Financial Mathematics

Multivariable polynomial approximation

Single variable linear approximation

$$f(x) = e^x \qquad f(3) = e^3 = 20.08553692$$

Problem: Approximate $f(3.01)$.

$$f'(x) = e^x \qquad f'(3) = e^3 = 20.08553692$$

$$f(3.01) \approx [20.08553692]+[20.08553692][0.01]$$
$$= 20.28639229$$

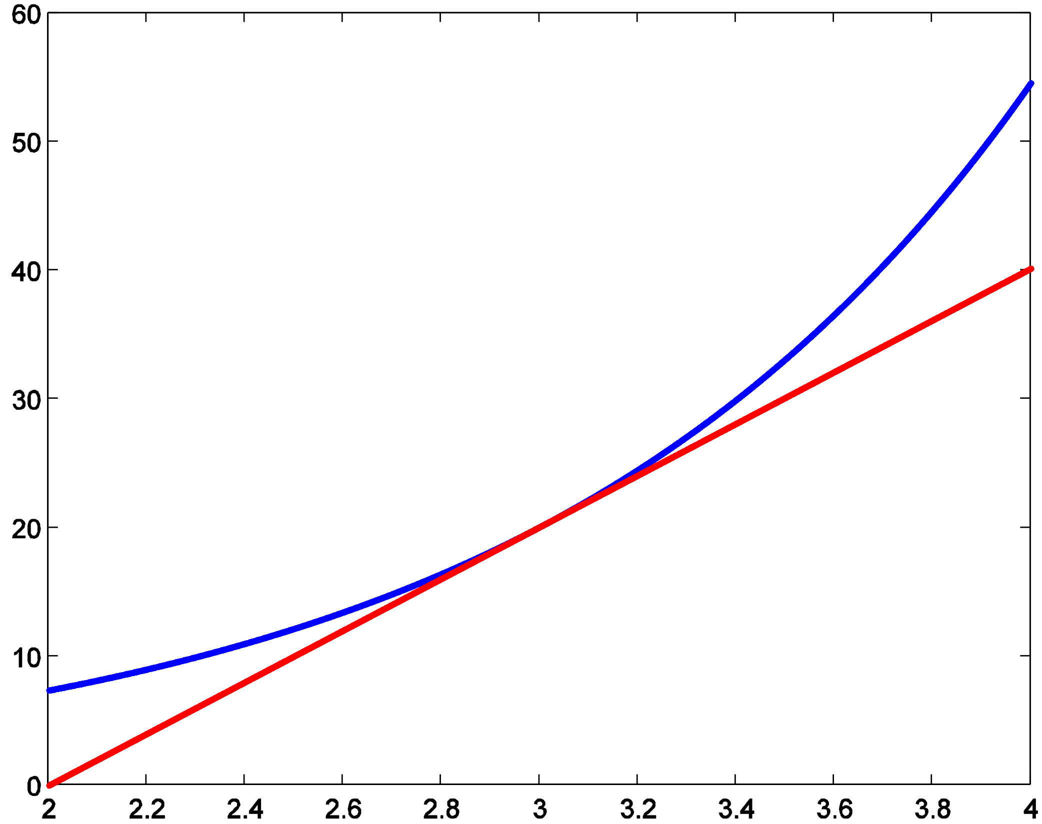
$$f(3.01) = e^{3.01} = 20.28739993$$

First order Taylor approx:

$$f(3.01) = f(3+0.01) \approx [f(3)] + [f'(3)][0.01]$$

$$y = f(x) = e^x$$

$$f(x) = e^x$$

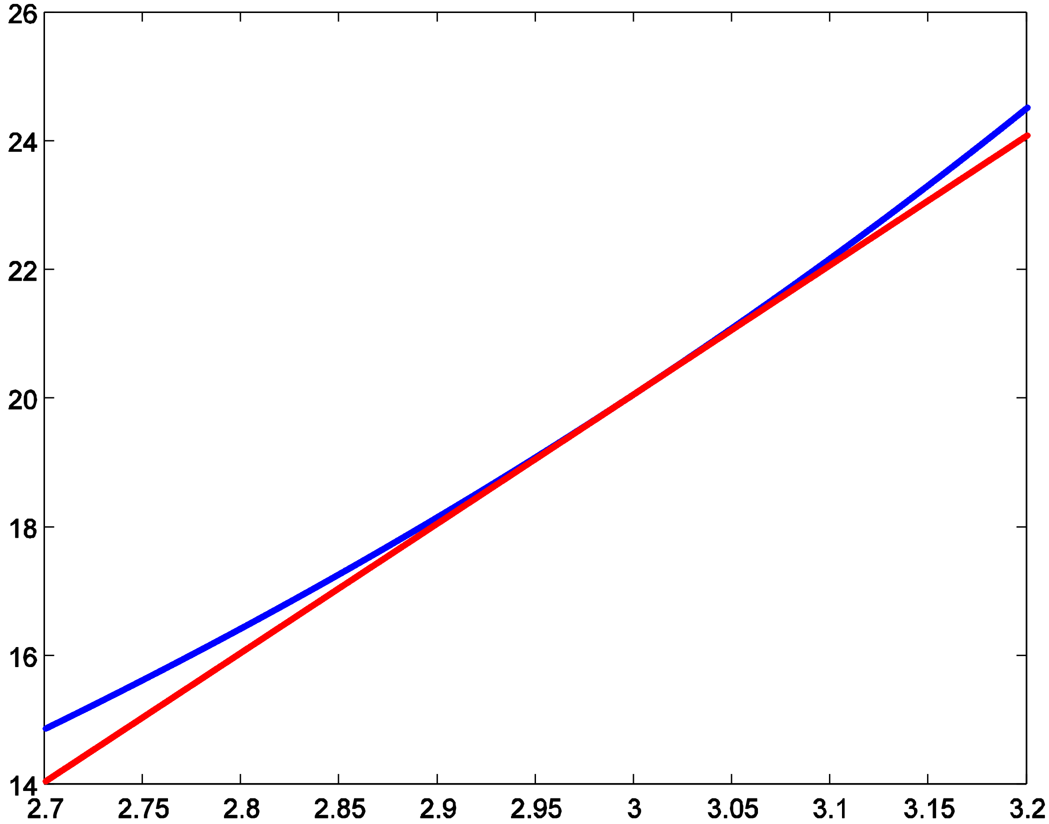


$$y = [f(3)] + [f'(3)][x]$$

$$f(3.01) = f(3 + 0.01) \approx [f(3)] + [f'(3)][0.01]$$

$$y = f(x) = e^x$$

$$f(x) = e^x$$



$$y = [f(3)] + [f'(3)][x]$$

$$f(3.01) = f(3 + 0.01) \approx [f(3)] + [f'(3)][0.01]$$

Multivariable linear approximation

$$f(x,y) = \sin((x+2y-2)e^{3y})$$
$$f(0,1) = \sin(0) = 0$$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear
s.t. $[f(0+h, 1+k)] - [f(0,1)] \approx L(h,k)$.

$$[f(0.01, 1.02)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0.01, 0.02)}_{?????}.$$

$$\underbrace{L(0.01, 0.02)}_{?????} = \underbrace{[L(0.01, 0)]}_{?????} + \underbrace{[L(0, 0.02)]}_{?????}.$$

Multivariable linear approximation

$f(x,y) = \sin((x+2y-2)e^{3y})$
$f(0,1) = \sin(0) = 0$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear

s.t. $[f(0+h, 1+k)] - [f(0,1)] \approx L(h,k)$.

$h = 0.01$, $k = 0$ (in both places)

$$[f(0.01,1)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0.01,0)}_{?????}.$$

$$\underbrace{L(0.01,0.02)}_{?????} = \underbrace{[L(0.01,0)]}_{?????} + \underbrace{[L(0,0.02)]}_{?????}.$$

Multivariable linear approximation

$f(x,y) = \sin((x+2y-2)e^{3y})$
$f(0,1) = \sin(0) = 0$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear
s.t. $[f(0+h, 1+\underset{0}{k})] - [f(0,1)] \approx L(h, \underset{0}{k})$.

$$[f(0.01,1)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0.01,0)}_{?????}.$$

$f_1(x) := f(x,1) = \sin(xe^3) = \sin(e^3x)$
$L_1(h) := L(h,0)$ Want: $L_1(0.01)$

$$[f_1(0+h)] - [f_1(0)] \approx L_1(h).$$

Multivariable linear approximation

Generally: $$[g(x+h)] - [g(x)] \approx [g'(x)]h$$

$$[f_1(0+h)] - [f_1(0)] \approx [f_1'(0)]h$$

$$L_1(h) = [f_1'(0)]h$$

$$f_1'(x) = [\cos(e^3x)][e^3]$$

$$f_1'(0) = e^3$$

$$L_1(h) = e^3h$$

$$L_1(0.01) = [20.08553692][0.01]$$

$$f_1(x) := f(x,1) = \sin(xe^3) = \sin(e^3x)$$

$$L_1(h) := L(h,0)$$

Want: $L_1(0.01)$

$$[f_1(0+h)] - [f_1(0)] \approx L_1(h).$$

Multivariable linear approximation

Generally: $$[g(x+h)] - [g(x)] \approx [g'(x)]h$$

$$[f_1(0+h)] - [f_1(0)] \approx [f_1'(0)]h$$

$$L_1(h) = [f_1'(0)]h$$

$$f_1'(x) = [\cos(e^3 x)][e^3]$$

$$f_1'(0) = e^3 = \left[\frac{d}{dx}[f_1(x)]\right]_{x:\to 0} = \left[\frac{d}{dx}[f(x,1)]\right]_{x:\to 0}$$

$$L_1(h) = e^3 h \qquad f_1(x) = f(x,1)$$

$$L_1(0.01) = [20.08553692][0.01]$$

$$\underbrace{L(0.01,0.02)}_{?????} = [\underbrace{L(0.01,0)}_{0.2008553692}] + [\underbrace{L(0,0.02)}_{?????}].$$

Multivariable linear approximation

$f(x,y) = \sin((x+2y-2)e^{3y})$
$f(0,1) = \sin(0) = 0$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear
s.t. $[f(0+h, 1+k)] - [f(0,1)] \approx L(h,k)$.

$$[f(0.01, 1.02)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0.01, 0.02)}_{?????}.$$

$$\underbrace{L(0.01, 0.02)}_{?????} = \underbrace{[L(0.01, 0)]}_{0.2008553692} + \underbrace{[L(0, 0.02)]}_{?????}.$$

Multivariable linear approximation

$$f(x,y) = \sin((x+2y-2)e^{3y})$$
$$f(0,1) = \sin(0) = 0$$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear

s.t. $[f(0+h, 1+k)] - [f(0,1)] \approx L(h,k)$.

$h = 0$, $k = 0.02$

$$[f(0, 1.02)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0, 0.02)}_{?????}.$$

$$\underbrace{L(0.01, 0.02)}_{?????} = \underbrace{[L(0.01, 0)]}_{0.2008553692} + \underbrace{[L(0, 0.02)]}_{?????}.$$

Multivariable linear approximation

$f(x,y) = \sin((x+2y-2)e^{3y})$

$f(0,1) = \sin(0) = 0$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear

s.t. $[f(0+\underset{0}{h}, 1+k)] - [f(0,1)] \approx L(\underset{0}{h}, k)$.

$$[f(0,1.02)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0,0.02)}_{?????}.$$

$f_2(y) := f(0,y) = \sin((2y-2)e^{3y})$

$L_2(k) := L(0,k)$ Want: $L_2(0.02)$

$$[f_2(1+k)] - [f_2(1)] \approx L_2(k).$$

Multivariable linear approximation

Generally: $$[g(y+k)] - [g(y)] \approx [g'(y)]k$$
$$[f_2(1+k)] - [f_2(1)] \approx [f_2'(1)]k$$

$$L_2(k) = [f_2'(1)]k$$

$$f_2'(y) = [\cos((2y-2)e^{3y})][2e^{3y} + (2y-2)3e^{3y}]$$
$$f_2'(1) = 2e^3$$
$$L_2(k) = 2e^3k$$
$$L_2(0.02) = [40.17107385][0.02]$$

$$f_2(y) := f(0,y) = \sin((2y-2)e^{3y})$$
$$L_2(k) := L(0,k)$$ Want: $L_2(0.02)$

$$[f_2(1+k)] - [f_2(1)] \approx L_2(k).$$

Multivariable linear approximation

Generally: $$[g(y+k)] - [g(y)] \approx [g'(y)]k$$

$$[f_2(1+k)] - [f_2(1)] \approx [f_2'(1)]k$$

$$L_2(k) = [f_2'(1)]k$$

$$f_2'(y) = [\cos((2y-2)e^{3y})][2e^{3y} + (2y-2)3e^{3y}]$$

$$f_2'(1) = 2e^3 = \left[\frac{d}{dy}[f_2(y)]\right]_{y:\to 1} = \left[\frac{d}{dy}[f(0,y)]\right]_{y:\to 1}$$

$$L_2(k) = 2e^3 k \qquad f_2(y) = f(0,y)$$

$$L_2(0.02) = [40.17107385][0.02]$$

$$\underbrace{L(0.01, 0.02)}_{?????} = \underbrace{[L(0.01, 0)]}_{0.2008553692} + \underbrace{[L(0, 0.02)]}_{0.8034214770}.$$

Multivariable linear approximation

$$f(x,y) = \sin((x+2y-2)e^{3y})$$
$$f(0,1) = \sin(0) = 0$$

Problem: Approximate $f(0.01, 1.02)$.

Idea: Find $L : \mathbb{R}^2 \to \mathbb{R}$ linear
s.t. $[f(0+h, 1+k)] - [f(0,1)] \approx L(h,k)$.

$$[f(0.01, 1.02)] - \underbrace{[f(0,1)]}_{0} \approx \underbrace{L(0.01, 0.02)}_{?????}.$$

$$\underbrace{L(0.01, 0.02)}_{?????} = \underbrace{[L(0.01, 0)]}_{0.2008553692} + \underbrace{[L(0, 0.02)]}_{0.8034214770}.$$

Multivariable linear approximation

Generally: $[g(0+h,1+k)] - [g(0,1)]$

$$\wr\wr$$

$$ah + bk,$$

$$a = \left[\frac{d}{dx}[g(x,1)]\right]_{x:\to 0} \qquad b = \left[\frac{d}{dy}[g(0,y)]\right]_{y:\to 1}$$

Next: Rephrase these,
in terms of "partial derivatives"...

Generally: $[g(x+h,y+k)] - [g(x,y)] \approx$?????

Generally: $[g(x+h)] - [g(x)] \approx [g'(x)]h$

Multivariable linear approximation

Let g be a function from (part of) $\mathbb{R}^2$ to $\mathbb{R}$.

Definition: The **partial derivative (or partial) of $g(x, y)$ with respect to x** is

$$\boxed{\frac{\partial}{\partial x}[g(x,y)]} \quad := \quad \lim_{h \to 0} \frac{[g(x+h,y)] - [g(x,y)]}{h}$$

Definition: The **partial derivative (or partial) of g with respect to the first variable** is the function $\partial_1 g$ defined by

$$\boxed{(\partial_1 g)(x,y)} \quad := \quad \lim_{h \to 0} \frac{[g(x+h,y)] - [g(x,y)]}{h}$$

Multivariable linear approximation

Let g be a function from (part of) $\mathbb{R}^2$ to $\mathbb{R}$.

Definition: The **partial derivative** **(or partial)** **of** $g(x,y)$ **with respect to** y is

$$\boxed{\frac{\partial}{\partial y}[g(x,y)]} \quad := \quad \lim_{k\to 0} \frac{[g(x,y+k)]-[g(x,y)]}{k}$$

Definition: The **partial derivative** **(or partial)** **of** g **with respect to the second variable** is the function $\partial_2 g$ defined by

$$\boxed{(\partial_2 g)(x,y)} \quad := \quad \lim_{k\to 0} \frac{[g(x,y+k)]-[g(x,y)]}{k}$$

Multivariable linear approximation

Generally: $[g(0+h,1+k)]-[g(0,1)]$

$$\approx$$

$$ah+bk,$$

the partial of $g(x,y)$ with respect to x

$$a = \left[\frac{d}{dx}[g(x,1)]\right]_{x:\to 0} = \left[\frac{\partial}{\partial x}[g(x,y)]\right]_{x:\to 0,y:\to 1}$$

the partial of $g(x,y)$ with respect to y

$$b = \left[\frac{d}{dy}[g(0,y)]\right]_{y:\to 1} = \left[\frac{\partial}{\partial y}[g(x,y)]\right]_{x:\to 0,y:\to 1}$$

Generally: $[g(x+h,y+k)]-[g(x,y)] \approx$?????

Generally: $[g(x+h)]-[g(x)] \approx [g'(x)]h$

Multivariable linear approximation

Generally: $[g(0+h,1+k)] - [g(0,1)]$

$$\approx$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right]_{x:\to 0, y:\to 1}[h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right]_{x:\to 0, y:\to 1}[k]$$

$$\|$$

$$\left[\left[\frac{\partial}{\partial x}[g(x,y)]\right][h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right][k]\right]_{\substack{x:\to 0,\\ y:\to 1}}$$

Generally: $[g(x+h,y+k)] - [g(x,y)] \approx$?????

Generally: $[g(x+h)] - [g(x)] \approx [g'(x)]h$

Multivariable linear approximation

Generally: $[g(0+h,1+k)] - [g(0,1)]$

$$\wr\wr$$

$$\left[\left[\frac{\partial}{\partial x}[g(x,y)]\right][h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right][k]\right]_{\substack{x:\to 0,\\ y:\to 1}}$$

Generally: $[g(x+h,y+k)] - [g(x,y)]$

$$\wr\wr$$

$$\left[\left[\frac{\partial}{\partial x}[g(x,y)]\right][h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right][k]\right]_{\substack{x:\to 0,\\ y:\to 1}}$$

Generally: $[g(x+h,y+k)] - [g(x,y)] \approx$?????

Generally: $[g(x+h)] - [g(x)] \approx [g'(x)]h$

Multivariable linear approximation

Generally: $[g(0+h,1+k)]-[g(0,1)]$

$$\wr\wr$$

$$\left[\left[\frac{\partial}{\partial x}[g(x,y)]\right][h]+\left[\frac{\partial}{\partial y}[g(x,y)]\right][k]\right]_{\substack{x:\to 0,\\ y:\to 1}}$$

Generally: $[g(x+h,y+k)]-[g(x,y)]$

$$\wr\wr$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right][h]+\left[\frac{\partial}{\partial y}[g(x,y)]\right][k]$$

Generally: $[g(x+h,y+k)]-[g(x,y)]\approx$?????

Generally: $[g(x+h)]-[g(x)]\approx[g'(x)]h$

Multivariable linear approximation

Generally: $[g(x+h,y+k)]-[g(x,y)]$

$$\approx$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right][h]+\left[\frac{\partial}{\partial y}[g(x,y)]\right][k]$$

General $p=(x,y), \qquad \Delta p=(h,k)x,y)]$

$$\approx$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right][h]+\left[\frac{\partial}{\partial y}[g(x,y)]\right][k]$$

Multivariable linear approximation

Generally: $[g(x+h,y+k)] - [g(x,y)]$

$$\approx$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right][h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right][k]$$

$$p = (x,y), \qquad \Delta p = (h,k)$$

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\approx$$

$$\left(\frac{\partial}{\partial x}[g(x,y)]\ ,\ \frac{\partial}{\partial y}[g(x,y)]\right) \cdot \underbrace{(h,k)}_{\Delta p}$$

Multivariable linear approximation

Generally: $[g(x+h,y+k)] - [g(x,y)]$

$$\wr\wr$$

$$\left[\frac{\partial}{\partial x}[g(x,y)]\right][h] + \left[\frac{\partial}{\partial y}[g(x,y)]\right][k]$$

$$p = (x,y), \qquad \Delta p = (h,k)$$

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\wr\wr$$

$$\underbrace{\left(\frac{\partial}{\partial x}[g(x,y)]\ ,\ \frac{\partial}{\partial y}[g(x,y)]\right)} \cdot [\Delta p]$$

Next: Rephrase this,
in terms of the "gradient" . . .

Multivariable linear approximation

Let g be a function from (part of) $\mathbb{R}^2$ to $\mathbb{R}$.

Definition: The **gradient of** g is the function from (part of) $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\boxed{(\nabla g)(x,y)} \quad := \quad \left(\frac{\partial}{\partial x}[g(x,y)], \frac{\partial}{\partial y}[g(x,y)]\right)$$

$$\| $$

$$((\partial_1 g)(x,y), (\partial_2 g)(x,y))$$

$$\boxed{\nabla g} = (\partial_1 g, \partial_2 g)$$

$$[\,\partial_1 g \quad \partial_2 g\,] =: \boxed{g'}$$

Multivariable linear approximation

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\wr\wr$$

$$[(\nabla g)(p)] \bullet [\Delta p]$$

$$p = (x,y), \qquad \Delta p = (h,k)$$

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\wr\wr$$

$$\underbrace{\left(\frac{\partial}{\partial x}[g(x,y)] \;,\; \frac{\partial}{\partial y}[g(x,y)] \right)}_{(\nabla g)(x,y)} \bullet [\Delta p]$$

Multivariable linear approximation

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\wr\wr$$

$$[(\nabla g)(p)] \bullet [\Delta p]$$

the gradient of g → (∇g)

$(\nabla g) := (\quad \partial_1 g \quad , \quad \partial_2 g \quad)$

---

cf: Single variable linear approximation

$$[g(x+\Delta x)] - [g(x)]$$

$$\wr\wr$$

$$[g'(x)] \quad [\Delta x]$$

Multivariable linear approximation

Generally: $[g(p+\Delta p)] - [g(p)]$

$$\wr\wr$$

$$[(\nabla g)(p)] \cdot [\Delta p]$$

the gradient of g → (∇g)

$(\nabla g) := (\quad \partial_1 g \quad , \quad \partial_2 g \quad)$

Definition: The **graph** of $z = g(x,y)$ is

$$\{(x,y,z) \,|\, z = g(x,y)\},$$

which is a subset of $\mathbb{R}^3$.

Question: If I'm standing on the graph of $z = e^y(2+\sin x)$ at the point $(0,0,2)$, and I seek the most uphill direction, what is it?

Multivariable linear approximation

Question: If I'm standing on the graph of $z = e^x(2 + \sin y)$ at the point $(0, 0, 2)$, and I seek the most uphill direction, what is it?

$g(x, y) := e^x(2 + \sin y)$ $\qquad$ $p := (0, 0)$

$(\nabla g)(x, y) := (e^x(2 + \sin y), e^x \cos y)$

$(\nabla g)(0, 0) := (2, 1)$

$$[g(p + \Delta p)] - [g(p)] \approx [(\nabla g)(p)] \cdot [\Delta p]$$

$$[g(0 + \Delta p)] - [g(0)] \approx (2, 1) \cdot [\Delta p]$$

$$= \sqrt{5}\, |\Delta p| \cos(\theta)$$

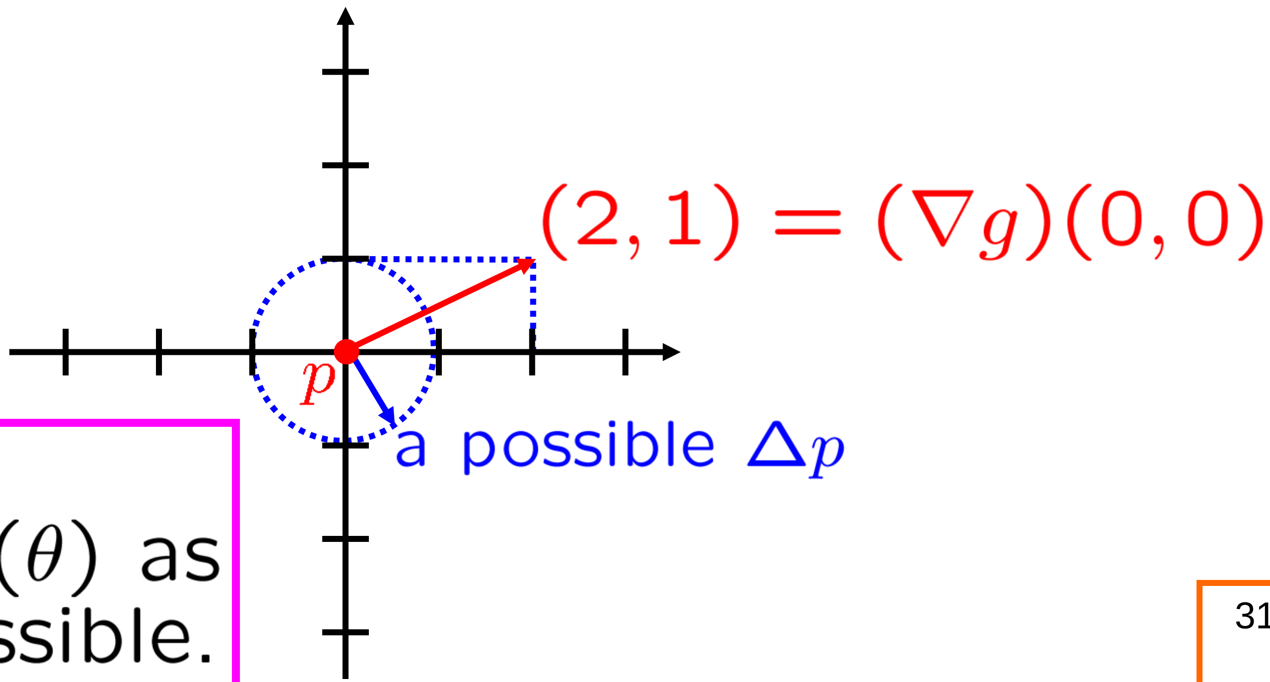
the angle between $(2, 1)$ and Δp

Multivariable linear approximation

Question: If I'm standing on the graph of $z = e^x(2 + \sin y)$ at the point $(0, 0, 2)$ and I seek the most uphill direction, what is it?

$g(x, y) := e^x(2 + \sin y)$ $\qquad$ $p := (0, 0)$

$[g(p + \Delta p)] - [g(p)] \approx \sqrt{5}\,|\Delta p|\,\cos(\theta)$



Make $\theta = 0$.

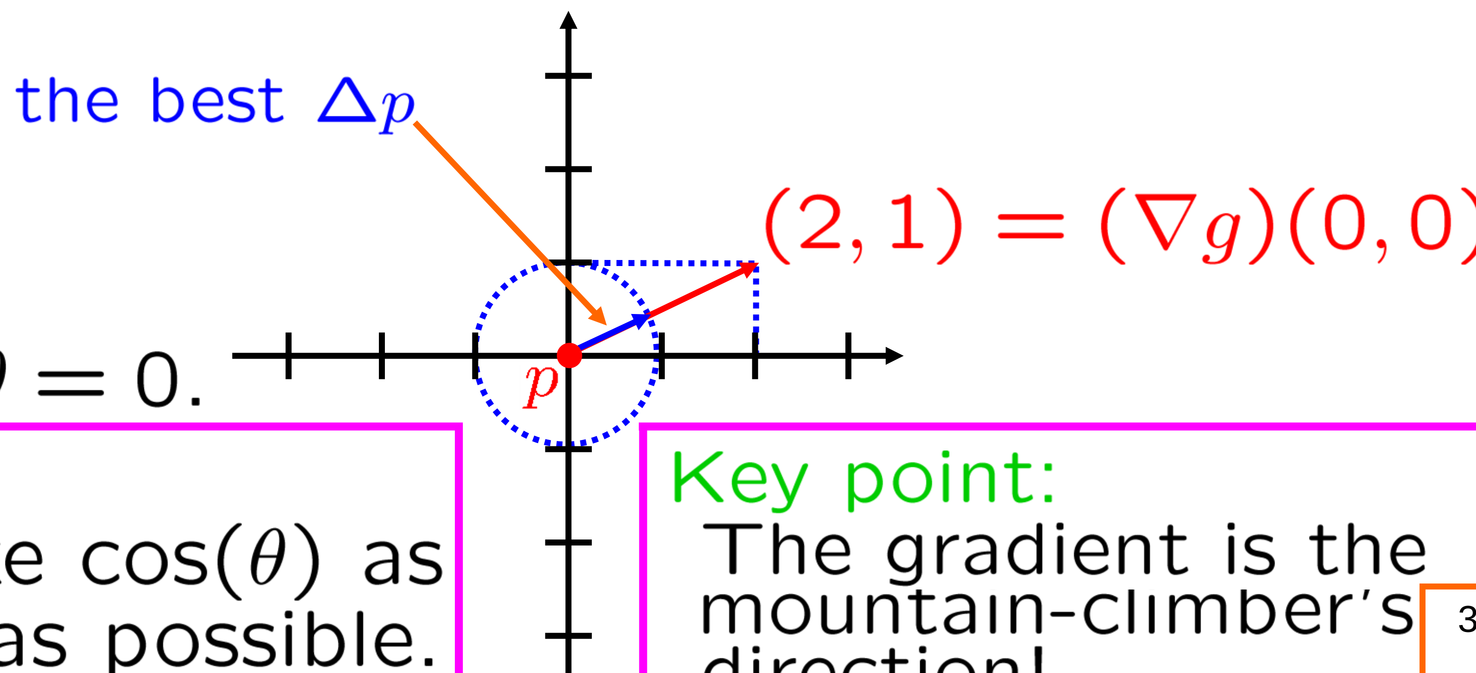
Goal:
Make $\cos(\theta)$ as big as possible.

Multivariable linear approximation

Question: If I'm standing on the graph of $z = e^x(2 + \sin y)$ at the point $(0, 0, 1)$, and I seek the most uphill direction, what is it?

$$g(x, y) := e^x(2 + \sin y) \qquad\qquad p := (0, 0)$$

$$[g(p + \Delta p)] - [g(p)] \approx \sqrt{5}\, |\Delta p| \cos(\theta)$$



Make $\theta = 0$.

Goal:
Make $\cos(\theta)$ as big as possible.

Key point:
The gradient is the mountain-climber's direction!

Practice partial derivatives:

$$f(x,y) = \sin((x + 2y - 2)e^{3y})$$

$$\frac{\partial}{\partial x}[f(x,y)] = (\partial_1 f)(x,y) =$$

$$[\cos((x + 2y - 2)e^{3y}][e^{3y}]$$

$$\frac{\partial}{\partial y}[f(x,y)] = (\partial_2 f)(x,y) =$$

$$[\cos((x + 2y - 2)e^{3y}][2e^{3y} + (x + 2y - 2)(3e^{3y})]$$

SKILL:

Given $g(x,y)$, compute

$$\frac{\partial}{\partial x}[g(x,y)] = (\partial_1 g)(x,y)$$

and

$$\frac{\partial}{\partial y}[g(x,y)] = (\partial_2 g)(x,y)$$

SKILL: Given $g(x,y)$, compute

$$(\nabla g)(x,y)$$

SKILL:

Given $g(x,y)$, compute

$$\frac{\partial^2}{\partial x^2}[g(x,y)] = (\partial_{11}g)(x,y),$$

$$\begin{array}{ccc} \dfrac{\partial^2}{\partial x \partial y}[g(x,y)] & = & (\partial_{12}g)(x,y) \\ \| & & \| \\ \dfrac{\partial^2}{\partial y \partial x}[g(x,y)] & = & (\partial_{21}g)(x,y), \end{array}$$

and $\dfrac{\partial^2}{\partial y^2}[g(x,y)] = (\partial_{22}g)(x,y)$,

Multivariable linear approximation

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function.

Definition: The **partial derivative** **(or partial)** **of** $g(x_1, \ldots, x_n)$ **with respect to** x_j is

$$\frac{\partial}{\partial x_j}[g(x_1, \ldots, x_n)]$$

jth entry

$$\varepsilon_j := (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$= \lim_{h \to 0} \frac{[g(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n)] - [g(x_1, \ldots, x_n)]}{h}$$

$$= \lim_{h \to 0} \frac{g((x_1, \ldots, x_n) + h\varepsilon_j) - g(x_1, \ldots, x_n)}{h}$$

Multivariable linear approximation

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function.

Definition: The **partial derivative (or partial) of g with respect to the jth variable** is

$$[(\partial_j g)(p)] = \lim_{h \to 0} \frac{g(p + h\varepsilon_j) - g(p)}{h}$$

jth entry

$$\varepsilon_j := (0, \ldots, 0, 1, 0, \ldots, 0)$$

Multivariable linear approximation

Notation:

∂_x is an abbreviation for $\dfrac{\partial}{\partial x}$.

∂_y is an abbreviation for $\dfrac{\partial}{\partial y}$.

∂_t is an abbreviation for $\dfrac{\partial}{\partial t}$.

etc.

Multivariable linear approximation

Notation:

∂_{x_1} is an abbreviation for $\dfrac{\partial}{\partial x_1}$.

∂_{x_2} is an abbreviation for $\dfrac{\partial}{\partial x_2}$.

etc.

∂_{x_n} is an abbreviation for $\dfrac{\partial}{\partial x_n}$.

Multivariable linear approximation

Notation:

∂_{y_1} is an abbreviation for $\dfrac{\partial}{\partial y_1}$.

∂_{y_2} is an abbreviation for $\dfrac{\partial}{\partial y_2}$.

etc.

∂_{y_k} is an abbreviation for $\dfrac{\partial}{\partial y_k}$.

Multivariable linear approximation

Notation:

∂_{z_1} is an abbreviation for $\dfrac{\partial}{\partial z_1}$.

∂_{z_2} is an abbreviation for $\dfrac{\partial}{\partial z_2}$.

etc.

∂_{z_j} is an abbreviation for $\dfrac{\partial}{\partial z_j}$.

Multivariable linear approximation

Notation:

∂_{s_1} is an abbreviation for $\dfrac{\partial}{\partial s_1}$.

∂_{s_2} is an abbreviation for $\dfrac{\partial}{\partial s_2}$.

etc.

∂_{s_p} is an abbreviation for $\dfrac{\partial}{\partial s_p}$.

Multivariable linear approximation

Let g be a function from (part of) $\mathbb{R}^n$ to $\mathbb{R}$.

Definition: The **gradient of** g is the function ∇g from (part of) $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$(\nabla g)(x_1, \ldots, x_n)$$

$$:= \left(\frac{\partial}{\partial x_1}[g(x_1, \ldots, x_n)], \ldots, \frac{\partial}{\partial x_n}[g(x_1, \ldots, x_n)] \right)$$

$$= ((\partial_1 g)(x_1, \ldots, x_n), \ldots, (\partial_n g)(x_1, \ldots, x_n))$$

$$\nabla g = (\partial_1 g, \ldots, \partial_n g)$$

$$[\, \partial_1 g \quad \cdots \quad \partial_n g \,] =: g'$$

SKILL:

Given $g(x_1, \ldots, x_n)$, and an integer $j \in [1, n]$, compute

$$\frac{\partial}{\partial x_j}[g(x_1, \ldots, x_n)] = (\partial_j g)(x_1, \ldots, x_j)$$

SKILL: Given $g(x_1, \ldots, x_n)$, compute

$$(\nabla g)(x_1, \ldots, x_n)$$

Definitions: $$\boxed{\frac{\partial^2}{\partial x\,\partial y}} := \frac{\partial}{\partial x}\frac{\partial}{\partial y} =: \boxed{\partial_{xy}}$$

$$\boxed{\frac{\partial^3}{\partial z\,\partial x\,\partial y}} := \frac{\partial}{\partial z}\frac{\partial}{\partial x}\frac{\partial}{\partial y} =: \boxed{\partial_{zxy}}$$

$$\boxed{\frac{\partial^3}{\partial y_4\,\partial w_2\,\partial z}} := \frac{\partial}{\partial y_4}\frac{\partial}{\partial w_2}\frac{\partial}{\partial z} =: \boxed{\partial_{y_4 w_2 z}}$$

etc., etc., etc., . . .

SKILL:

Given $g(x_1, \ldots, x_n)$, and

two integers $j, k \in [1, n]$, compute

$$\frac{\partial^2}{\partial x_j \, \partial x_k}[g(x_1, \ldots, x_n)] = (\partial_{jk} g)(x_1, \ldots, x_n)$$

Fact $\longrightarrow$ $\|$ $\qquad\qquad$ $\|$

$$\frac{\partial^2}{\partial x_k \, \partial x_j}[g(x_1, \ldots, x_n)] = (\partial_{kj} g)(x_1, \ldots, x_n)$$

Multivariable Maclaurin approximation

The **second order Macl. approximation of** $f(x)$ **w.r.t.** x is the polynomial of degree ≤ 2

$$p(x) = a + bx + cx^2$$

such that

$$f(0) = p(0),\ f'(0) = p'(0) \text{ and } f''(0) = p''(0).$$

The **second order Macl. approximation of** $f(x,y)$ **w.r.t.** (x,y) is the poly. of degree ≤ 2

$$p(x,y) = a + bx + cy + sx^2 + txy + uy^2$$

such that

$$f(\mathbf{0}) = p(\mathbf{0}),$$
$$(\partial_1 f)(\mathbf{0}) = (\partial_1 p)(\mathbf{0}),\ (\partial_2 f)(\mathbf{0}) = (\partial_2 p)(\mathbf{0}),$$
$$(\partial_{11} f)(\mathbf{0}) = (\partial_{11} p)(\mathbf{0}),\ (\partial_{12} f)(\mathbf{0}) = (\partial_{12} p)(\mathbf{0}),$$

$\mathbf{0} := (0,0)$

$$\text{and } (\partial_{22} f)(\mathbf{0}) = (\partial_{22} p)(\mathbf{0}).$$

$$p(x, y) = a + bx + cy + sx^2 + txy + uy^2$$

$$f(\mathbf{0}) = p(\mathbf{0}),$$

$$p(x, y) = a + bx + cy + sx^2 + txy + uy^2$$

$$p(x,y) = a + bx + cy + sx^2 + txy + uy^2$$

$$f(\mathbf{0}) = \overbrace{p(\mathbf{0})}^{a},$$

$$(\partial_1 f)(\mathbf{0}) = \overbrace{(\partial_1 p)(\mathbf{0})}^{b}, \quad (\partial_2 f)(\mathbf{0}) = \overbrace{(\partial_2 p)(\mathbf{0})}^{c},$$

$$(\partial_{11} f)(\mathbf{0}) = \overbrace{(\partial_{11} p)(\mathbf{0})}^{2s}, \quad (\partial_{12} f)(\mathbf{0}) = \overbrace{(\partial_{12} p)(\mathbf{0})}^{t},$$

and $(\partial_{22} f)(\mathbf{0}) = \overbrace{(\partial_{22} p)(\mathbf{0})}^{2u}$.

$$(\partial_1 p)(x,y) = (\partial/\partial x)(p(x,y)) = b + 2sx + ty$$

$$(\partial_2 p)(x,y) = (\partial/\partial y)(p(x,y)) = c + tx + 2uy$$

$$(\partial_{11} p)(x,y) = (\partial/\partial x)^2(p(x,y)) = 2s$$

$$(\partial_{12} p)(x,y) = (\partial^2/\partial x\,\partial y)(p(x,y)) = t$$

$$(\partial_{22} p)(x,y) = (\partial/\partial y)^2(p(x,y)) = 2u$$

$$p(x,y) = a + bx + cy + sx^2 + txy + uy^2$$

$$f(\mathbf{0}) = a,$$

$$(\partial_1 f)(\mathbf{0}) = b, \qquad (\partial_2 f)(\mathbf{0}) = c,$$

$$(\partial_{11} f)(\mathbf{0}) = 2s, \qquad (\partial_{12} f)(\mathbf{0}) = t,$$

and $(\partial_{22} f)(\mathbf{0}) = 2u.$

$$(\partial_1 p)(x,y) = (\partial/\partial x)(p(x,y)) = b + 2sx + ty$$
$$(\partial_2 p)(x,y) = (\partial/\partial y)(p(x,y)) = c + tx + 2uy$$
$$(\partial_{11} p)(x,y) = (\partial/\partial x)^2(p(x,y)) = 2s$$
$$(\partial_{12} p)(x,y) = (\partial^2/\partial x\,\partial y)(p(x,y)) = t$$
$$(\partial_{22} p)(x,y) = (\partial/\partial y)^2(p(x,y)) = 2u$$

$$p(x,y) = a + bx + cy + sx^2 + txy + uy^2$$

$$f(\mathbf{0}) = a,$$

$$(\partial_1 f)(\mathbf{0}) = b, \qquad (\partial_2 f)(\mathbf{0}) = c,$$

$$(\partial_{11} f)(\mathbf{0}) = 2s, \qquad (\partial_{12} f)(\mathbf{0}) = t,$$

and $(\partial_{22} f)(\mathbf{0}) = 2u.$

$$\begin{aligned} p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + \\ [(\partial_{11} f)(\mathbf{0})][x^2/2] + \\ [(\partial_{12} f)(\mathbf{0})][xy] + \\ [(\partial_{22} f)(\mathbf{0})][y^2/2] \end{aligned}$$

Multivariable Maclaurin approximation

The **second order Macl. approximation of** $f(x,y)$ **w.r.t.** x,y is the poly. of degree ≤ 2

$$p(x,y) = a + bx + cy + sx^2 + txy + uy^2$$

such that

$$f(\mathbf{0}) = p(\mathbf{0}),$$
$$(\partial_1 f)(\mathbf{0}) = (\partial_1 p)(\mathbf{0}),\ (\partial_2 f)(\mathbf{0}) = (\partial_2 p)(\mathbf{0}),$$
$$(\partial_{11} f)(\mathbf{0}) = (\partial_{11} p)(\mathbf{0}),\ (\partial_{12} f)(\mathbf{0}) = (\partial_{12} p)(\mathbf{0}),$$

and $(\partial_{22} f)(\mathbf{0}) = (\partial_{22} p)(\mathbf{0})$.

$$\begin{aligned} p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + \\ [(\partial_{11} f)(\mathbf{0})][x^2/2] + \\ [(\partial_{12} f)(\mathbf{0})][xy] + \\ [(\partial_{22} f)(\mathbf{0})][y^2/2] \end{aligned}$$

Multivariable Maclaurin approximation

The **second order Macl. approximation of** $f(x,y)$ **w.r.t.** x,y is the poly. of degree ≤ 2

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + [(\partial_{11} f)(\mathbf{0})][x^2/2] + [(\partial_{12} f)(\mathbf{0})][xy] + [(\partial_{22} f)(\mathbf{0})][y^2/2]$$

Exercise:

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + [(\partial_{11} f)(\mathbf{0})][x^2/2] + [(\partial_{12} f)(\mathbf{0})][xy] + [(\partial_{22} f)(\mathbf{0})][y^2/2]$$

Multivariable Maclaurin approximation

The **second order Macl. approximation of** $f(x,y)$ **w.r.t.** x,y is the poly. of degree ≤ 2

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + [(\partial_{11} f)(\mathbf{0})][x^2/2] + [(\partial_{12} f)(\mathbf{0})][xy] + [(\partial_{22} f)(\mathbf{0})][y^2/2]$$

Exercise:

Write out the third order Maclaurin approximation of $f(x,y)$.

Exercise:

Write out the second order Maclaurin approximation of $f(x,y,z)$.

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + [(\partial_{11} f)(\mathbf{0})][x^2/2] + [(\partial_{12} f)(\mathbf{0})][xy] + [(\partial_{22} f)(\mathbf{0})][y^2/2]$$

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y + [(\partial_{11} f)(\mathbf{0})][x^2/2] + [(\partial_{12} f)(\mathbf{0})][xy] + [(\partial_{22} f)(\mathbf{0})][y^2/2]$$

the gradient of f

$$p(x,y) = [f(\mathbf{0})] + [(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y +$$
$$[(\partial_{11} f)(\mathbf{0})][x^2/2] +$$
$$[(\partial_{12} f)(\mathbf{0})][xy] +$$
$$[(\partial_{22} f)(\mathbf{0})][y^2/2]$$

the gradient of f

$$\nabla f := (\partial_1 f, \partial_2 f)$$

the Hessian of f

$$Hf := \begin{bmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{21} f & \partial_{22} f \end{bmatrix}$$

$$(\nabla f)(x,y) = ((\partial_1 f)(x,y), (\partial_2 f)(x,y))$$
$$= ([\partial/\partial x][f(x,y)], [\partial/\partial y][f(x,y)])$$

$$(\nabla f)(\mathbf{0}) = ((\partial_1 f)(\mathbf{0}), (\partial_2 f)(\mathbf{0}))$$

$$f' = [\partial_1 f \quad \partial_2 f]$$

$$[(\partial_1 f)(\mathbf{0})]x + [(\partial_2 f)(\mathbf{0})]y$$
$$= [(\nabla f)(\mathbf{0})] \cdot (x,y) = L_{f'(0)}(x,y)$$

$$f'' := Hf := \begin{bmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{21} f & \partial_{22} f \end{bmatrix}$$

$$[(\partial_{11} f)(\mathbf{0})][x^2/2] +$$
$$[(\partial_{12} f)(\mathbf{0})][xy] +$$
$$[(\partial_{22} f)(\mathbf{0})][y^2/2]$$

$$Hf := \begin{bmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{21} f & \partial_{22} f \end{bmatrix}$$

$$[(\partial_{11} f)(\mathbf{0})][x^2/2] +$$
$$[(\partial_{12} f)(\mathbf{0})][xy] +$$
$$[(\partial_{22} f)(\mathbf{0})][y^2/2]$$

$$f'' := Hf := \begin{bmatrix} \partial_{11}f & \partial_{12}f \\ \partial_{21}f & \partial_{22}f \end{bmatrix}$$

Note:
The Hessian
is symmetric.

$$f''(x,y) = \begin{bmatrix} (\partial_{11}f)(x,y) & (\partial_{12}f)(x,y) \\ (\partial_{21}f)(x,y) & (\partial_{22}f)(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} ([\partial/\partial x]^2[f(x,y)] & [\partial^2/\partial x\,\partial y][f(x,y)] \\ [\partial^2/\partial y\,\partial x][f(x,y)] & [\partial/\partial y]^2[f(x,y)] \end{bmatrix}$$

$$\left.\begin{array}{l} [(\partial_{11}f)(\mathbf{0})][x^2/2]+ \\ [(\partial_{12}f)(\mathbf{0})][xy]+ \\ [(\partial_{22}f)(\mathbf{0})][y^2/2] \end{array}\right\} = \frac{Q_{f''(\mathbf{0})}(x,y)}{2!}$$

The preceding is for *real*-valued

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Here, $f'(0)$ is a $1 \times n$ matrix and $f''(0)$ is an $n \times n$ matrix.

For *vector*-valued

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^k,$$

first-order Macl. approx:
$f \approx [f(0)] + L_{f'(0)}$

$f'(0)$ is a $k \times n$ matrix
and $f''(0)$ would be a $k \times n \times n$ "tensor",
but we'll avoid that, and not refer to $f''(0)$.

Instead, we'll write $f = (f_1, \ldots, f_k)$, where each $f_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *real*-valued, so $f_j'(0)$ and $f_j''(0)$ are $1 \times n$ and $n \times n$, respectively.

SKILL:

Given $f(x_1, \ldots, x_n)$, compute the gradient, Hessian and 2nd order Macl. approximation of f.

Notation:

The gradient of f is denoted

gradient matrix → $\boxed{f'}$ or $\boxed{\nabla f}$ or $\boxed{\text{grad}(f)}$.

Notation:

The $\underbrace{\text{Hessian}}_{\text{or } n \times n \text{ Hessian}}$ of f is denoted

$\boxed{f''}$ or $\boxed{\nabla\nabla f}$ or $\boxed{\text{Hess}(f)}$ or $\boxed{Hf}$.

Multivariable jets

Def'n: Let $f = (f_1, \ldots, f_q)$ be an $\mathbb{R}^q$-valued fn defined and smooth on a neighborhood of $(0, \ldots, 0)$ in $\mathbb{R}^n$.

Let $k \geq 0$ be an integer.

Let S be the set of monomials in $\partial_1, \ldots, \partial_n$ of degree $\leq k$.

The k-**jet of** f **at** $(0, \ldots, 0)$ is the function

$$\boxed{J_0^k f} : \quad S \times \{1, \ldots, q\} \rightarrow \mathbb{R}$$

$$(\ \partial_1^{j_1} \cdots \partial_n^{j_n}\ , \qquad p \qquad) \mapsto (\partial_1^{j_1} \cdots \partial_n^{j_n} f_p)(0, \ldots, 0).$$

Note: Let $M := (\#S) \cdot q = \binom{n+k}{k} \cdot q.$

Fixing an ordering of $S \times \{1, \ldots, q\}$, a k-jet can be thought of as an element of $\mathbb{R}^M$.

Multivariable Maclaurin approximation

Def'n: Let $f = (f_1, \ldots, f_q)$ be an $\mathbb{R}^q$-valued fn defined and smooth on a neighborhood of $(0, \ldots, 0)$ in $\mathbb{R}^n$.

Let $k \geq 0$ be an integer.

Let S be the set of monomials in $\partial_1, \ldots, \partial_n$ of degree $\leq k$.

The k**th order Maclaurin approx. of** f is the poly. $P = (P_1, \ldots, P_q) : \mathbb{R}^n \to \mathbb{R}^q$ of degree $\leq k$

s.t. $J_0^k f = J_0^k P$,

i.e., s.t., $\forall \partial_1^{j_1} \cdots \partial_n^{j_n} \in S$, $\forall j \in \{1, \ldots, q\}$,

$$(\partial_1^{j_1} \cdots \partial_n^{j_n} f_j)(\mathbf{0}) = (\partial_1^{j_1} \cdots \partial_n^{j_n} P_j)(\mathbf{0}).$$

SKILLS: Find the gradient and $n \times n$ Hessian of a function of n variables.

Find the k-jet at $(0, \ldots, 0)$ of a function of n variables.

Find the kth order Macl. approx. of a function of n variables.

Count the number of terms in the kth order Macl. approx. of a function of n variables.

Count the number of entries in the k-jet at $(0, \ldots, 0)$ of a function of n variables.

Denote this function by F.
Say we've computed $F(100, 97, 0.01, 0.2)$.

There are
a constant $C \in \mathbb{R}^2$,
a homogeneous linear $L : \mathbb{R}^4 \to \mathbb{R}^2$,
& a homogeneous quadratic $Q : \mathbb{R}^4 \to \mathbb{R}^2$,
s.t. $C \quad + \quad L(w, x, y, z) \quad + \quad Q(w, x, y, z)$
agrees, at $(0, 0, 0, 0)$, to order two, with
$$F(100 + w, 97 + x, 0.01 + y, 0.2 + z).$$

meaning?

---

e.g.: Black-Scholes gives a function that maps
(spot, strike, risk-free rate, volatility)
four input
$\overset{F}{\longmapsto}$ (price, Delta)
two output

$$C \quad + \quad L(w,x,y,z) \quad + \quad Q(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order two, with

$$F(100+w, 97+x, 0.01+y, 0.2+z).$$

Meaning:

$$C \quad + \quad L(w,x,y,z) \quad + \quad Q(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order two, with

$$F(100+w, 97+x, 0.01+y, 0.2+z).$$

$$C + L(w,x,y,z) + Q(w,x,y,z) =: A(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order zero, with

$$F(100+w, 97+x, 0.01+y, 0.2+z). =: B(w,x,y,z)$$

Meaning:

$$\left[C + L(w,x,y,z) + Q(w,x,y,z) \right]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

$$\|$$

$$\left[F(100+w, 97+x, 0.01+y, 0.2+z) \right]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

ordered pairs

one vector equation

two scalar equations

$$\underbrace{C + L(w,x,y,z) + Q(w,x,y,z)}_{=:A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order zero, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=:B(w,x,y,z)}.$$

Meaning:

$$\Big[\, A(w,x,y,z) \,\Big]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

$$\|$$

$$\Big[\, B(w,x,y,z) \,\Big]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

ordered pairs

one vector equation

two scalar equations

$$\underbrace{C + L(w,x,y,z) + Q(w,x,y,z)}_{=: A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order zero, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=: B(w,x,y,z)}.$$

Meaning:

$$A(0,0,0,0)$$

$$||$$

$$B(0,0,0,0)$$

ordered pairs

one vector equation

two scalar equations

$$\underbrace{C \quad + \quad L(w,x,y,z) \quad + \quad Q(w,x,y,z)}_{=: A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order zero, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=: B(w,x,y,z)}.$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

$$A(0,0,0,0)$$

$$\parallel$$

$$B(0,0,0,0)$$

$$\underline{C + L(w,x,y,z) + Q(w,x,y,z)} =: A(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order one, with

$$\underline{F(100+w, 97+x, 0.01+y, 0.2+z)}. \quad =: B(w,x,y,z)$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

ordered pairs

$$(\partial_1 A)(0,0,0,0) = (\partial_1 B)(0,0,0,0)$$
$$(\partial_2 A)(0,0,0,0) = (\partial_2 B)(0,0,0,0)$$
$$(\partial_3 A)(0,0,0,0) = (\partial_3 B)(0,0,0,0)$$
$$(\partial_4 A)(0,0,0,0) = (\partial_4 B)(0,0,0,0)$$

ordered pairs

four vector equations

eight scalar equations

$$\underbrace{C + L(w,x,y,z) + Q(w,x,y,z)}_{=: A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order one, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=: B(w,x,y,z)}.$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

$$(\partial_1 A)(0,0,0,0) = (\partial_1 B)(0,0,0,0)$$

$$(\partial_2 A)(0,0,0,0) = (\partial_2 B)(0,0,0,0)$$

$$(\partial_3 A)(0,0,0,0) = (\partial_3 B)(0,0,0,0)$$

$$(\partial_4 A)(0,0,0,0) = (\partial_4 B)(0,0,0,0)$$

$$\left[\frac{\partial}{\partial z}(A(w,x,y,z))\right]_{\substack{w=0\\x=0\\y=0\\z=0}} = \left[\frac{\partial}{\partial z}(B(w,x,y,z))\right]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

too much space

$$\underbrace{C + L(w,x,y,z) + Q(w,x,y,z)}_{=: A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order one, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=: B(w,x,y,z)}.$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

$$(\partial_1 A)(0,0,0,0) = (\partial_1 B)(0,0,0,0)$$
$$(\partial_2 A)(0,0,0,0) = (\partial_2 B)(0,0,0,0)$$
$$(\partial_3 A)(0,0,0,0) = (\partial_3 B)(0,0,0,0)$$
$$(\partial_4 A)(0,0,0,0) = (\partial_4 B)(0,0,0,0)$$

$\forall$integers $j \in [1,4]$,

$$(\partial_j A)(0,0,0,0) = (\partial_j B)(0,0,0,0)$$

$$\underline{C + L(w,x,y,z) + Q(w,x,y,z)} =: A(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order one, with

$$\underline{F(100+w, 97+x, 0.01+y, 0.2+z)}. =: B(w,x,y,z)$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

and

$\forall$integers $j \in [1,4]$,

$$(\partial_j A)(0,0,0,0) = (\partial_j B)(0,0,0,0)$$

$\forall$integers $j \in [1,4]$,

$$(\partial_j A)(0,0,0,0) = (\partial_j B)(0,0,0,0)$$

$$\underbrace{C + L(w,x,y,z) + Q(w,x,y,z)}_{=:A(w,x,y,z)}$$

agrees, at $(0,0,0,0)$, to order two, with

$$\underbrace{F(100+w, 97+x, 0.01+y, 0.2+z)}_{=:B(w,x,y,z)}.$$

Meaning:

$$A(0,0,0,0) = B(0,0,0,0)$$

and

$\forall$integers $j \in [1,4]$,

$$(\partial_j A)(0,0,0,0) = (\partial_j B)(0,0,0,0)$$

and

$\forall$integers $j, k \in [1,4]$,

$$(\partial_j \partial_k A)(0,0,0,0) = (\partial_j \partial_k B)(0,0,0,0)$$

twice as many scalar eq'ns as vector eq'ns

$$\underline{C + L(w,x,y,z) + Q(w,x,y,z)} =: A(w,x,y,z)$$

agrees, at $(0,0,0,0)$, to order two, with

$$\underline{F(100+w, 97+x, 0.01+y, 0.2+z)} =: B(w,x,y,z)$$

two scalar eq'ns

e.g.:

$$(\partial_2 \partial_3 A)(0,0,0,0) = (\partial_2 \partial_3 B)(0,0,0,0)$$

too much space

$$\left[\frac{\partial^2}{\partial x\, \partial y}(A(w,x,y,z)) \right]_{\substack{w=0\\x=0\\y=0\\z=0}} = \left[\frac{\partial^2}{\partial x\, \partial y}(B(w,x,y,z)) \right]_{\substack{w=0\\x=0\\y=0\\z=0}}$$

$\forall$integers $j, k \in [1, 4]$,

$$(\partial_j \partial_k A)(0,0,0,0) = (\partial_j \partial_k B)(0,0,0,0)$$

2 3 2 3

STOP