Financial Mathematics

Principal component analysis

Also: $\text{dist}\ (\ v_p\ ,\ \langle v_1, \ldots, v_{p-1} \rangle)\ =\ |v_p|$

For most matrices, to know if one row is nearly a linear combination of the rest is a hard problem. Not so if the rows are pw-orthog.

Discussion:

Can a collection of pw-orthogonal vectors be linearly dependent?

Yes, if some of them are zero . . .

How about if they're all nonzero? NO.

$$c_1 v_1 + \cdots + c_p v_p = 0$$

$$(c_1 v_1 + \cdots + c_p v_p) \cdot v_j = 0$$

$$c_j \cancel{(v_j \cdot v_j)} = 0$$

$$c_j = 0$$

Principal Component Analysis

Singular Value Decomposition

(a.k.a. Singular Value Decomposition)

Principal Component Analysis Theorem:

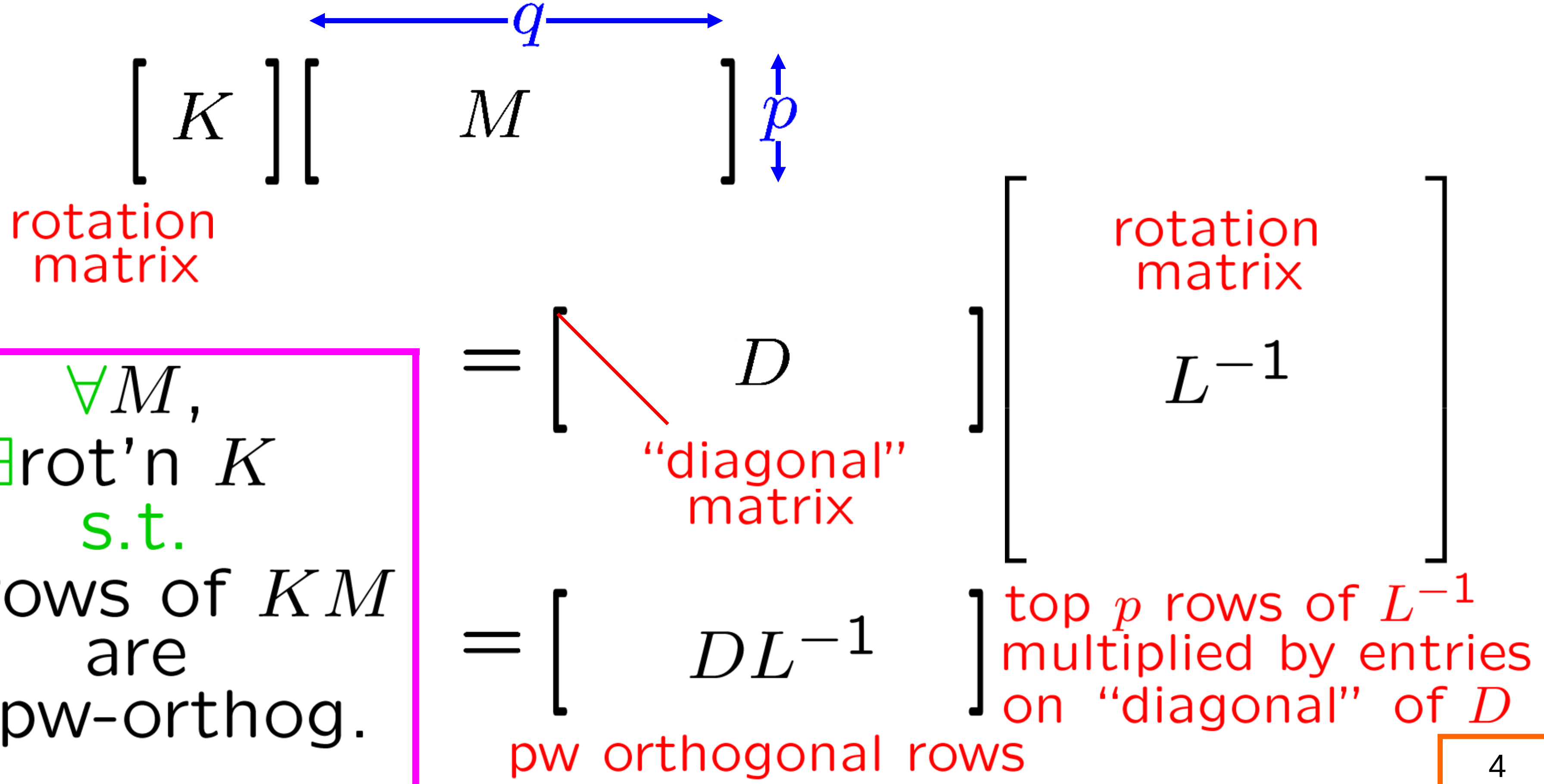
Let $M \in \mathbb{R}^{p \times q}$.

Then there are rotation matrices

$K \in \mathbb{R}^{p \times p}$ and $L \in \mathbb{R}^{q \times q}$

s.t. KML is "diagonal".

but not square
$p \times q$



$$[\,K\,][\,M\,] = [\,D\,][\,L^{-1}\,] = [\,DL^{-1}\,]$$

$\forall M$,
$\exists$rot'n K
s.t.
rows of KM
are
pw-orthog.

(a.k.a. Singular Value Decomposition)

Principal Component Analysis Theorem:

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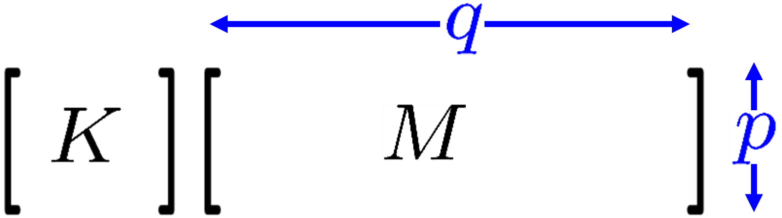

pw orthogonal rows

rotation
matrix

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Any matrix, after some post-processing, can be made to have pw orthogonal rows.

(a.k.a. Singular Value Decomposition)

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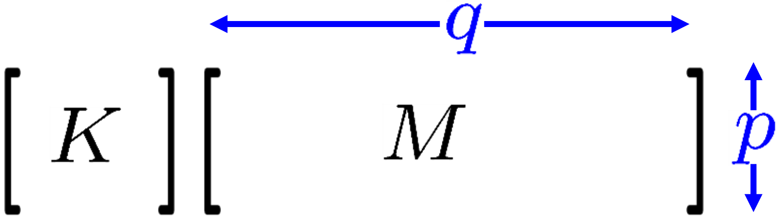

pw orthogonal rows

rotation
matrix

$\forall M$,
$\exists$rot'n K
s.t.
rows of KM
are
pw-orthog.

"post-processing" =
left multiplication by
a rotation matrix

Any matrix, after some post-processing, can be made to have pw orthogonal rows.

(a.k.a. Singular Value Decomposition)

Principal Component Analysis Theorem:

Let $M \in \mathbb{R}^{p \times q}$.

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---

dimensionless

Say we make 17 measurements at time 0, then again at time 1, then again at time 2, etc., until time $4,999$.

Assemble data into a matrix $M \in \mathbb{R}^{17 \times 5000}$.

Each row is a "time series".

If we changed our instruments to produce KM, then the 17 "time series" would be pw orthog.

(uncorrelated)

If 12 of the 17 are zero, or close to zero,
we'd say that the other 5
are the "important factors".

(a.k.a. Singular Value Decomposition)

Principal Component Analysis Theorem:

Let $M \in \mathbb{R}^{p \times q}$.

Then there are rotation matrices

$K \in \mathbb{R}^{p \times p}$ and $L \in \mathbb{R}^{q \times q}$

s.t. KML is "diagonal".

but not square
$p \times q$

Say we track 17 asset returns at time 0, then again at time 1, then again at time 2, etc., until time $4,999$.

Assemble data into a matrix $M \in \mathbb{R}^{17 \times 5000}$.

Each row of K describes an index.

If we watch the returns of the 17 indexes, then the 17 "time series" would be pw orthog.
(uncorrelated)

If 12 of the 17 are zero, or close to zero,
we'd say that the other 5
are the "important factors".

PCA / SVD Theorem: Let $M \in \mathbb{R}^{p \times q}$.

Then $\exists$rotation matrices

$$K \in \mathbb{R}^{p \times p} \quad \text{and} \quad L \in \mathbb{R}^{q \times q}$$

s.t. KML is "diagonal".

---

e.g.:

$$\begin{bmatrix} -0.98 & 0.01 & 1.03 & -9.21 & 3.43 \\ -1.02 & -0.03 & 1 & -9.24 & 3.42 \end{bmatrix}$$

---

ordinary row & col. operations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We'll see, via PCA/SVD, in a moment. . .

rotational left & right multiplications

$$\begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha \approx \sqrt{\Sigma_{i=1}^{10} (\cdots)^2} \qquad \varepsilon \approx 0$$

$$M := \begin{bmatrix} -0.98 & 0.01 & 1.03 & -9.21 & 3.43 \\ -1.02 & -0.03 & 1 & -9.24 & 3.42 \end{bmatrix}$$

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$$K \approx \begin{bmatrix} 0.71 & 0.71 \\ -0.71 & 0.71 \end{bmatrix}$$

Let's try to use this to show that the data in M is driven by only one "factor", but distorted by "noise"...

$$L \approx \begin{bmatrix} -0.1 & -0.56 & -0.09 & -0.8 & 0.19 \\ 0 & -0.6 & -0.54 & 0.37 & -0.46 \\ 0.1 & -0.49 & 0.83 & 0.2 & -0.17 \\ -0.93 & -0.1 & 0.05 & 0.24 & 0.26 \\ 0.34 & -0.28 & -0.13 & 0.36 & 0.81 \end{bmatrix}$$

L and K are rotation matrices.

$$KML \approx \begin{bmatrix} 14.06 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 & 0 \end{bmatrix}$$

Thanks to Carl Hagen

$$M := \begin{bmatrix} -0.98 & 0.01 & 1.03 & -9.21 & 3.43 \\ -1.02 & -0.03 & 1 & -9.24 & 3.42 \end{bmatrix}$$

$$K \approx \begin{bmatrix} 0.71 & 0.71 \\ -0.71 & 0.71 \end{bmatrix}$$

Let's try to use this to show that the data in M is driven by only one "factor", but distorted by "noise" . . .

$$KM \approx \begin{bmatrix} -1.42 & -0.01 & 1.44 & -13.10 & 4.86 \\ \boxed{-0.03} & \boxed{-0.03} & \boxed{-0.02} & \boxed{-0.02} & \boxed{-0.01} \end{bmatrix}$$

zero out the small rows

$$\approx \begin{bmatrix} -1.42 & -0.01 & 1.44 & -13.10 & 4.86 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M \approx K^{-1} \begin{bmatrix} -1.42 & -0.01 & 1.44 & -13.10 & 4.86 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \\ -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \end{bmatrix}$$

$$M := \begin{bmatrix} -0.98 & 0.01 & 1.03 & -9.21 & 3.43 \\ -1.02 & -0.03 & 1 & -9.24 & 3.42 \end{bmatrix}$$

$$M \approx \begin{bmatrix} -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \\ -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \end{bmatrix}$$

Rows only span one dimension.

$$M \approx$$

$$\begin{bmatrix} -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \\ -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \end{bmatrix}$$

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$$M \approx \begin{bmatrix} -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \\ -1.01 & -0.01 & 1.02 & -9.30 & 3.45 \end{bmatrix}$$

Rows only span one dimension.

Only “factor” is important;
everything else is “noise”.

PCA (SVD) Theorem: Let $M \in \mathbb{R}^{p\times q}$.
Then ∃rotation matrices
$K \in \mathbb{R}^{p\times p}$ and $L \in \mathbb{R}^{q\times q}$
s.t. KML is "diagonal".

Easier version: Let $M \in \mathbb{R}^{p\times q}$.
Then ∃a rotation matrix $K \in \mathbb{R}^{p\times p}$
s.t. $\underline{KM \text{ has pairwise orthogonal rows}}$.

Pf of easier version:

MM^t is symmetric.

Spectral Theorem

$\boxed{\text{Choose a rotation matrix } K \text{ s.t. } K(MM^t)K^t \text{ is diagonal.}}$

$K(MM^t)K^t$ diagonal

//

All off-diagonal entries of $(KM)(KM)^t$ are 0.
If $j,k \in [1,p]$ are integers, and $j \neq k$,
then [jth row of KM]·[kth col. of $(KM)^t$] $= 0$,
so [jth row of KM]·[kth row of KM] $= 0$.
KM has pw-orthogonal rows. QED

PCA (SVD) Theorem: Let $M \in \mathbb{R}^{p\times q}$.
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Easier version: Let $M \in \mathbb{R}^{p\times q}$.
Then $\exists$a rotation matrix $K \in \mathbb{R}^{p\times p}$
s.t. KM has pairwise orthogonal rows,
with all zero rows at bottom.

---

e.g.: Say $\overset{\text{rotation}}{K_0}M = \begin{bmatrix} -1 & 0 & 1 \\ \multicolumn{3}{c}{\boxed{0 \quad\quad 0 \quad\quad 0}} \\ 0 & 2 & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \multicolumn{3}{c}{\boxed{0 \quad\quad 1 \quad\quad 0}} \end{bmatrix}}_{K \leftarrow \text{orthogonal, but not rotation}} K_0 M = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ \multicolumn{3}{c}{\boxed{0 \quad\quad 0 \quad\quad 0}} \end{bmatrix}$$

PCA (SVD) Theorem: Let $M \in \mathbb{R}^{p\times q}$.

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$$\text{e.g.:} \quad \text{Say } \overset{\text{rotation}}{\searrow} K_0 M = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \boxed{0 \quad -1 \quad 0} \end{bmatrix}}_{K \leftarrow \text{rotation}} K_0 M = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

QED

PCA (SVD) Theorem: Let $M \in \mathbb{R}^{p \times q}$.
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s.t. KM has pairwise orthogonal rows,
with all zero rows at bottom.

Remark: $X \in \mathbb{R}^{p \times q}$ has pw orthog. non0 rows
$\Rightarrow$ $\exists$diagonal $D \in \mathbb{R}^{p \times p}$, $\exists Y \in \mathbb{R}^{p \times q}$
s.t. Y has orthonormal rows and $X = DY$.

---

e.g.:

$$\overset{X}{\overset{\|}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}}} = \overbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}}^{\overset{D}{\|}} \overbrace{\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}}^{\overset{Y}{\|}}$$

QED

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Remark: $X \in \mathbb{R}^{p\times q}$ has pw orthog. non0 rows
$\Rightarrow$ $\exists$ "diagonal" $D_1 \in \mathbb{R}^{p\times q}$, $\exists Y_1 \in \mathbb{R}^{q\times q}$
s.t. Y_1 has orthonormal rows and $X = D_1Y_1$.
Y_1 is orthogonal.

$p \leq q$

$$\underset{X}{\underset{/\!/}{\begin{bmatrix}-1&0&1\\0&2&0\end{bmatrix}}} = \begin{bmatrix}\sqrt{2}&0\\0&2\end{bmatrix} \begin{bmatrix}-1/\sqrt{2}&0&1/\sqrt{2}\\0&1&0\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}\sqrt{2}&0&\boxed{0}\\0&2&\boxed{0}\end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix}-1/\sqrt{2}&0&1/\sqrt{2}\\0&1&0\\1/\sqrt{2}&0&1/\sqrt{2}\end{bmatrix}}_{Y_1}$$

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Y_1 is orthogonal.

$p \leq q$

$\begin{matrix}-1 & 0 & 1\end{matrix}$

$$\begin{bmatrix}-1 & 0 & 1\\ 0 & 2 & 0\end{bmatrix} = \begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}$$

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QED

still “diagonal”

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{-\sqrt{2}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} +1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Remark: $X \in \mathbb{R}^{p\times q}$ has pw orthog. non0 rows

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Y_1 is orthogonal.

Can choose Y_1 to be a rotation matrix.

PCA (SVD) Theorem: Let $M \in \mathbb{R}^{p\times q}$.
Then $\exists$rotation matrices
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s.t. KML is "diagonal".

Pf of PCA Theorem: $[\,KM =: X = D_1 Y_1\,] \times Y_1^{-1}$

$$KML = KMY_1^{-1} = D_1$$

$$L := Y_1^{-1}$$

QED

Easier version: Let $M \in \mathbb{R}^{p\times q}$.
Then $\exists$a rotation matrix $K \in \mathbb{R}^{p\times p}$
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$$\forall A, B \in \mathbb{R}^{p \times q},\ \boxed{\mathrm{dist}(A, B)} := \sqrt{\sum_{i,j} (A_{ij} - B_{ij})^2}$$

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$K \in \mathbb{R}^{p \times p}$ orthogonal

$$\Rightarrow \forall A, B \in \mathbb{R}^{p \times q},\ \text{dist}(KA, KB) = \text{dist}(A, B)$$

$M' := KM$ has pairwise $\overset{\text{uncorrelated}}{\overline{\text{orthogonal}}}$ rows

$$\forall \text{integers } i \in [1, p],\ s_i := \sqrt{\sum_j (M'_{ij})^2}$$ the singular values of M

e.g.: $$M = \begin{bmatrix} -0.98 & 0.01 & 1.03 & -9.21 & 3.43 \\ -1.02 & -0.03 & 1 & -9.24 & 3.42 \end{bmatrix}$$

Thanks to Carl Hagen:
Singular values are 14.06, 0.05.

Easier version: Let $M \in \mathbb{R}^{p\times q}$.
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the singular values of M

$\tilde{M}'$ obtained from M' by zeroing "small" rows

noise

$\text{dist}(M', \tilde{M}')$ small

$$\text{dist}(\underbrace{K^{-1}M'}_{\substack{\| \\ M}}, \underbrace{K^{-1}\tilde{M}'}_{\substack{\| \\ .. \\ \tilde{M}}}) \text{ small}$$

$p_0 :=$ number of non0 rows in $\tilde{M}'$

$$\tilde{M} \longleftarrow \boxed{p_0 \text{ uncorrelated ``factors''}}$$

KRL “diag”

$$\begin{array}{lc|ccccc|l}
 & \text{movie 0} & \text{movie 1} & \text{movie 2} & \text{movie 3} & \cdots & \text{movie } q & \\
\text{person 0} & ?? & r_{01} & r_{02} & r_{03} & \cdots & r_{0q} & \\
\text{person 1} & r_{10} & r_{11} & r_{12} & r_{13} & \cdots & r_{1q} & \\
\text{person 2} & r_{20} & r_{21} & r_{22} & r_{23} & \cdots & r_{2q} & \\
\text{person 3} & r_{30} & r_{31} & r_{32} & r_{33} & \cdots & r_{3q} & =: R \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
\text{person } p & r_{p0} & r_{p1} & r_{p2} & r_{p3} & \cdots & r_{pq} &
\end{array}$$

$$\text{person' } t \;:=\; \sum_i K_{ti} \cdot (\text{person } i) \qquad r'_{0u} := \sum_j (r_{0j}) \cdot L_{ju}$$

$t = 1, \ldots, p$

$$\text{movie' } u := \sum_j (\text{movie } j) \cdot L_{ju}$$

$u = 1, \ldots, q$

KRL "diag"

$R' = KRL$

$$\begin{array}{ccccccc}
 & \text{movie 0} & \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie' } q \\
\text{person 0} & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\text{person' 1} & r'_{10} & r'_{11} & r'_{12} & r'_{13} & \cdots & r'_{1q} \\
\text{person' 2} & r'_{20} & r'_{21} & r'_{22} & r'_{23} & \cdots & r'_{2q} \\
\text{person' 3} & r'_{30} & r'_{31} & r'_{32} & r'_{33} & \cdots & r'_{3q} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & r'_{p1} & r'_{p2} & r'_{p3} & \cdots & r'_{pq}
\end{array}$$

$$\left[\begin{array}{ccccc}
r'_{11} & r'_{12} & r'_{13} & \cdots & r'_{1q} \\
r'_{21} & r'_{22} & r'_{23} & \cdots & r'_{2q} \\
r'_{31} & r'_{32} & r'_{33} & \cdots & r'_{3q} \\
\vdots & \vdots & \vdots & & \vdots \\
r'_{p1} & r'_{p2} & r'_{p3} & \cdots & r'_{pq}
\end{array}\right] =: R'$$

$$\text{person' } t := \sum_i K_{ti} \cdot (\text{person } i)$$

$$r'_{0u} := \sum_j (r_{0j}) \cdot L_{ju}$$

$t = 1, \ldots, p$

$$r'_{t0} := \sum_i K_{ti} \cdot (r_{i0})$$

$$\text{movie' } u := \sum_j (\text{movie } j) \cdot L_{ju}$$

$u = 1, \ldots, q$

$$\begin{array}{l c c c c c c}
 & \text{movie 0} & \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie' } q \\
\text{person 0} & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\text{person' 1} & r'_{10} & s_1 & 0 & 0 & \cdots & 0 \\
\text{person' 2} & r'_{20} & 0 & s_2 & 0 & \cdots & 0 \\
\text{person' 3} & r'_{30} & 0 & 0 & s_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & 0 & 0 & 0 & \cdots & ??
\end{array}$$

singular values

$$\text{person' } t \underset{t=1,\dots,p}{} := \sum_i K_{ti} \cdot (\text{person } i)$$

$$\boxed{K,\ L\ \text{rotation}}$$

$$\text{movie' } u \underset{u=1,\dots,q}{} := \sum_j (\text{movie } j) \cdot L_{ju}$$

$$
\begin{array}{l|c|cccccc}
 & \text{movie } 0 & \text{movie' } 1 & \text{movie' } 2 & \text{movie' } 3 & \cdots & \text{movie' } q \\
\text{person } 0 & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\text{person' } 1 & r'_{10} & s_1 & 0 & 0 & \cdots & 0 \\
\text{person' } 2 & r'_{20} & 0 & s_2 & 0 & \cdots & 0 \\
\text{person' } 3 & r'_{30} & 0 & 0 & s_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & 0 & 0 & 0 & \cdots & ??
\end{array}
$$

$$
\underbrace{\begin{array}{l}\text{person } 0 \\ \text{on movie' } u\end{array}}_{r'_{0u}} \underset{?}{=} \sum_t \frac{r'_{0t}}{s_t} \cdot \underbrace{\left(\begin{array}{l}\text{person' } t \\ \text{on movie' } u\end{array}\right)}_{s_t\delta_t^u}
$$

$$
\begin{array}{l c c c c c c}
 & \text{movie 0} & \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie' } q \\
\text{person 0} & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\text{person' 1} & r'_{10} & s_1 & 0 & 0 & \cdots & 0 \\
\text{person' 2} & r'_{20} & 0 & s_2 & 0 & \cdots & 0 \\
\text{person' 3} & r'_{30} & 0 & 0 & s_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & 0 & 0 & 0 & \cdots & ??
\end{array}
$$

$$
\underbrace{\begin{array}{l}\text{person 0}\\ \text{on movie' } u\end{array}}_{r'_{0u}} \underset{?}{=} \sum_t \frac{r'_{0t}}{\cancel{s_t}} \cdot \cancel{s_t}\delta_t^u = r'_{0u}
$$

QED

$s_t\delta_t^u$

$$\begin{array}{lcccccc}
 & \text{movie 0} & \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie' } q \\
\text{person 0} & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\text{person' 1} & r'_{10} & s_1 & 0 & 0 & \cdots & 0 \\
\text{person' 2} & r'_{20} & 0 & s_2 & 0 & \cdots & 0 \\
\text{person' 3} & r'_{30} & 0 & 0 & s_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & 0 & 0 & 0 & \cdots & ??
\end{array}$$

$$\begin{matrix}\text{person 0} \\ \text{on movie' } \boxed{u}\end{matrix} = \sum_t \frac{r'_{0t}}{s_t} \cdot \left(\begin{matrix}\text{person' } t \\ \text{on movie' } \boxed{u}\end{matrix} \right)$$

$u = 1, \ldots, q$ $\qquad$ $u :\to 0$

$$\begin{array}{lc|ccccc}
 & \text{movie 0} & \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie' } q \\
\text{person 0} & ?? & r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \\
\hline
\text{person' 1} & r'_{10} & s_1 & 0 & 0 & \cdots & 0 \\
\text{person' 2} & r'_{20} & 0 & s_2 & 0 & \cdots & 0 \\
\text{person' 3} & r'_{30} & 0 & 0 & s_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\text{person' } p & r'_{p0} & 0 & 0 & 0 & \cdots & ??
\end{array}$$

$$\begin{matrix}\text{person 0} \\ \text{on movie 0}\end{matrix} \approx \sum_t \frac{r'_{0t}}{s_t} \cdot \left(\underbrace{\boxed{\begin{matrix}\text{person' } t \\ \text{on movie 0}\end{matrix}}}_{r'_{t0}} \right)$$

$$\begin{array}{lcc}
 & \text{movie 0} & \begin{array}{ccccc} \text{movie' 1} & \text{movie' 2} & \text{movie' 3} & \cdots & \text{movie'}\ q \end{array} \\
\text{person 0} & ?? & \begin{array}{ccccc} r'_{01} & r'_{02} & r'_{03} & \cdots & r'_{0q} \end{array} \\
\begin{array}{l} \text{person' 1} \\ \text{person' 2} \\ \text{person' 3} \\ \vdots \\ \text{person'}\ p \end{array} & \begin{array}{c} r'_{10} \\ r'_{20} \\ r'_{30} \\ \vdots \\ r'_{p0} \end{array} & \begin{bmatrix} s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ 0 & 0 & s_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & ?? \end{bmatrix}
\end{array}$$

person 0
on movie 0 $\approx \sum_t \frac{r'_{0t}}{s_t} \cdot r'_{t0}$

||

??

DONE!

What if some $s_t = 0$?
What if some $s_t \approx 0$?