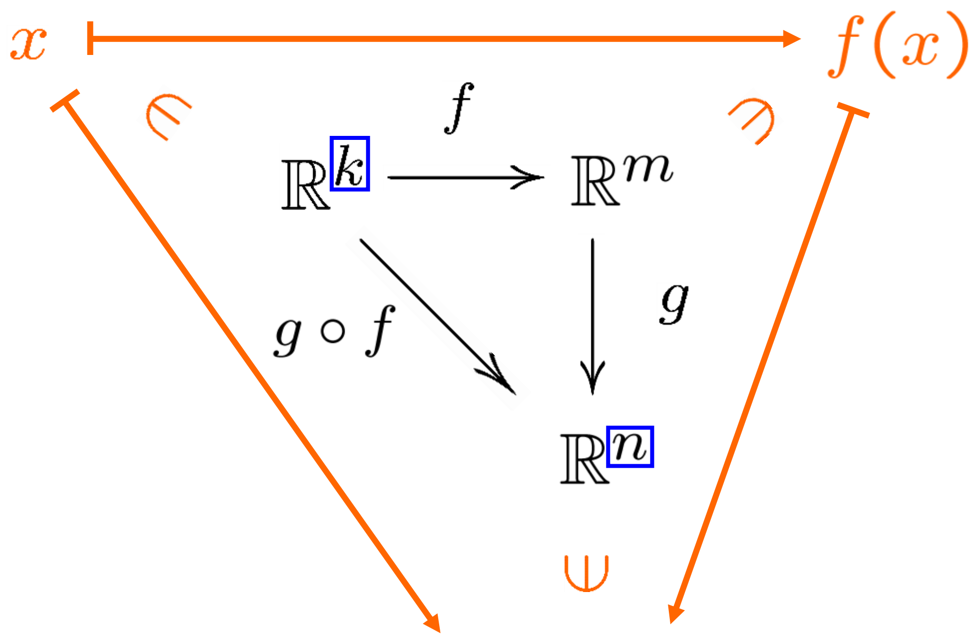
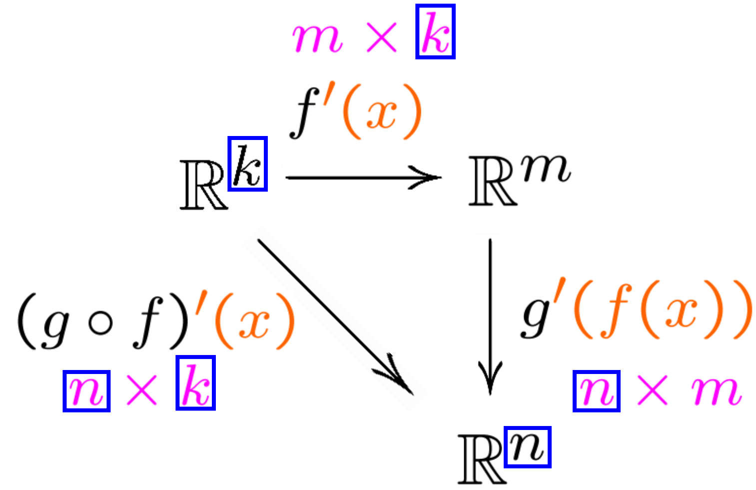
Financial Mathematics

The multivariable chain rule



$$(g \circ f)(x) = g(f(x))$$



$$\underset{n \times k}{(g \circ f)'(x)} = \underset{n \times m}{[\; g'(f(x)) \;]} \; \underset{m \times k}{[\; f'(x) \;]}$$

$k = 1$

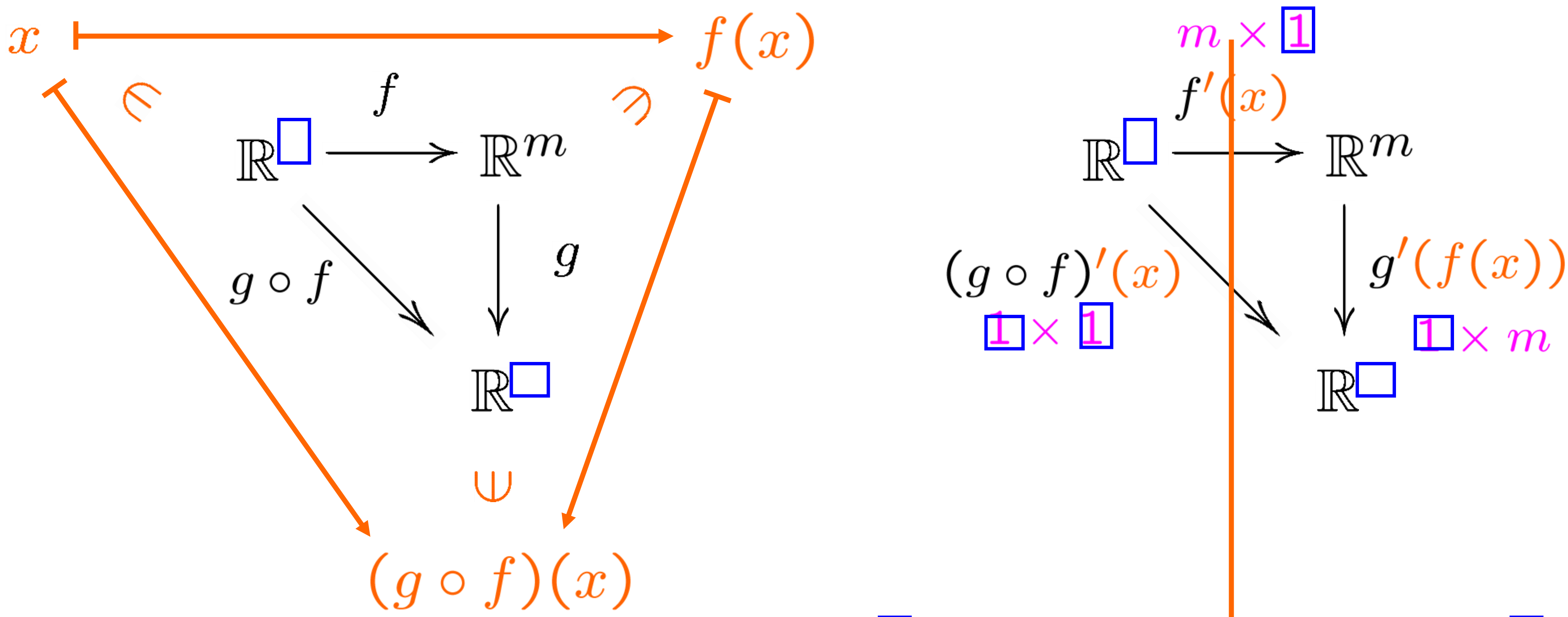
$n = 1$

$(g \circ f)''(x)$

$$= \underset{k \times m}{[f'(x)]^t} \underset{m \times m}{[g''(f(x))]} \underset{m \times k}{[f'(x)]}$$

$$+ \underset{1 \times m}{[g'(f(x))]} \circledast \underset{m \text{ of } k \times k}{[f''(x)]}$$

take derivative of the fn
plug in the expression
multiply by
the derivative of
the expression



$\boxed{1} \times \boxed{1}$ $\quad$ $\boxed{1} \times m$ $\quad$ $m \times \boxed{1}$

$$(g \circ f)'(x) = [\ g'(f(x))\ ]\ [\ f'(x)\ ]$$

$k = 1$

$n = 1$

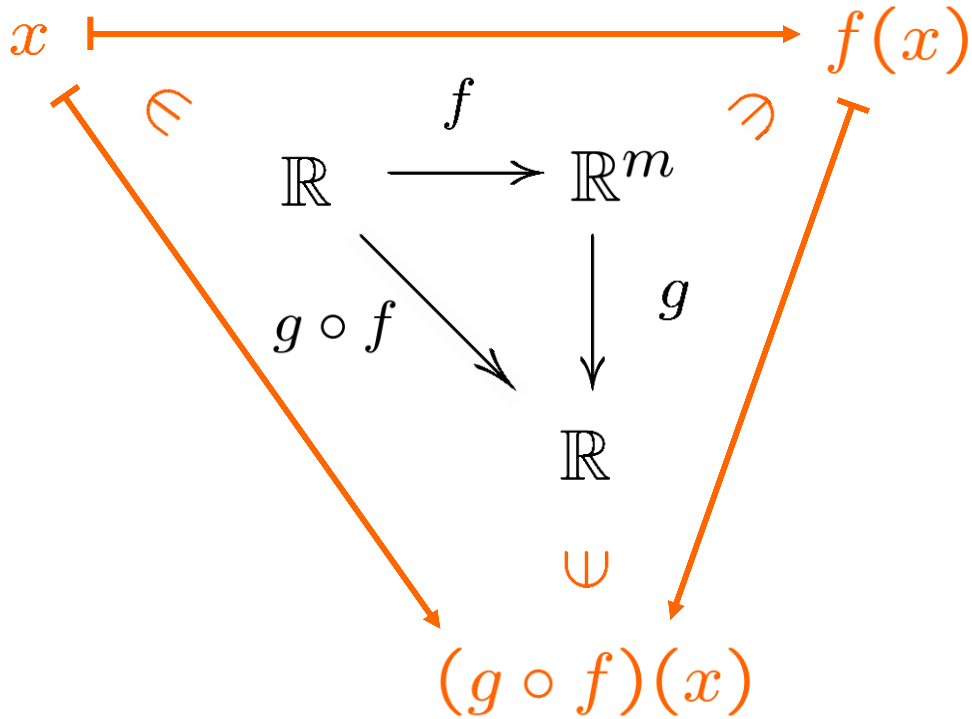
take derivative of the fn
plug in the expression
multiply by
the derivative of
the expression

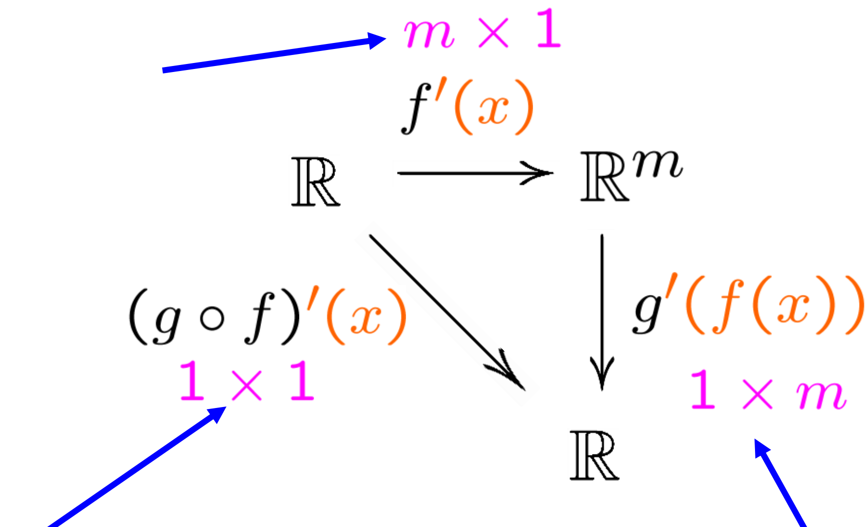
$$f'(x) \in \mathbb{R}^m$$

$$(\nabla g)(f(x)) \in \mathbb{R}^m$$

$$(g \circ f)'(x) \in \mathbb{R}$$

$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$





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$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$

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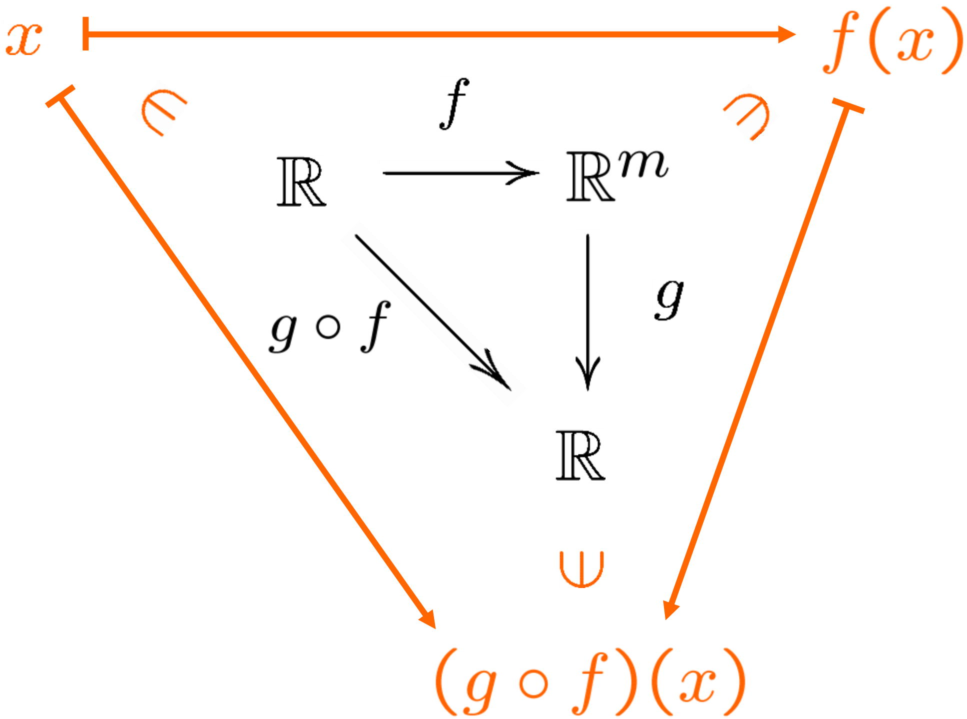
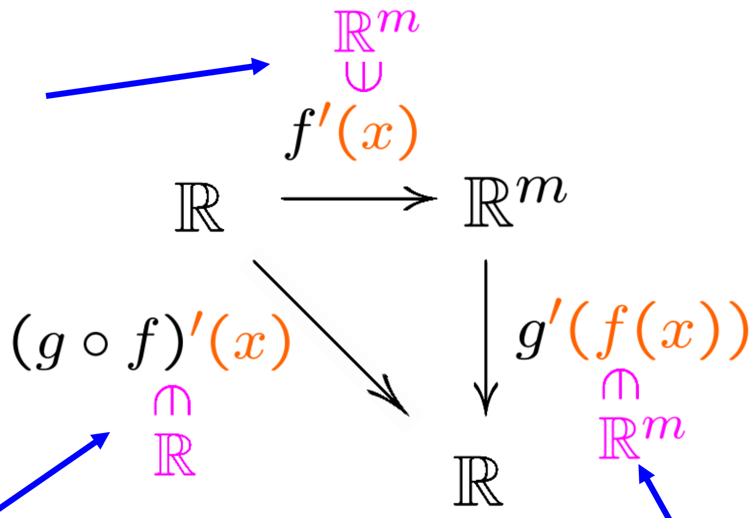




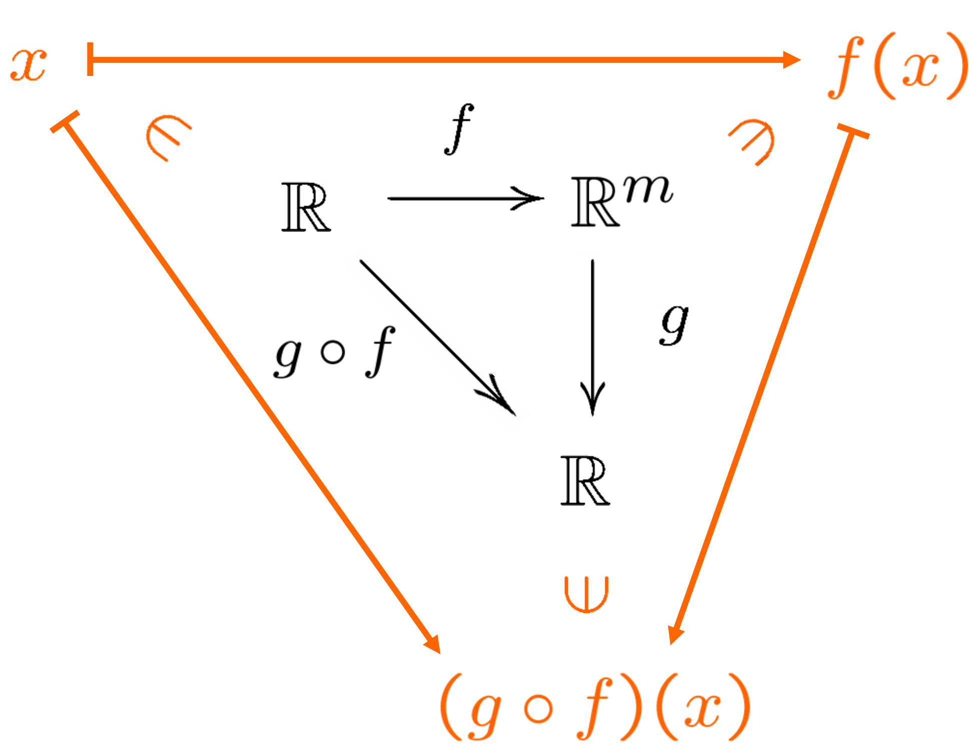
take derivative of the fn
plug in the expression
multiply by the derivative of
the expression

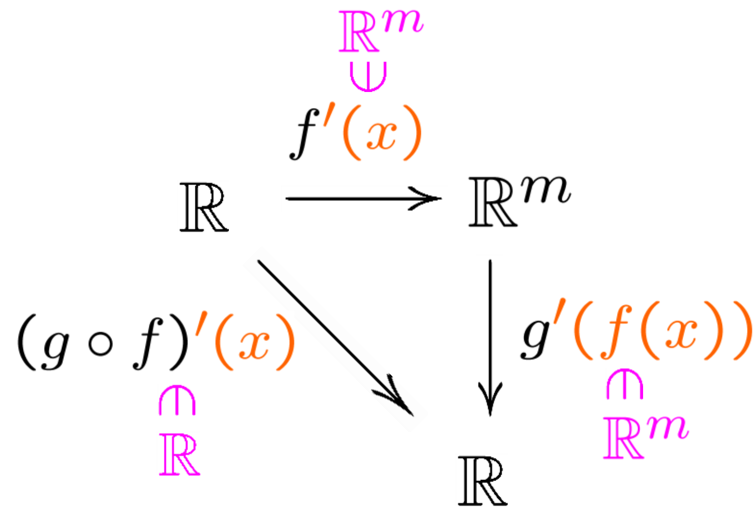
$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$

Fix $a, w \in \mathbb{R}^m$. $\quad f(t) := a + tw \quad f'(t) = w$

$$(g \circ f)'(t) = [\ (\nabla g)(f(t))\ ] \bullet [f'(t)]$$

$$\frac{d}{dt}[g(f(t))]$$





take derivative of the fn
plug in the expression
multiply by the derivative of the expression

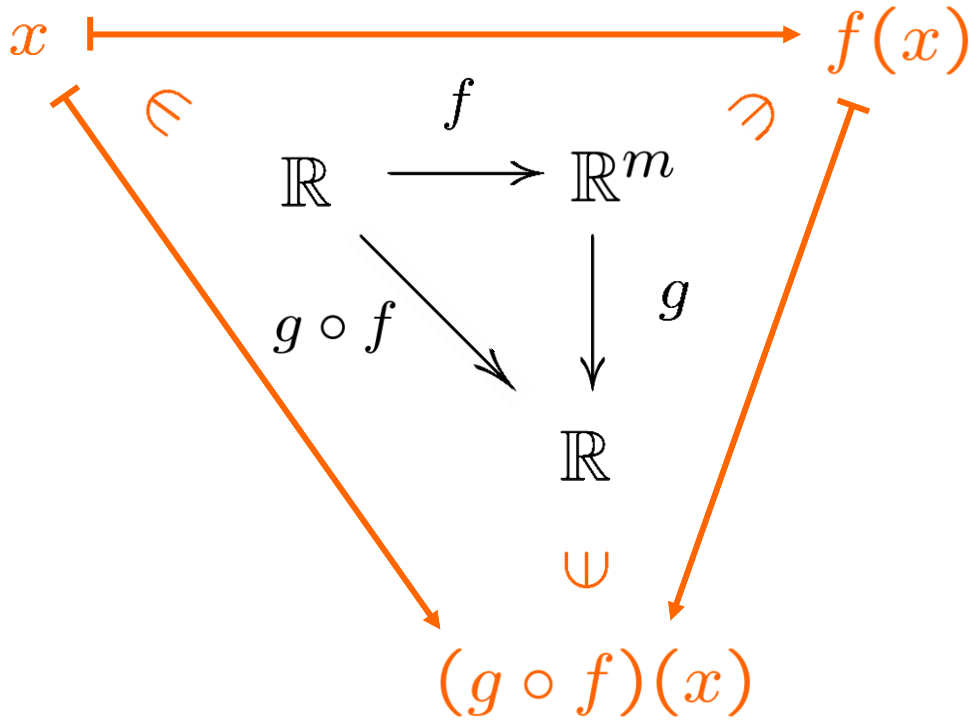
$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$

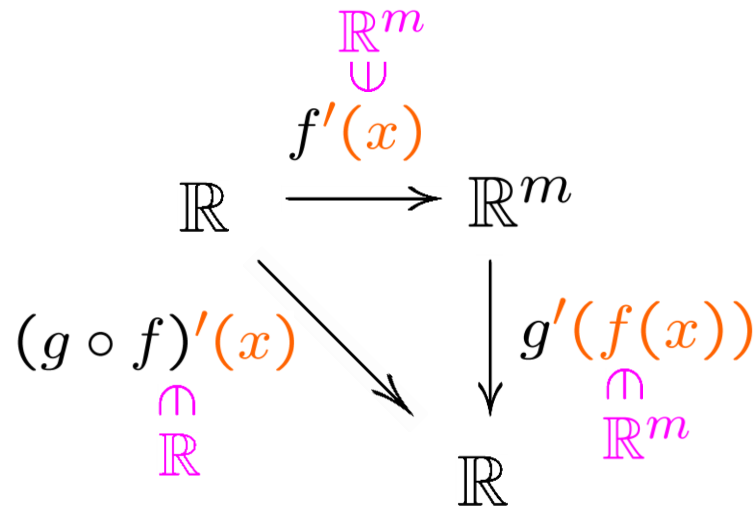
Fix $a, w \in \mathbb{R}^m$. $\qquad f(t) := a + tw \qquad f'(t) = w$

$$(g \circ f)'(t) = [\ (\nabla g)(f(t))\ ] \bullet [f'(t)]$$

$$\frac{d}{dt}[g(a + tw)] = [\ (\nabla g)(a + tw)\ ] \bullet w$$

$t = 0$





take derivative of the fn
plug in the expression
multiply by the derivative of
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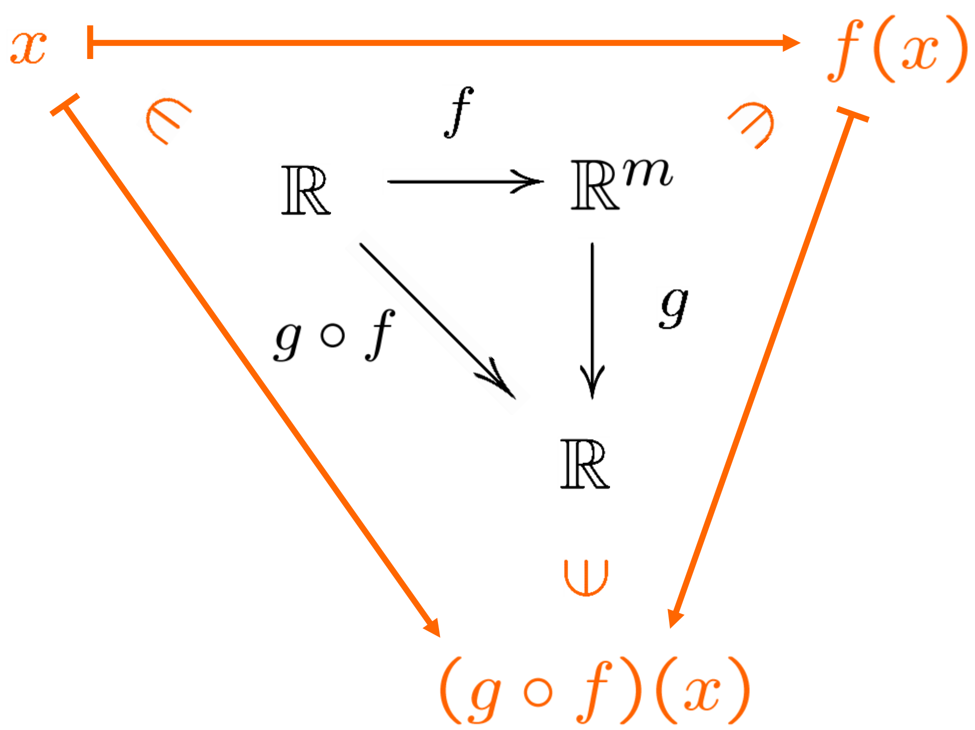
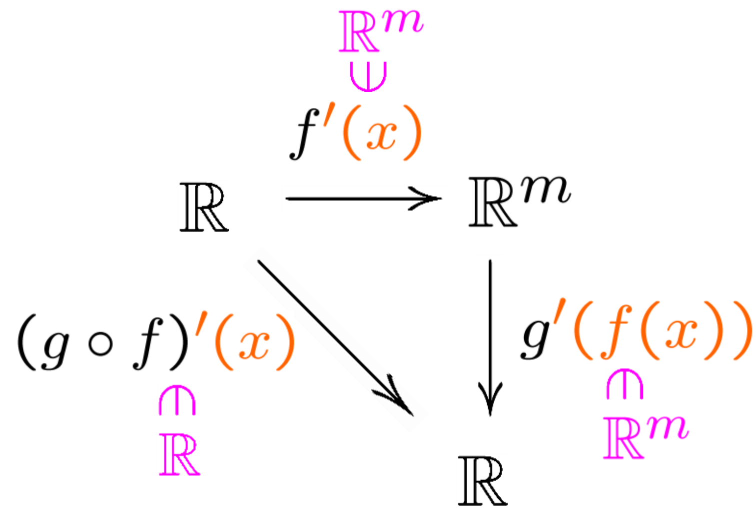
$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$

Fix $a, w \in \mathbb{R}^m$. $\quad f(t) := a + tw \quad f'(t) = w$

$$(g \circ f)'(t) = [\ (\nabla g)(f(t))\ ] \bullet [f'(t)]$$

$$\left[\frac{d}{dt}\right]_{t=0} [g(a + tw)] = [\ (\nabla g)(a + \cancel{0w})\ ] \bullet w$$





take derivative of the fn
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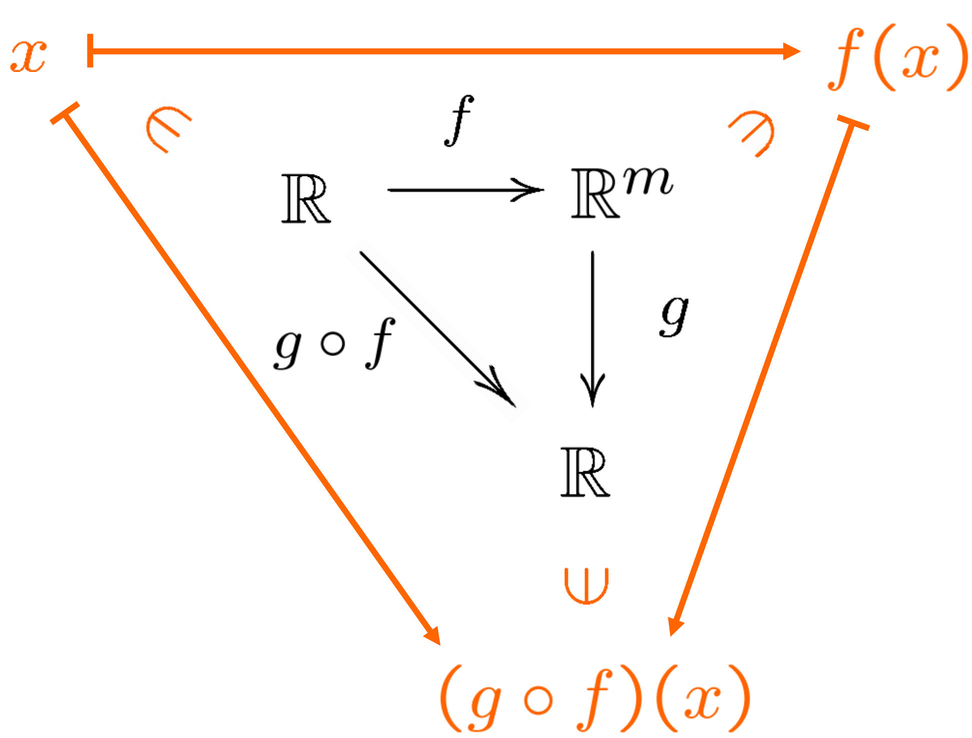
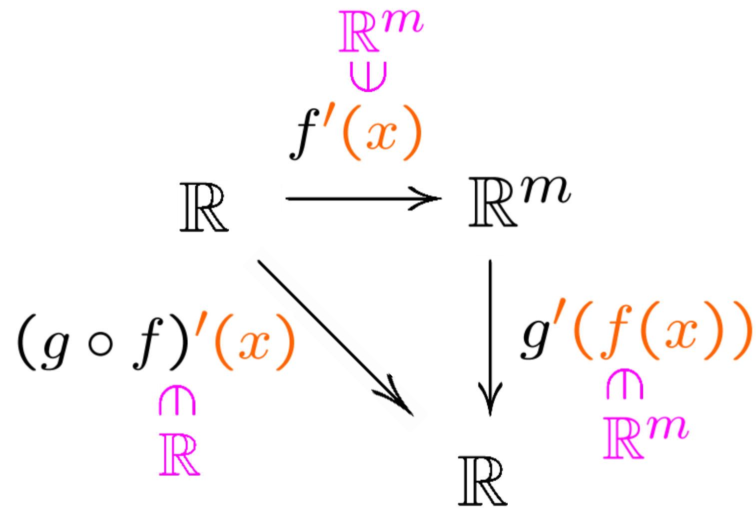
$$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$$

Fix $a, w \in \mathbb{R}^m$. $\qquad f(t) := a + tw \qquad f'(t) = w$

$$(g \circ f)'(t) = [\ \ (\nabla g)(f(t))\ \ ] \bullet [f'(t)]$$

$$\left[\frac{d}{dt}\right]_{t=0} [g(a + tw)] = [\ (\nabla g)(v)\ \ \ \ \ \ ] \bullet w$$





take derivative of the fn
plug in the expression
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$(g \circ f)''(x) = ?$

$(g \circ f)'(x) = [\ (\nabla g)(f(x))\ ] \bullet [\ f'(x)\ ]$

$$\left[\frac{d^2}{dt^2}\right]_{t=0} [g(a + tw)] = ?$$

$$\left[\frac{d}{dt}\right]_{t=0} [g(a + tw)] = [\ (\nabla g)(a)\ ] \bullet w$$

Next subtopic: Application to multivariable Macl. approx.

Problem: Find the second-order Maclaurin approximation, $q(x,y)$, (w.r.t. (x,y)) to the expression $g(x,y) := [\sin(e^{x+3y})] + 4xy + 4y - 2$.

Unassigned exercise:

With $q(x,y)$ and $g(x,y)$ as above, see if you can show that

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - q(x,y)}{x^2 + y^2} = 0,$$

i.e., that $g(x,y) = (q(x,y)) + (o(x^2 + y^2))$ near $(x,y) = (0,0)$.

Know: q is the 2nd order Macl. approx of g

Want: $$\lim_{(x,y)\to(0,0)} \frac{g(x,y)-q(x,y)}{x^2+y^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)-q(x,y)}{x^2+y^2} = 0$$

Know: q is the 2nd order Macl. approx of g

Want: $$\lim_{(x,y)\to(0,0)} \left| \frac{g(x,y) - q(x,y)}{x^2 + y^2} \right| = 0$$

Let $B :=$ ball of radius 100 about $(0,0)$.

Estimate $|(g(x,y)) - (q(x,y))|$ for $(x,y) \in B$,

then divide by $x^2 + y^2$,

then show quotient $\to 0$, as $(x,y) \to (0,0)$.

$(-7,9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$f(t) := g(-7t, 9t)$$
$$p(t) := q(-7t, 9t)$$

$$|(g(-7,9)) - (q(-7,9))| = |(f(1)) - (p(1))|$$

p is the 2nd order M. approx of f

$$f(1) - p(1) = [f'''(s)] \left[\frac{1^3}{3!}\right],$$ some $s \in (0,1)$.

$$f'(t) = [(\partial_1 g)(-7t, 9t)][(d/dt)(-7t)] + [(\partial_2 g)(-7t, 9t)][(d/dt)(9t)]$$

$$= [-7][(\partial_1 g)(-7t, 9t)] + [9][(\partial_2 g)(-7t, 9t)]$$

$$= (-7(\partial_1 g) + 9(\partial_2 g))(-7t, 9t)$$

Chain rule

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$\|$$

$$|(f(1)) - (p(1))|$$

$f(t) := g(-7t, 9t)$

$p(t) := q(-7t, 9t)$

p is the 2nd order M. approx of f

$$f(1) - p(1) = [f'''(s)]\left[\frac{1^3}{3!}\right], \text{ some } s \in (0,1).$$

$$f'(t) = (-7(\partial_1 g) + 9(\partial_2 g))(-7t, 9t)$$

$$f''(t) = (\ \ (-7)^2(\partial_{11} g) \ + \ 2(-7)(9)(\partial_{12} g) + 9^2(\partial_{22} g) \ )\ (-7t, 9t)$$

$$= (-7(\partial_1 g) + 9(\partial_2 g))(-7t, 9t)$$

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$f(t) := g(-7t, 9t)$$
$$p(t) := q(-7t, 9t)$$

$$\| \quad |(f(1)) - (p(1))|$$

p is the 2nd order M. approx of f

$f(1) - p(1) = [f'''(s)] \left[\dfrac{1^3}{3!}\right]$, some $s \in (0, 1)$.

$$f'(t) = (-7(\partial_1 g) + 9(\partial_2 g))(-7t, 9t)$$

$$f''(t) = (\ (-7)^2(\partial_{11} g) \ + \ 2(-7)(9)(\partial_{12} g)$$
$$+ \ 9^2(\partial_{22} g) \ )\ (-7t, 9t)$$

$$f'''(t) = (\ (-7)^3(\partial_{111} g) \ + \ 3(-7)^2(9)(\partial_{112} g)$$
$$3(-7)(9)^2(\partial_{122} g) \ + \ 9^3(\partial_{222} g) \ )\ (-7t, 9t)$$

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$\|$$

$$|(f(1)) - (p(1))|$$

$f(t) := g(-7t, 9t)$

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p is the 2nd order M. approx of f

$$f(1) - p(1) = [f'''(s)]\left[\frac{1^3}{3!}\right], \text{ some } s \in (0,1).$$

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$$3(-7)(9)^2(\partial_{122}g)\ +\ 9^3(\partial_{222}g)\ )\ (-7t, 9t)$$

Choose $M > 0$ so large that: $|\partial_{111}g| < M$, $|\partial_{112}g| < M$, $|\partial_{122}g| < M$, $|\partial_{222}g| < M$ on B.

$$f'''(t) = (\ (-7)^3(\partial_{111}g)\ +\ 3(-7)^2(9)(\partial_{112}g)$$
$$3(-7)(9)^2(\partial_{122}g)\ +\ 9^3(\partial_{222}g)\ )\ (-7t, 9t)$$

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$||$$

$$|(f(1)) - (p(1))|$$

$$f(t) := g(-7t, 9t)$$
$$p(t) := q(-7t, 9t)$$

p is the 2nd order M. approx of f

$$f(1) - p(1) = [f'''(s)]\left[\frac{1^3}{3!}\right],$$ some $s \in (0, 1)$.

$$f'''(t) = \left(\ (-7)^3(\partial_{111}g) \ + \ 3(-7)^2(9)(\partial_{112}g) \right.$$
$$\left. 3(-7)(9)^2(\partial_{122}g) \ + \ 9^3(\partial_{222}g) \ \right) (-7t, 9t)$$

Choose $M > 0$ so large that: $|\partial_{111}g| < M$, $|\partial_{112}g| < M$, $|\partial_{122}g| < M$, $|\partial_{222}g| < M$ on B.

$$|f'''(s)| < ((7^3) + 3(7)^2(9) + 3(7)(9)^2 + (9)^3)M$$
$$= (7+9)^3 M$$

$$|f(1) - p(1)| = |f'''(s)|/6 < (7+9)^3 M/6$$

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$\|$$

$$|(f(1)) - (p(1))|$$

$$f(t) := g(-7t, 9t)$$
$$p(t) := q(-7t, 9t)$$

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$$f(1) - p(1) = [f'''(s)] \left[\frac{1^3}{3!} \right], \text{ some } s \in (0,1).$$

$$f'''(t) = (\ (-7)^3(\partial_{111}g)\ +\ 3(-7)^2(9)(\partial_{112}g)$$

$$3(-7)(9)^2(\partial_{122}g)\ +\ 9^3(\partial_{222}g)\ )\ (-7t, 9t)$$

Choose $M > 0$ so large that: $|\partial_{111}g| < M$, $|\partial_{112}g| < M$, $|\partial_{122}g| < M$, $|\partial_{222}g| < M$ on B.

$$|f'''(s)| < ((7^3) + 3(7)^2(9) + 3(7)(9)^2 + (9)^3)M$$

$$= (7+9)^3 M$$

$$|f(1) - p(1)| = |f'''(s)|/6 < (7+9)^3 M/6$$

$(-7, 9) \in B$ Estimate $|(g(-7,9)) - (q(-7,9))|$

$$\| $$

$$|(f(1)) - (p(1))|$$

$$\wedge$$

$$(7+9)^3 M/6$$

$\forall (x, y) \in B$,

$$|(g(x,y)) - (q(x,y))| < (|x| + |y|)^3 M/6$$

Let $B :=$ ball of radius 100 about $(0,0)$,
Estimate $|(g(x,y)) - (q(x,y))|$ for $(x,y) \in B$,
then divide by $x^2 + y^2$,
then show quotient $\to 0$, as $(x,y) \to (0,0)$.

Want: $\dfrac{(|x| + |y|)^3}{x^2 + y^2} \to 0$, as $(x,y) \to (0,0)$.

IOU
Fact: $(|x| + |y|)^2 \le 2(x^2 + y^2)$

$$\frac{(|x| + |y|)^3}{x^2 + y^2} \le 2(|x| + |y|) \to 0, \text{ as } (x,y) \to (0,0)$$

QED

$\forall (x,y) \in B$,

$$|(g(x,y)) - (q(x,y))| < (|x| + |y|)^3 M/6$$

Let $B :=$ ball of radius 100 about $(0,0)$,
Estimate $|(g(x,y)) - (q(x,y))|$ for $(x,y) \in B$,
then divide by $x^2 + y^2$,
then show quotient $\to 0$, as $(x,y) \to (0,0)$.

Want: $$\frac{(|x| + |y|)^3}{x^2 + y^2} \to 0, \quad \text{as } (x,y) \to (0,0).$$

IOU

Fact: $(|x| + |y|)^2 \leq 2(x^2 + y^2)$

Proof: $0 \leq (|x| - |y|)^2 = x^2 + y^2 - 2|x||y|$

$$2|x||y| \leq x^2 + y^2$$

$$(|x| + |y|)^2 = x^2 + y^2 + 2|x||y|$$
$$\leq x^2 + y^2 + x^2 + y^2$$

QED

STOP