1 \leq k$. Contradiction. QED

Corollary:

Any set of $k+1$ vectors in $\mathbb{R}^k$ is l.d.

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Any subspace of $\mathbb{R}^k$ has a basis.

l.i. spanning set

Proof: Let S be a subspace of $\mathbb{R}^k$.

Want: S has a l.i. spanning set.

Every l.i. subset of S has size $\leq k$.

Let B be a l.i. subset of S of maximal size.

Want: B is a spanning set of S.

Want: $\langle B \rangle = S$ Know: $\langle B \rangle \subseteq S$

Suppose $v \in S$ and $v \notin \langle B \rangle$.

Want: Contradiction.

Fact: Let F be a finite subset of $\mathbb{R}^n$.
Suppose F is l.i.
Say $v \in \mathbb{R}^n \backslash \langle F \rangle$, *i.e.*, $v \in \mathbb{R}^n$ and $v \notin \langle F \rangle$.
Then $F \cup \{v\}$ is l.i.

Corollary:
Any subspace of $\mathbb{R}^k$ has a basis.

l.i. spanning set

Proof: Let S be a subspace of $\mathbb{R}^k$.
Want: S has a l.i. spanning set.
Every l.i. subset of S has size $\leq k$.
Let B be a l.i. subset of S
of maximal size.
Want: B is a spanning set of S.
Want: $\langle B \rangle = S$ Know: $\langle B \rangle \subseteq S$
Suppose $v \in S$ and $v \notin \langle B \rangle$.
Want: Contradiction.
Then $B \cup \{v\}$ is a l.i. subset of S.
Contradiction. QED

Definition:

Let V be a subspace of some Euclidean space.

The **dimension of** V, denoted $\dim(V)$,

is the size of any basis of V.

e.g.:

$$S := \langle (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \rangle$$

$\{(1,3,4,2), \quad (2,1,2,-1)\}$ is a basis of S.

Then $\dim(S) = 2$.

SKILL: Determine if vect is in span of others.

Question:
$(1,2,3,4) \in \langle (1,1,1,1),(2,3,4,5),(1,0,-1,2)\rangle$?

Solution:
Make vectors the rows of a matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

Key point:
Elementary row ops don't change row span.
Bring to row canonical form, then re-ask.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Question:

$(1, 2, 3, 4) \in \langle (1, 0, -1, 0), (0, 1, 2, 0), (0, 0, 0, 1) \rangle$?

Coefficients: 1 2 4

Result: $(1, 2, 3, 4)$

Answer: Yes.

Fact: Let F be a finite subset of $\mathbb{R}^n$.
Suppose F is l.i.
Say $v \in \mathbb{R}^n \backslash \langle F \rangle$, *i.e.*, $v \in \mathbb{R}^n$ and $v \notin \langle F \rangle$.
Then $F \cup \{v\}$ is l.i.

SKILL: Extract a basis from a spanning set.

Want to extract a basis from $v_1, \ldots, v_m$.

ALGORITHM:

1. Throw out all zero vectors.
2. Determine if $v_2 \in \langle v_1 \rangle$.
   If yes, throw it out; otherwise, keep it.
3. Determine if $v_3 \in \langle v_1, v_2 \rangle$.
   If yes, throw it out; otherwise, keep it.
4. Determine if $v_4 \in \langle v_1, v_2, v_3 \rangle$.
   If yes, throw it out; otherwise, keep it.

etc.

Fact: Let F be a finite subset of $\mathbb{R}^n$.
Suppose F is l.i.
Say $v \in \mathbb{R}^n \backslash \langle F \rangle$, *i.e.*, $v \in \mathbb{R}^n$ and $v \notin \langle F \rangle$.
Then $F \cup \{v\}$ is l.i.

SKILL: Determine if a set of vects is l.i.
and, if not, express one of them
as a l.c. of the rest.

Want to check if $v_1, \ldots, v_m$ are l.i.

ALGORITHM:

1. If there's a zero vector, then write it as a l.c. of the others, with 0 coefficients.
2. Check if $v_2 \in \langle v_1 \rangle$.
   If so, write v_2 as a l.c. of v_1.
   If not, v_1, v_2 are l.i.
3. Check if $v_3 \in \langle v_1, v_2 \rangle$.
   If so, write v_3 as a l.c. of v_1, v_2.
   If not, v_1, v_2, v_3 are l.i.

etc.

SKILL: Find the ker and im of a lin. transf.

KEY POINT:
Elementary row ops don't change ker.
Elementary col. ops don't change image.

Note: If E is invertible, then

$$\ker(L_{EA}) = \ker(L_A)$$

and $$\mathrm{im}(L_{AE}) = \mathrm{im}(L_A).$$

SKILL: Find the ker and im of a lin. transf.

KEY POINT:

Elementary row ops don't change ker.
Elementary col. ops don't change image.

Find ker of a row canonical matrix.
Find image of a column canonical matrix.

e.g.:

Find kernel of $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find image of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

Find kernel of $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A :=$

Find kernel of $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find kernel of $A := \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$L_A(a,b,c,d) = (a-c, b+2c, d)$$

$$\ker(L_A) = \{(a,b,c,d) \,|\, a = c, b = -2c, d = 0\}$$

basis: $(1,-2,1,0)$ ← $c = 1$

Find image of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} =: B$

$$L_B(a, b, c, d) = (a, b, 2b)$$

$$L_B(\mathbb{R}^4) = \{(a, b, 2b) \mid a, b \in \mathbb{R}\}$$

basis: $(1, 0, 0), (0, 1, 2)$

$a = 1$, $b = 0$ → $(1, 0, 0)$

$a = 0$, $b = 1$ → $(0, 1, 2)$

SKILL: Find dim of ker and im of a lin. transf.

KEY POINT:
Elementary row and col. ops change neither dim of kernel nor dim of image.

Note: If E is invertible, then

$$\ker(L_{EA}) = \ker(L_A),$$

$$\mathrm{im}(L_{AE}) = \mathrm{im}(L_A),$$

$$L_E(\ker(L_{AE})) = \ker(L_A)$$

and

$$\mathrm{im}(L_{EA}) = L_E(\mathrm{im}(L_A)).$$

SKILL: Find dim of ker and im of a lin. transf.

KEY POINT:
Elementary row and col. ops change neither dim of kernel nor dim of image.

Find dimensions of kernel and image of a fully canonical matrix.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dim ker $=$ number of zero columns in $C = 1$

dim im $=$ number of nonzero rows in $C = 2$

$\|$

number of nonzero columns in C

Theorem: Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be linear.
Then $[\dim(\ker(L))] + [\dim(\mathrm{im}(L))] = n$.

Intuition: L kills off some dimensions from $\mathbb{R}^n$.
The remaining dimensions cover the image.

$$\dim(\mathrm{im}(L)) = n - [\dim(\ker(L))]$$

Find dimensions of kernel and image of a fully canonical matrix.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Then $[\dim(\ker(L))] + [\dim(\operatorname{im}(L))] = n$.

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The remaining dimensions cover the image.

$$\dim(\operatorname{im}(L)) = n - [\dim(\ker(L))]$$

Proof: Choose $M \in \mathbb{R}^{k \times n}$ s.t. $L = L_M$.
Reduce M to a fully canonical matrix $C \in \mathbb{R}^{k \times n}$.

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dim ker = number of zero columns in $C = 1$
dim im = number of nonzero rows in $C = 2$
||
number of nonzero columns in C

Theorem: Let $L:\mathbb{R}^n \to \mathbb{R}^k$ be linear.
Then [dim(ker(L))] + [dim(im(L))] $= n$.

Intuition: L kills off some dimensions from $\mathbb{R}^n$.
The remaining dimensions cover the image.

$$\text{dim(im}(L)) = n - [\text{dim(ker}(L))]$$

Proof: Choose $M \in \mathbb{R}^{k\times n}$ s.t. $L = L_M$.

Reduce M to a fully canonical matrix $C \in \mathbb{R}^{k\times n}$.

dim ker = number of zero columns in C

dim im = number of nonzero rows in C

||

number of nonzero columns in C

[dim ker]+[dim im]

dim ker = number of zero columns in C

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||

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Theorem: Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be linear.
Then $[\dim(\ker(L))] + [\dim(\mathrm{im}(L))] = n$.

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$$\dim(\mathrm{im}(L)) = n - [\dim(\ker(L))]$$

Proof: Choose $M \in \mathbb{R}^{k \times n}$ s.t. $L = L_M$.

Reduce M to a fully canonical matrix $C \in \mathbb{R}^{k \times n}$.

dim ker = number of zero columns in C

dim im = number of nonzero rows in C

||

number of nonzero columns in C

[dim ker]+[dim im] = number of columns in C

$= n$

QED

Theorem: Let $M \in \mathbb{R}^{n \times n}$.
Then the following are equivalent:

(•) M is a product of elem. matrices
(•) M is reducible to I via elem. row & col. ops
(•) M is reducible to I via elem. row ops
(•) M is reducible to I via elem. col. ops
(•) M is invertible
(•) M has a left inverse
(•) M has a right inverse
(•) $L_M : \mathbb{R}^n \to \mathbb{R}^n$ is onto
(•) $\ker(M) = \{0\}$
(•) L_M is 1-1
(•) $\text{im}(M) = \mathbb{R}^{n \times 1}$

Proof: All eleven properties are equivalent to:
(•) The fully canon. form of M is the identity.

QED

Fact: If a matrix has a left inverse and a right inverse, then they are equal.

Proof: Say A is a left inverse for B, and C is a right inverse for B.

Want: $A = C$

identity matrix

$$A = AI = A(BC) = (AB)C = IC = C$$

QED

Fact: $X, Y \in \mathbb{R}^{n\times n}$, $\quad XY = I \quad \Rightarrow \quad YX = I$

Proof: X has a right inverse, namely Y.

X has a left inverse.

If a matrix has a left inverse and a right inverse, then they are equal.

Y is a left inverse for X. QED

$\exists$left inverse iff $\exists$right inverse

DISCUSSION

Can (short wide)$\times$(tall thin)$= I$?

Can (tall thin)$\times$(short wide)$= I$?

Fact: If a matrix has a left inverse
and a right inverse,
then they are equal.

DISCUSSION

Can a non-square matrix have a left inverse?
Can a non-square matrix have a right inverse?
Can a non-square matrix have both at once?

Fact: $X, Y \in \mathbb{R}^{n \times n}, \quad XY = I \quad \Rightarrow \quad YX = I$

Proof: X has a right inverse, namely Y.
X has a left inverse.
If a matrix has a left inverse
and a right inverse,
then they are equal.
Y is a left inverse for X. QED

∃left inverse
iff
∃right inverse

DISCUSSION

Can (short wide)×(tall thin)$= I$?
Can (tall thin)×(short wide)$= I$?

Inversion of matrices

RECALL: Elem. matrices are invertible

Add second row to third

e.g.:

$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =: A$$

Elem. matrix

Subtract second row from third

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} =: B$$

Exercise: Check that $AB = BA = I$.

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 4 & -9 & -1 \\ 6 & -13 & 4 \end{bmatrix}}_{A} \xrightarrow{E_1} \xrightarrow{E_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -13 \\ 0 & -1 & -14 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & -1 & -14 \end{bmatrix}$$

E_1: $-4\times$R1 $\rightarrow$R2

E_2: $-6\times$R1 $\rightarrow$R3

E_3: $-1\times$R2

E_4: $1\times$R2$\rightarrow$R3

$$\xrightarrow{E_4} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{E_5} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_6} \begin{bmatrix} 1 & 0 & 29 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{bmatrix}$$

E_5: $-1\times$R3

E_6: $2\times$R2 $\rightarrow$R1

E_7: $-13\times$R3 $\rightarrow$R2

E_8: $-29\times$R3 $\rightarrow$R1

$$\xrightarrow{E_7} \xrightarrow{E_8} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I$$

$E_8E_7E_6E_5E_4E_3E_2E_1I$ is a left inverse to A

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{I} \xrightarrow{E_1} \xrightarrow{E_2} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$$

E_1: $-4\times$R1 $\rightarrow$R2 E_2: $-6\times$R1 $\rightarrow$R3 E_3: $-1\times$R2

E_4: $1\times$R2$\rightarrow$R3

$$\begin{bmatrix} 9 & -2 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \xleftarrow{E_6} \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \xleftarrow{E_5} \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

E_6: $2\times$R2 $\rightarrow$R1 E_5: $-1\times$R3

E_7: $-13\times$R3 $\rightarrow$R2

E_8: $-29\times$R3 $\rightarrow$R1

$$\underbrace{\begin{bmatrix} -49 & -31 & 29 \\ -22 & -14 & 13 \\ 2 & 1 & -1 \end{bmatrix}}_{\text{left inverse to } A}$$

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I$$

$E_8E_7E_6E_5E_4E_3E_2E_1I$ is a left inverse to A

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 4 & -9 & -1 \\ 6 & -13 & 4 \end{bmatrix}}_{A}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \xrightarrow{-4\times R1\rightarrow R2} \xrightarrow{-6\times R1\rightarrow R3} \left[\begin{matrix} 1 & -2 & 3 \\ 0 & -1 & -13 \\ 0 & -1 & -14 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -6 & 0 & 1 \end{matrix}\right]$$

$$\downarrow -1\times R2$$

$$\left[\begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & 0 & -1 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & -1 & 1 \end{matrix}\right] \xleftarrow{1\times R2\rightarrow R3} \left[\begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & -1 & -14 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -6 & 0 & 1 \end{matrix}\right]$$

$$\downarrow -1\times R3$$

$$\left[\begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{matrix}\right] \xrightarrow{2\times R2\rightarrow R1} \left[\begin{matrix} 1 & 0 & 29 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{matrix}\right|\left.\begin{matrix} 9 & -2 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{matrix}\right]$$

$$\downarrow -13\times R3 \rightarrow R2$$

$$\downarrow -29\times R3\rightarrow R1$$

$$\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\underbrace{\left.\begin{matrix} -49 & -31 & 29 \\ -22 & -14 & 13 \\ 2 & 1 & -1 \end{matrix}\right]}_{B}$$

$$\overbrace{\begin{bmatrix}1 & -2 & 3\\ 4 & -9 & -1\\ 6 & -13 & 4\end{bmatrix}}^{A}$$

$$\begin{bmatrix} & & 29\\ & & 13\\ 2 & 1 & -1\end{bmatrix}$$

$$\overbrace{\begin{bmatrix}1 & -2 & 3\\ 4 & -9 & -1\\ 6 & -13 & 4\end{bmatrix}}^{A}$$

$$\underbrace{\begin{bmatrix}-49 & -31 & 29\\ -22 & -14 & 13\\ 2 & 1 & -1\end{bmatrix}}_{B}$$

$$\overbrace{\begin{bmatrix} -49 & -31 & 29 \\ -22 & -14 & 13 \\ 2 & 1 & -1 \end{bmatrix}}^{B} \overbrace{\begin{bmatrix} 1 & -2 & 3 \\ 4 & -9 & -1 \\ 6 & -13 & 4 \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{I}$$

$$A = (E_1)^{-1} \cdots (E_8)^{-1}$$

so A is invertible, with inverse $E_8 \cdots E_1$.

$$AB = A\,B\,A\,A^{-1} = A\,I\,A^{-1} = I$$

$$AB = BA = I$$

A is the (two-sided) inverse of B.

SKILLS:

Determine if a matrix is invertible.

Given an invertible matrix, find its inverse.

SKILL:

Given a system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

with all the a_{jk} fixed, but with, each day, a new choice of $b_1, \ldots, b_n$, find an efficient algorithm for finding solutions.

$$A := [a_{jk}]_{k=1,\ldots,n}^{j=1,\ldots,n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$b := [b_k]_{k=1,\ldots,n} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad x := [x_k]_{k=1,\ldots,n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

SKILL: matrix equation

Given a system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$Ax = b$$

with all the a_{jk} fixed, but with, each day, a new choice of $b_1, \ldots, b_n$, find an efficient algorithm for finding solutions.

$$A := [a_{jk}]_{k=1,\ldots,n}^{j=1,\ldots,n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$b := [b_k]_{k=1,\ldots,n} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$x := [x_k]_{k=1,\ldots,n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

SKILL: matrix equation

Given a matrix equation

$$Ax = b \qquad Ax = b$$

with all the a_{jk} (A) fixed, but with, each day, a new choice of $b_1, \ldots, b_n$ (b), find an efficient algorithm for finding solutions.

$$A := [a_{jk}]_{k=1,\ldots,n}^{j=1,\ldots,n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$b := [b_k]_{k=1,\ldots,n} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$x := [x_k]_{k=1,\ldots,n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

SKILL:

Given a matrix equation

$$Ax = b$$

A

with A fixed, but with, but with, each day, a new choice of b, find an b find an efficient algorithm for finding solutions.

$$A := [a_{jk}]_{k=1,\ldots,n}^{j=1,\ldots,n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$b := [b_k]_{k=1,\ldots,n} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$x := [x_k]_{k=1,\ldots,n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

SKILL:
Given a matrix equation

$$Ax = b$$

with A fixed, but with, each day, a new choice of b, find an efficient algorithm for finding solutions.

Method:
Find A^{-1}, assuming A is invertible.
Use $x = A^{-1}b$.

STOP