Financial Mathematics

Lagrange multipliers and constrained approximation

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Problem: Find the major and minor axes of

$$E := \{(x,y)\,|\,Q(x,y) = 8\}$$

$$M := \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$Q = Q_M$$

$$Q_M(v) = [L_M(v)] \cdot v$$

$$M \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$R := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{N} := \boxed{R^{-1} M R} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$R^t = R^{-1}$$

$$(Q \circ L_R)(x,y) = 2x^2 + 4y^2$$

$$\begin{aligned} (Q \circ L_R)(v) &= [L_M(L_R(v))] \cdot (\boxed{L_R}(v)) \\ &= [L_{R^t}(L_M(L_R(v)))] \cdot v \\ &= [L_{\boxed{R^{-1} M R}}(v)] \cdot v \\ &= [L_{\boxed{N}}(v)] \cdot v = Q_N(v) \end{aligned}$$

$$\begin{aligned} (Q \circ L_R)(x,y) \\ \| \\ Q_N(x,y) \\ /\!/ \\ 2x^2 + 4y^2 \end{aligned}$$

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$$F = (Q \circ L_R)^{-1}(8) = {L_R}^{-1}(Q^{-1}(8))$$

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is what we want to graph.

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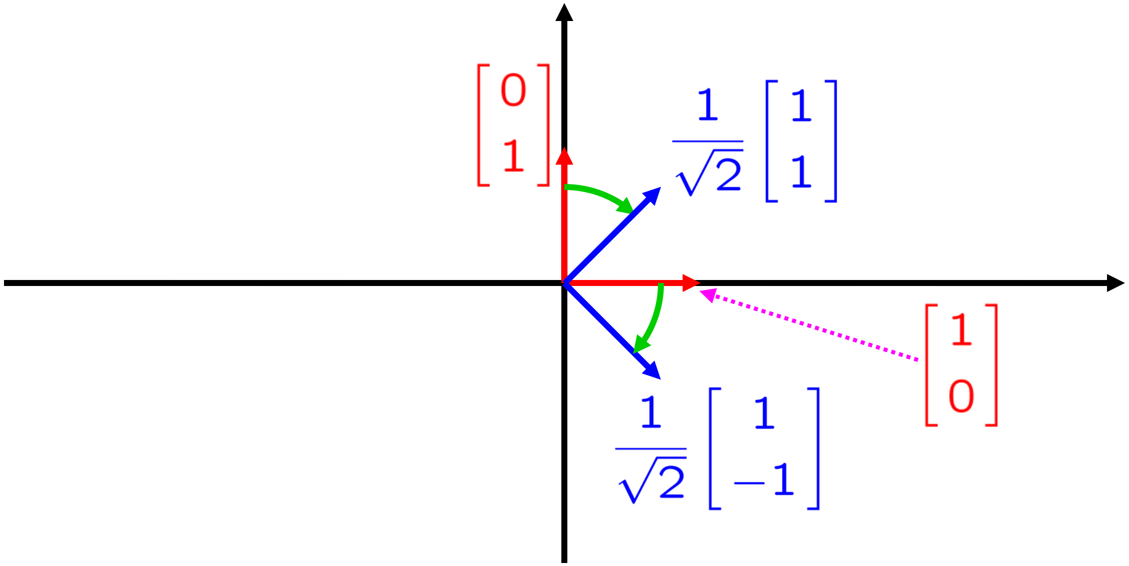

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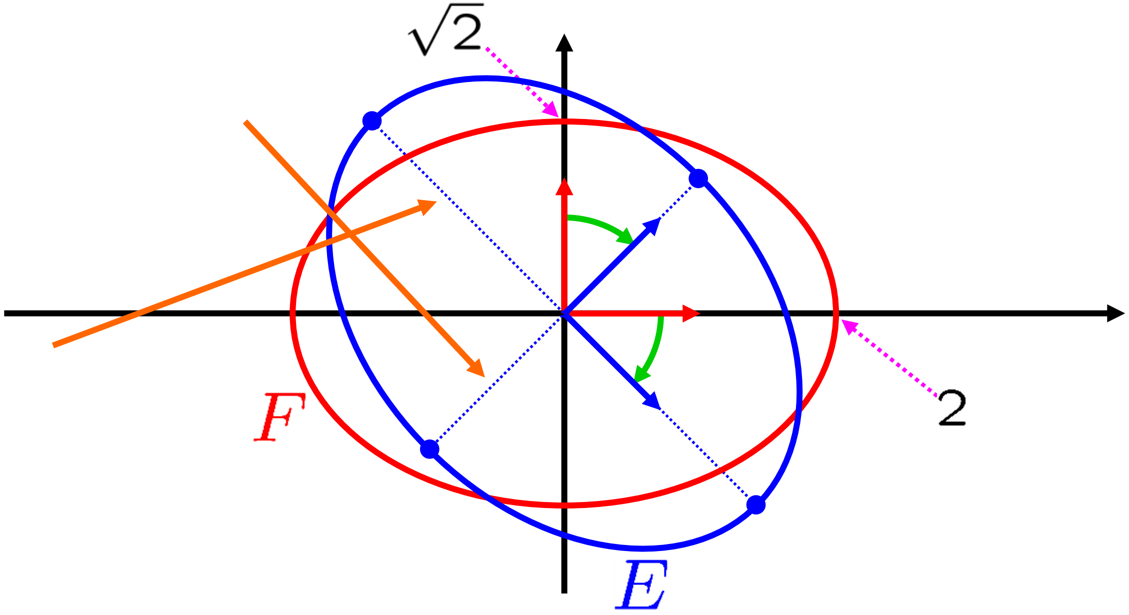

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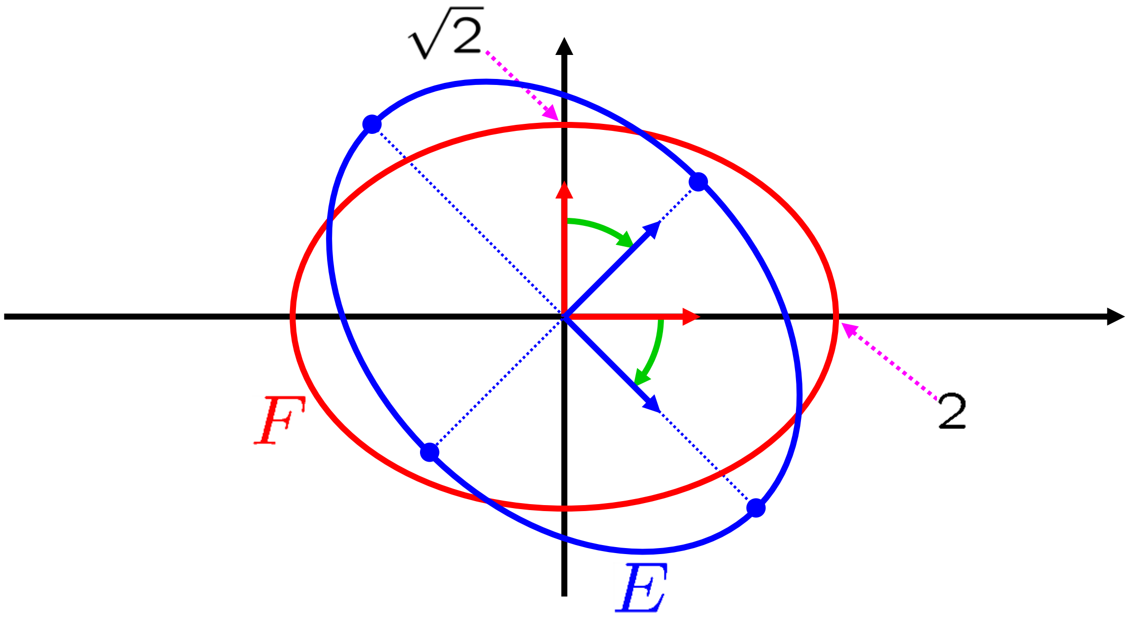

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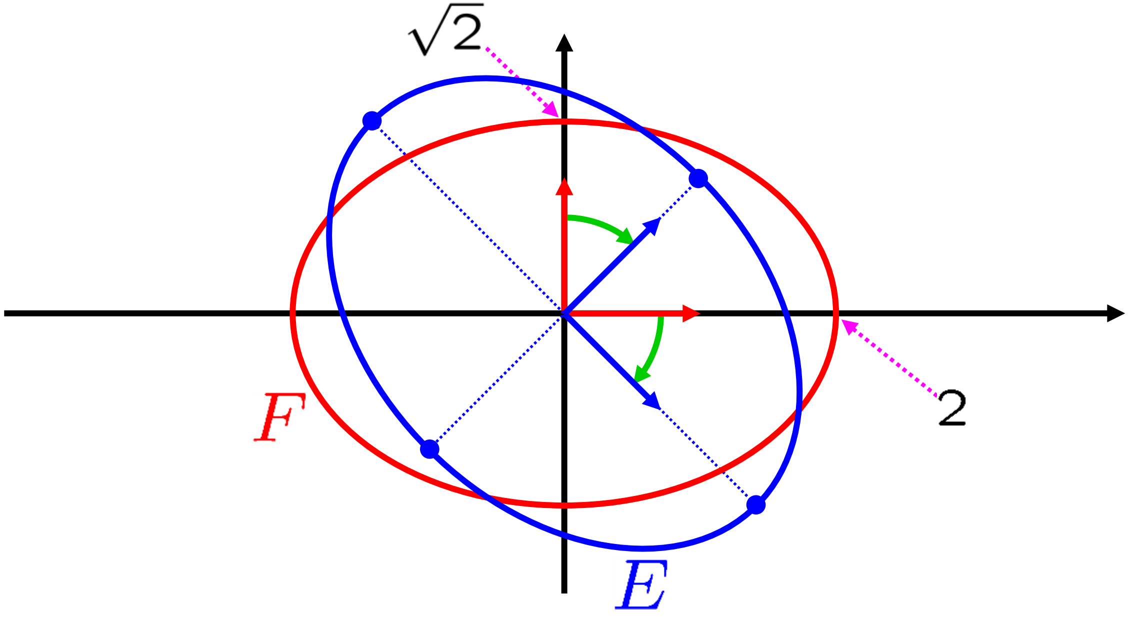

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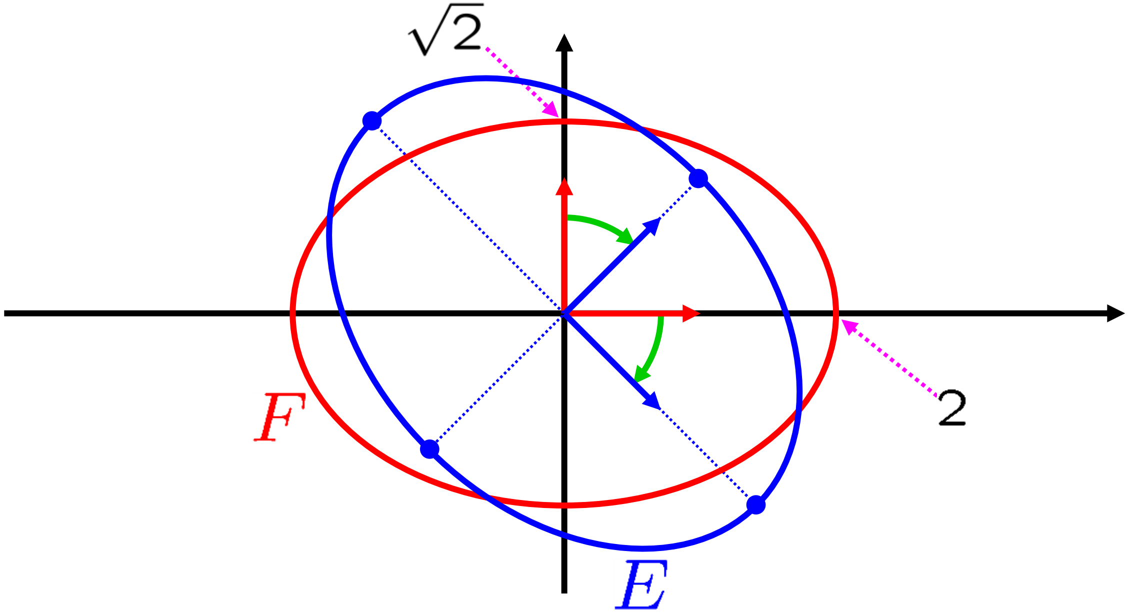

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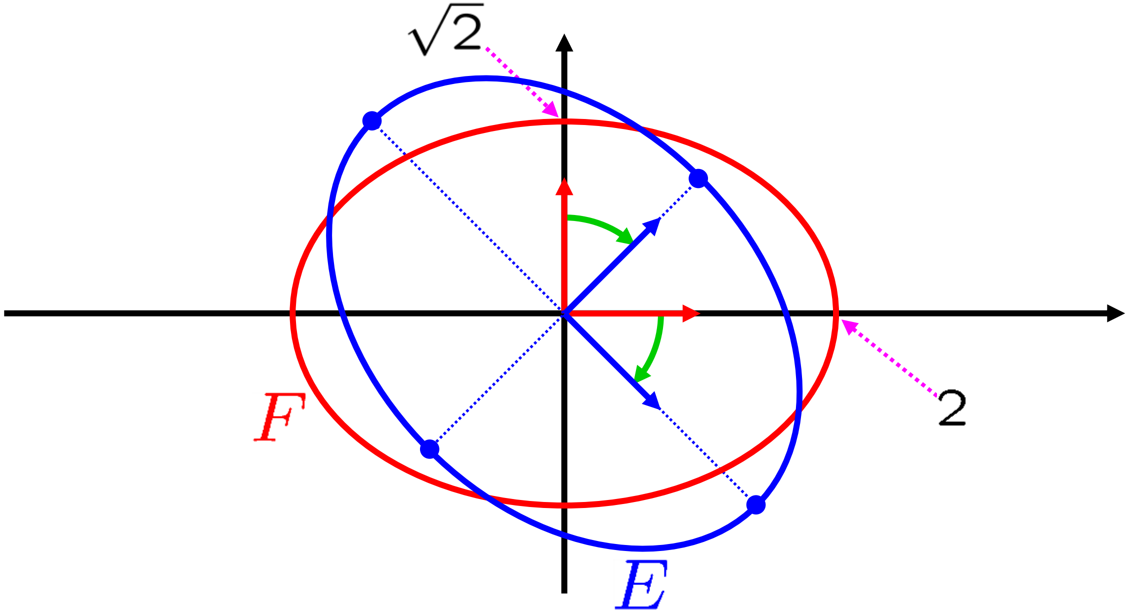

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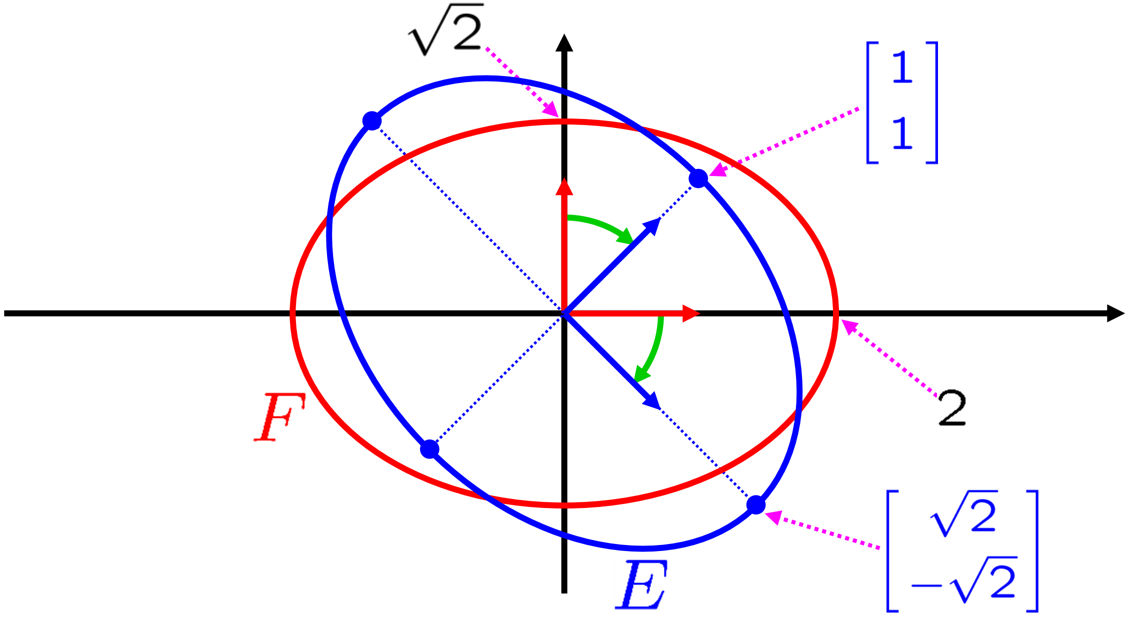

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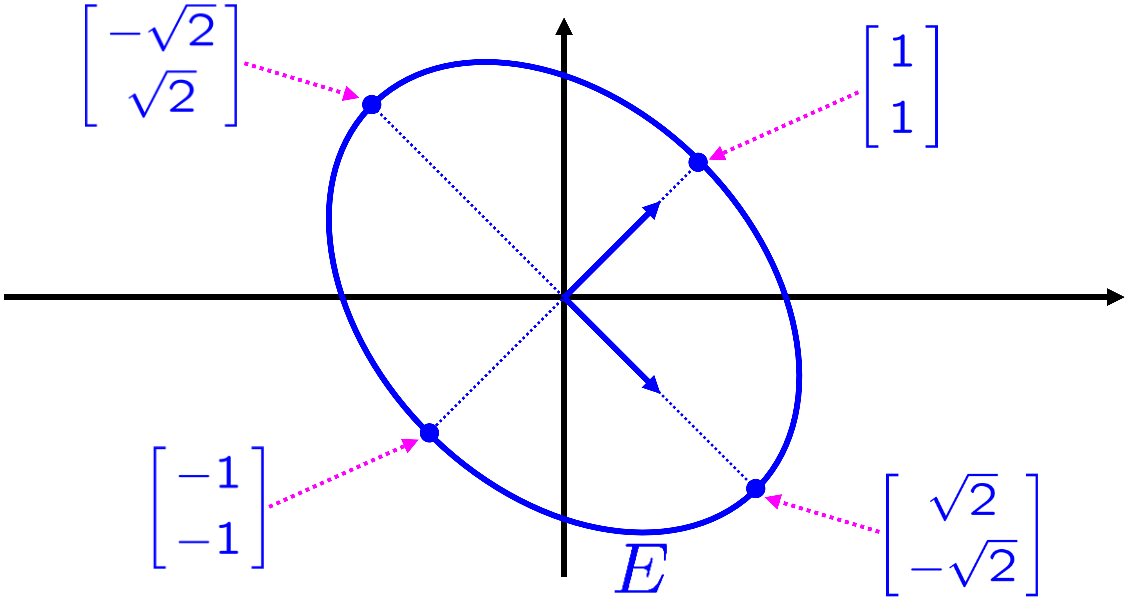

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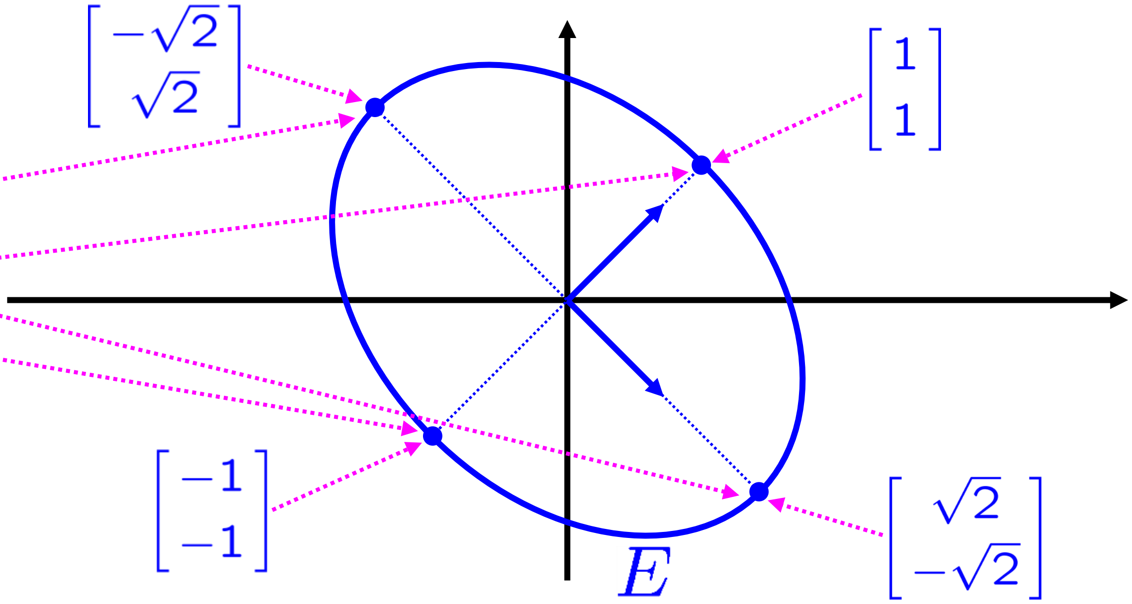
Alternate sol'n:

$f(x,y) := \sqrt{x^2 + y^2}$

(Maximize)

Minimize $f(x,y)$, subject to the constraint that $Q(x,y) = 8$.



That is, find a point (a,b) s.t.

$Q(a,b) = 8$ and $f(a,b) = \min_{p \in E} f(p)$

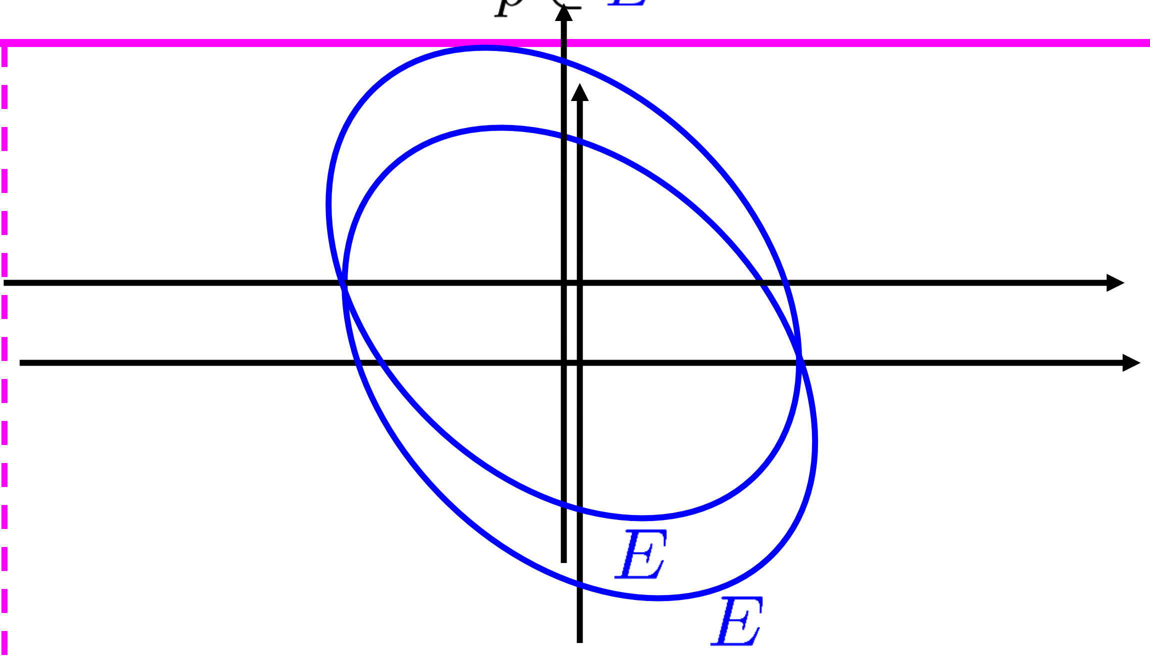
Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$$Q(a,b) = 8 \quad \text{and} \quad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := \boxed{\sqrt{x^2+y^2}}$$



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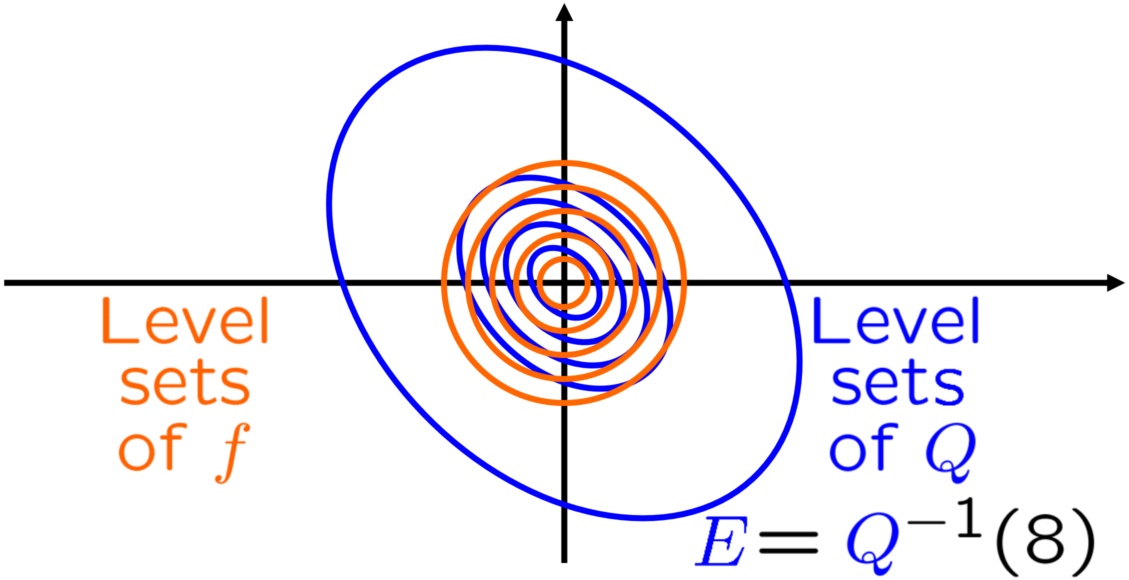
Find a point (a, b) s.t.

$$Q(a, b) = 8 \quad \text{and} \quad f(a, b) = \min_{p \in E} f(p)$$

$$f(x, y) := \boxed{x^2 + y^2}$$

$$r := (0.25, -0.15)$$



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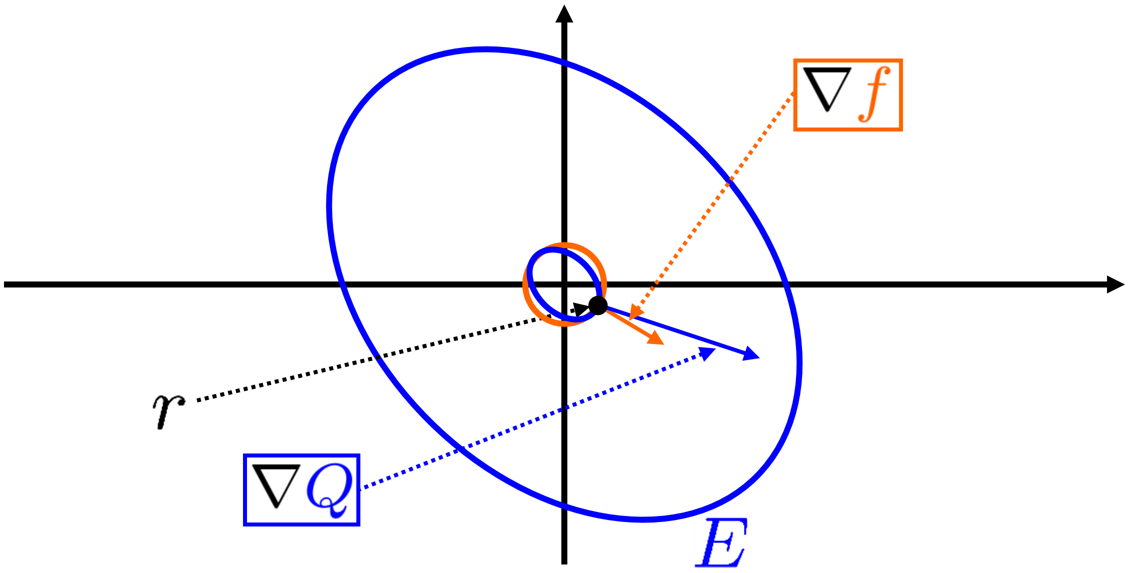

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla f)(r) = (.5, -.3)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$$(\nabla Q)(r) = (1.2, -0.4)$$

KEY POINT:
The gradient is perpendicular to the level set.

Expect two answers: $(a,b) = \pm(1,1)$

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Find a point (a,b) s.t.

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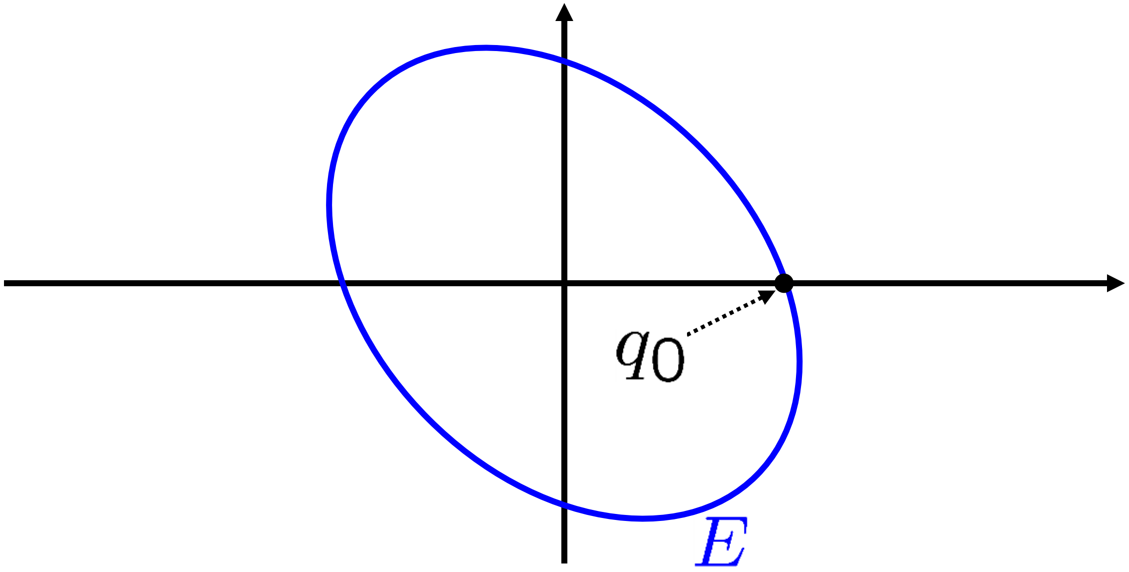
Pick a point $q_0 \in E$



Start at q_0.

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

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Find a point (a,b) s.t.

$$Q(a,b) = 8 \quad \text{and} \quad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := x^2 + y^2$$

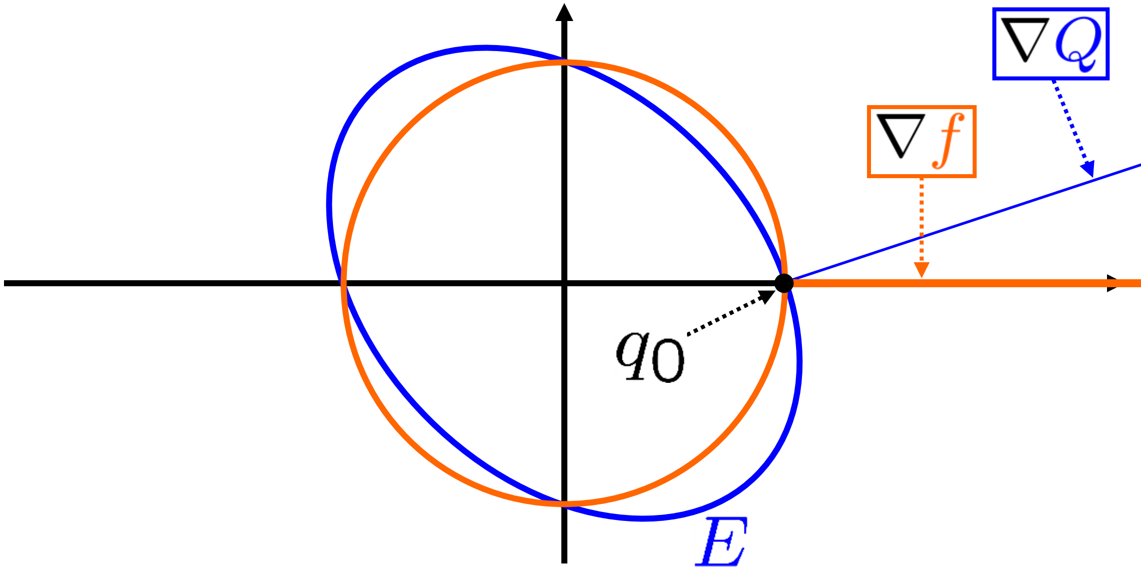
$$s = \sqrt{8/3} \approx 1.633$$

$$q_0 = (s,0) \in E$$

Start at q_0.

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla f)(q_0) = (2s, 0)$$



$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

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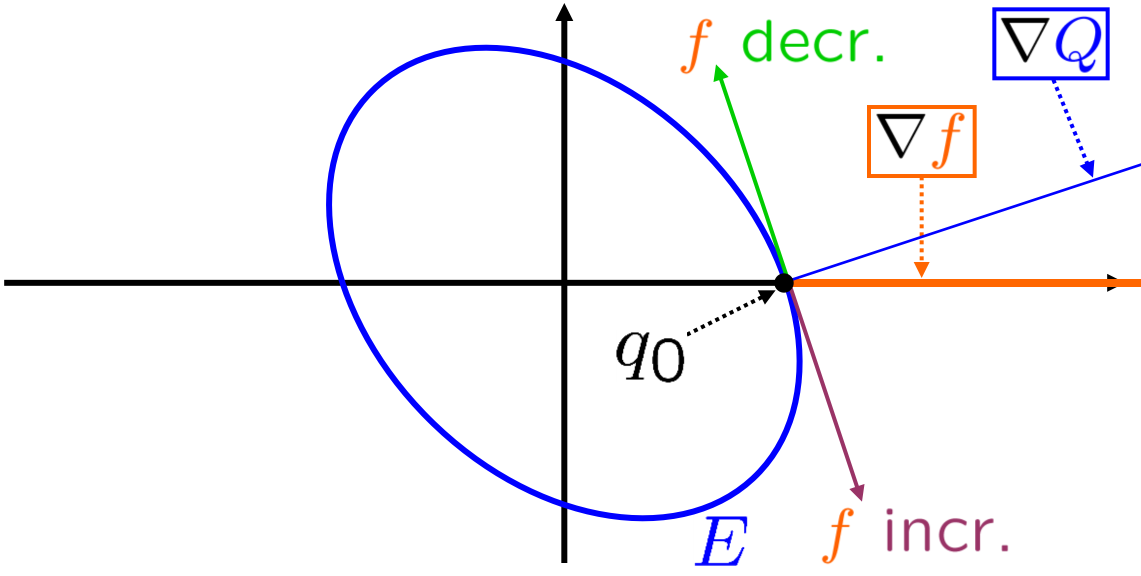

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

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Follow blue constraint. To decrease f, go in the direction of the green arrow.

Expect two answers: $(a,b) = \pm(1,1)$

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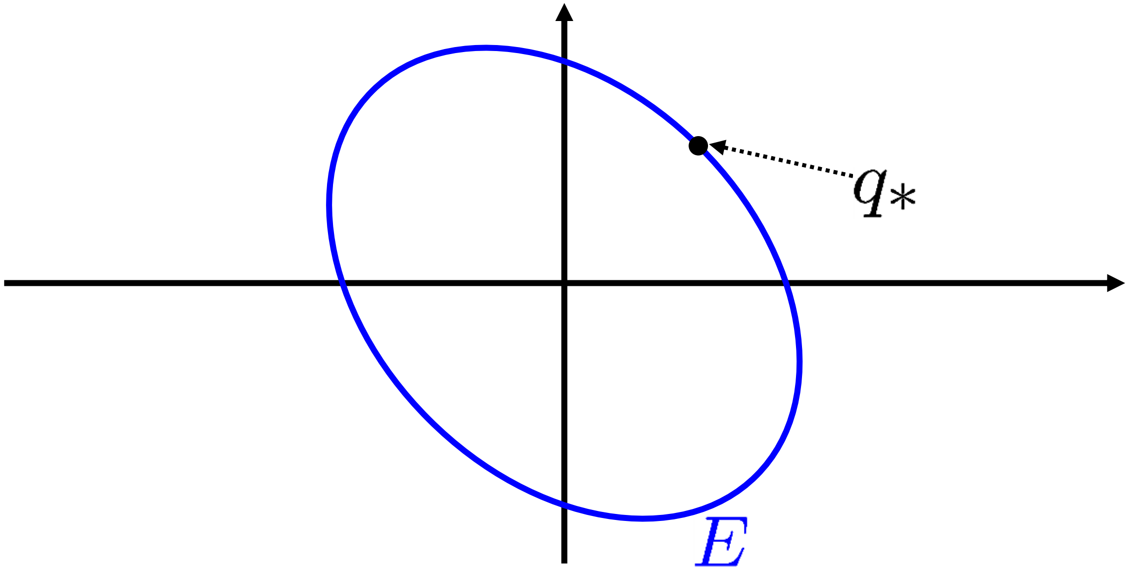

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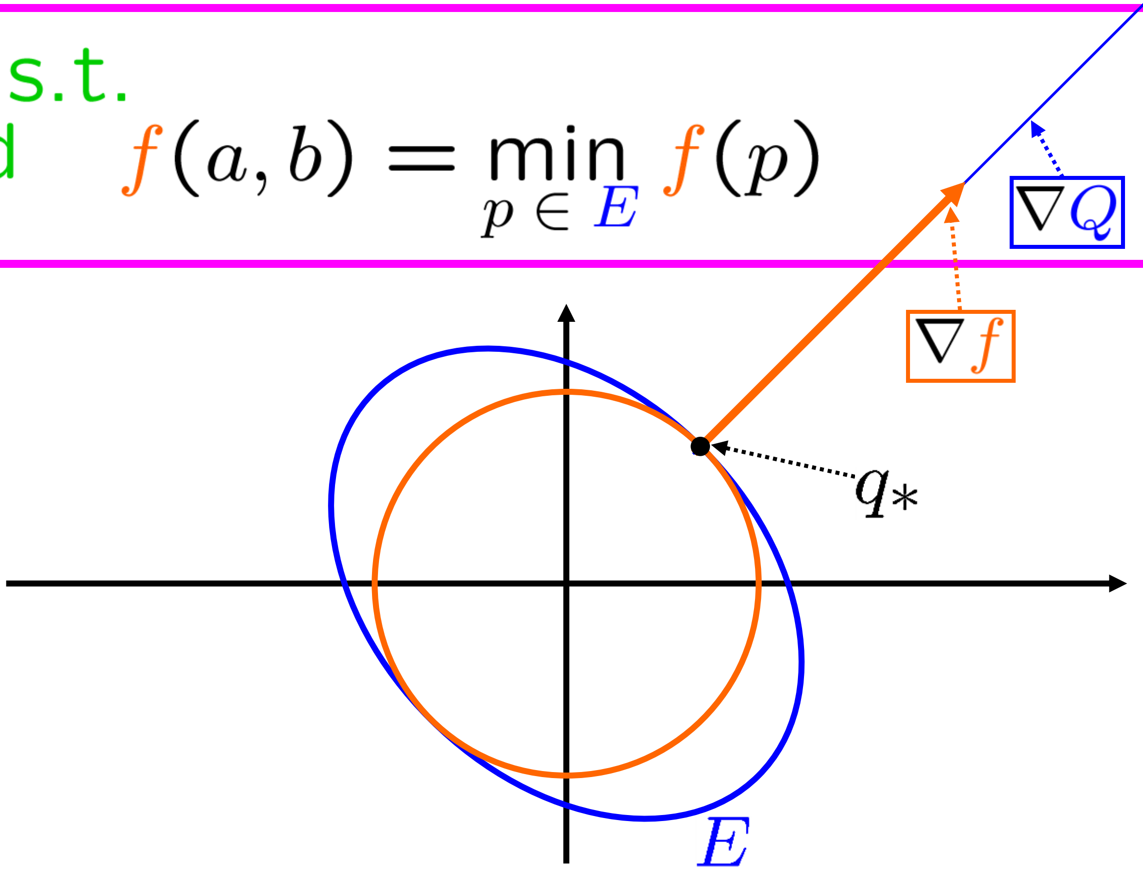

$$q_* = (1,1) \in E$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla f)(q_*) = (2,2)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$$(\nabla Q)(q_*) = (8,8)$$

Follow blue constraint.
To first order, can't
incr. or decr. f.
q_* is a CRITICAL PT.

Expect two answers: $(a,b) = \pm(1,1)$

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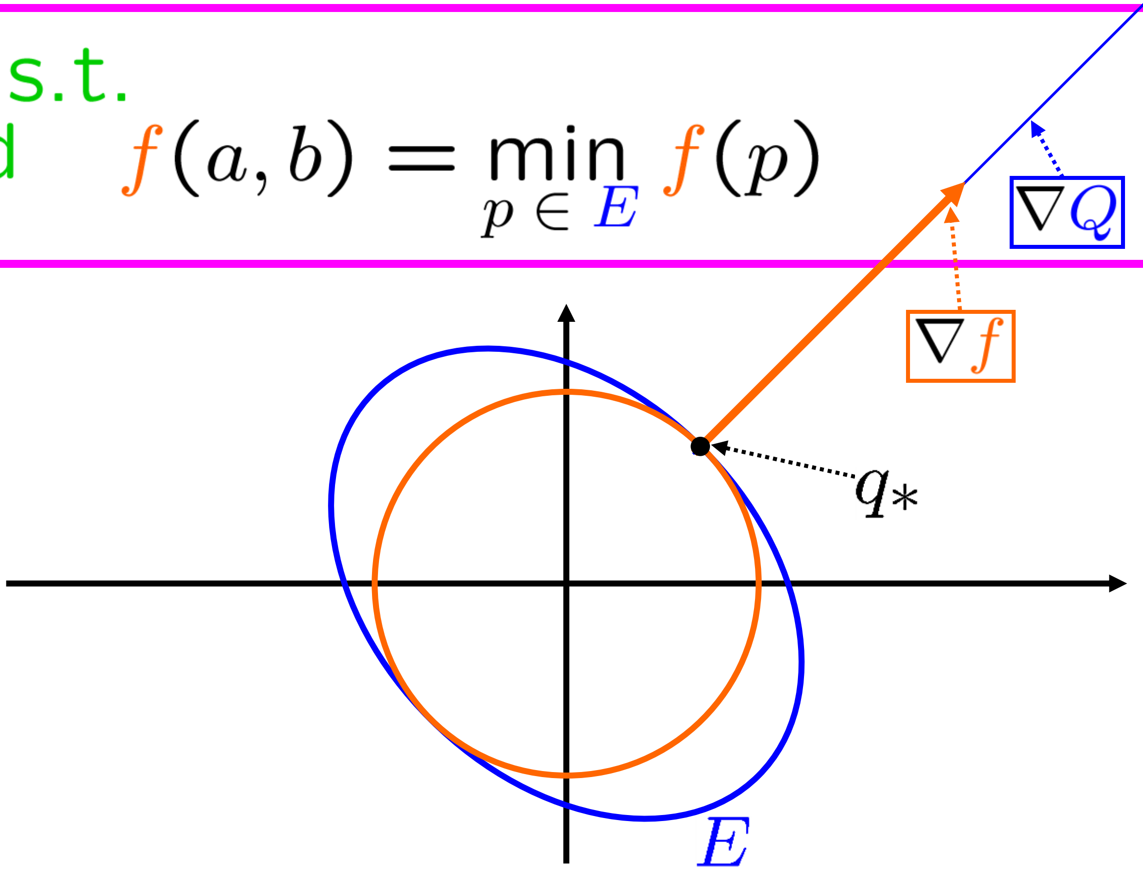

A critical point occurs when ∇Q is parallel to ∇f.

p is a critical point if $\exists \lambda \in \mathbb{R}$ s.t. $(\nabla f)(p) = \lambda[(\nabla Q)(p)]$.

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$$(\nabla f)(x,y) = (2x, 2y)$$
$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

To find critical points, solve

$$Q(x,y) = 8$$
$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many equations? 3

scalar

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$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many $\underbrace{\text{equations}}_{\text{scalar}}$? 3

$$3x^2 + 2xy + 3y^2 = 8$$

$$2x = \lambda[6x + 2y]$$

$$2y = \lambda[2x + 6y]$$

multiply by $[2x + 6y]$, get this

$$\boxed{[2x][2x + 6y] = \lambda[2x + 6y][6x + 2y]}$$

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$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many equations? 3

$\underbrace{\qquad\qquad\qquad}_{\text{scalar}}$

$$3x^2 + 2xy + 3y^2 = 8$$

$$2x = \lambda[6x + 2y]$$

$$\boxed{2y} = \boxed{\lambda[2x + 6y]}$$

$$[2x][2x + 6y] = \boxed{\lambda[2x + 6y]}[6x + 2y]$$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$$Q(a,b) = 8 \qquad \text{and} \qquad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := x^2 + y^2$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

To find critical points, solve

$$Q(x,y) = 8$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many equations? 3

scalar

$$3x^2 + 2xy + 3y^2 = 8$$

$$2x = \lambda[6x + 2y]$$

$$\boxed{2y} = \boxed{\lambda[2x + 6y]}$$

$$4x^2 + \cancel{12xy} = \cancel{12xy} + 4y^2$$

$$[2x][2x + 6y]$$

$$= \boxed{[2y]}\,[6x + 2y]$$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

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$$4x^2 + \cancel{12xy} = \cancel{12xy} + 4y^2$$

$$x^2 = y^2$$

$$x = \pm y$$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$$Q(a,b) = 8 \quad \text{and} \quad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := x^2 + y^2$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$x = -y$

LATER

To find critical points, solve

$$Q(x,y) = 8$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

$$\text{How many } \underbrace{\text{equations}}_{\text{scalar}}\text{?} \quad 3$$

$$3x^2 + 2xy + 3y^2 = 8$$

$$x = y$$

$$3x^2 + 2x^2 + 3x^2 = 8$$

$$4x^2 + \cancel{12xy} = \cancel{12xy} + 4y^2$$

$$x^2 = y^2$$

$$x = \pm y$$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$Q(a,b) = 8$ and $f(a,b) = \min\limits_{p \in E} f(p)$

$$f(x,y) := x^2 + y^2$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$x = -y$

LATER

To find critical points, solve

$$Q(x,y) = 8$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many $\underbrace{\text{equations}}_{\text{scalar}}$? 3

$$3x^2 + 2xy + 3y^2 = 8$$

$$x = y$$

$$3x^2 + 2x^2 + 3x^2 = 8$$

$$8x^2 = 8$$

$$x^2 = 1$$

$$y = x = \pm 1$$

$(x,y) = (1,1)$

$f(x,y) = 2$

$(x,y) = (-1,-1)$

$f(x,y) = 2$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$$Q(a,b) = 8 \quad \text{and} \quad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := x^2 + y^2$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$x = -y$

LATER

$(x,y) = (1,1)$
$f(x,y) = 2$

$(x,y) = (-1,-1)$
$f(x,y) = 2$

To find critical points, solve

$$Q(x,y) = 8$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many $\underbrace{\text{equations}}_{\text{scalar}}$? 3

$$3x^2 + 2xy + 3y^2 = 8$$

$$x = \boxed{y}$$

$$3x^2 + 2x^2 + 3x^2 = 8$$

$$8x^2 = 8$$

$$x^2 = 1$$

$$y = x = \pm 1$$

$(x,y) = (1,1)$
$f(x,y) = 2$

$(x,y) = (-1,-1)$
$f(x,y) = 2$

Expect two answers: $(a,b) = \pm(1,1)$

$$Q(x,y) = 3x^2 + 2xy + 3y^2$$

Find a point (a,b) s.t.

$$Q(a,b) = 8 \quad \text{and} \quad f(a,b) = \min_{p \in E} f(p)$$

$$f(x,y) := x^2 + y^2$$

$$(\nabla f)(x,y) = (2x, 2y)$$

$$(\nabla Q)(x,y) = (6x + 2y, 2x + 6y)$$

$x = -y$

LATER

$(x,y) = (1,1)$

$f(x,y) = 2$

$(x,y) = (-1,-1)$

$f(x,y) = 2$

To find critical points, solve

$$Q(x,y) = 8$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)]$$

How many unknowns? 3

How many $\underbrace{\text{equations}}_{\text{scalar}}$? 3

$$3x^2 + 2xy + 3y^2 = 8$$

$$x = \boxed{-y}$$

$$3x^2 - 2x^2 + 3x^2 = 8$$

$$4x^2 = 8$$

$$x^2 = 2$$

$$-y = x = \pm\sqrt{2}$$

$(x,y) = (\sqrt{2}, -\sqrt{2})$

$f(x,y) = 4$

$(x,y) = (-\sqrt{2}, \sqrt{2})$

$f(x,y) = 4$

Expect two answers: $(a,b) = \pm(1,1)$ ☺

objective

To minimize $\boxed{f}(x_1, \ldots, x_n)$,

constraint

subject to $\boxed{Q(x_1, \ldots, x_n) = C}$,

find critical points, by solving

$$Q(x_1, \ldots, x_n) = C$$

$$(\nabla f)(x_1, \ldots, x_n) = \boxed{\lambda}[(\nabla Q)(x_1, \ldots, x_n)],$$

then minimize f

over these critical points.

Lagrange multiplier

$n + 1$

How many unknowns?

How many equations?

scalar

$1 + n$

objective

constraint

To minimize $\boxed{f}(x_1, \ldots, x_n)$,

subject to $\boxed{Q(x_1, \ldots, x_n) = C}$,

find critical points, by solving

$$Q(x_1, \ldots, x_n) = C$$

$$(\nabla f)(x_1, \ldots, x_n) = \lambda[(\nabla Q)(x_1, \ldots, x_n)],$$

then minimize f

over these critical points.

Also need to check all "non-smooth" points for the constraint,

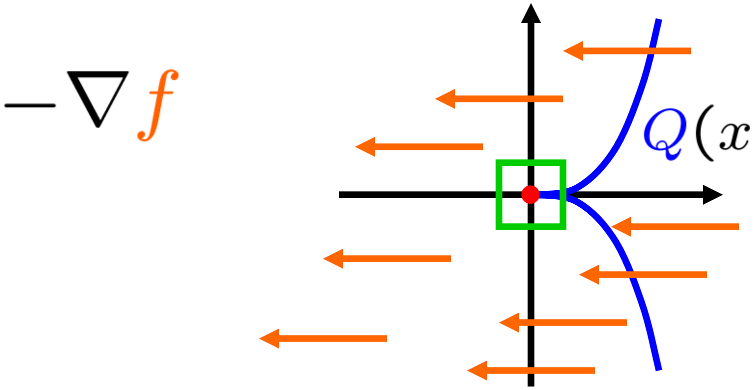
i.e., all the points $(x_1, \ldots, x_n)$ s.t.

$$(\nabla Q)(x_1, \ldots, x_n) = (0, \ldots, 0).$$

Example: Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $Q(x,y) = x^3 - y^2$.

Minimize $f(x,y)$ subject to $Q(x,y) = 0$.

Minimum occurs at $(x,y) = (0,0)$, where we have $f(0,0) = 0$.



Also need to check all "non-smooth" points for the constraint, *i.e.*, all the points $(x_1, \ldots, x_n)$ s.t. $(\nabla Q)(x_1, \ldots, x_n) = (0, \ldots, 0)$.

Example: Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $Q(x,y) = x^3 - y^2$.

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Minimum occurs at $(x,y) = (0,0)$, where we have $f(0,0) = 0$.

However, $(\nabla f)(x,y) = (1,0)$, and $(\nabla Q)(x,y) = (3x^2, -2y)$, so we cannot solve

$$Q(x,y) = 0$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)].$$

$1 = \lambda[3x^2] = 0$

$\lambda \neq 0$

$0 = \lambda[2y]$

$y = 0$

$x^3 - y^2 = 0$

$x = 0$

Also need to check all "non-smooth" points for the constraint, i.e., all the points $(x_1, \ldots, x_n)$ s.t.

$$(\nabla Q)(x_1, \ldots, x_n) = (0, \ldots, 0).$$

Example: Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $Q(x,y) = x^3 - y^2$.

Minimize $f(x,y)$ subject to $Q(x,y) = 0$.

Minimum occurs at $(x,y) = (0,0)$, where we have $f(0,0) = 0$.

However, $(\nabla f)(x,y) = (1,0)$, and $(\nabla Q)(x,y) = (3x^2, -2y)$, so we cannot solve

$$Q(x,y) = 0$$

$$(\nabla f)(x,y) = \lambda[(\nabla Q)(x,y)].$$

Also need to check all "non-smooth" points for the constraint, *i.e.*, all the points (x,y) where $(3x^2, -2y) = (\nabla Q)(x,y) = (0,0)$.

We find $(x,y) = (0,0)$ *this* way.

To minimize $f(x_1, \dots, x_n)$,

subject to $Q_1(x_1, \dots, x_n) = C_1, \ \dots$

$Q_k(x_1, \dots, x_n) = C_k,$

find critical points, by solving

$$Q_1(x_1, \dots, x_n) = C_1, \ \dots$$

$$Q_k(x_1, \dots, x_n) = C_k,$$

$$(\nabla f)(x_1, \dots, x_n) = \lambda_1[(\nabla Q_1)(x_1, \dots, x_n)] + \cdots$$
$$\lambda_k[(\nabla Q_k)(x_1, \dots, x_n)],$$

then minimize f over these critical points.

$n + k$

How many unknowns?

How many $\underbrace{\text{equations}}_{\text{scalar}}$? $k + n$

To minimize $f(x_1,\ldots,x_n)$,

subject to $Q_1(x_1,\ldots,x_n) = C_1, \ \ldots$

$$Q_k(x_1,\ldots,x_n) = C_k,$$

find critical points, by solving

$$Q_1(x_1,\ldots,x_n) = C_1, \ \ldots$$

$$Q_k(x_1,\ldots,x_n) = C_k,$$

STOP

$$(\nabla f)(x_1,\ldots,x_n) = \lambda_1[(\nabla Q_1)(x_1,\ldots,x_n)] + \cdots$$
$$\lambda_k[(\nabla Q_k)(x_1,\ldots,x_n)],$$

then minimize f over these critical points.

Also need to check all "non-smooth" points for the constraint,

i.e., all the points $(x_1,\ldots,x_n)$ where

$(\nabla Q_1)(x_1,\ldots,x_n)$, ... , $(\nabla Q_k)(x_1,\ldots,x_n)$ are linearly dependent.