Financial Mathematics
Multivariable change of variables
START OF MULTIVARIABLE INTEGRAL CALCULUS
Single variable change of variables formula

**Theorem:**

Let \( D, E \subseteq \mathbb{R} \) be open.
Assume \( \psi : D \to E \) is smooth and bijective.
Assume \( f : E \to \mathbb{R} \) is continuous.

Then
\[
\int_E f(x) \, dx = \int_D \left[ f(\psi(s)) \right] \left| \psi'(s) \right| \, ds
\]

**Special case:**

\( D = (a, b) \), \( \psi \) increasing, \( E = (\psi(a), \psi(b)) \)

\[
\int_{\psi(a)}^{\psi(b)} f(x) \, dx = \int_a^b \left[ f(\psi(s)) \right] \left| \psi'(s) \right| \, ds
\]
Single variable change of variables formula

**Theorem:**

Let \( D, E \subseteq \mathbb{R} \) be open.

Assume \( \psi : D \to E \) is smooth and bijective.

Assume \( f : E \to \mathbb{R} \) is continuous.

Then

\[
\int_{E} f(x) \, dx = \int_{D} \left[ f(\psi(s)) \right] \left| \psi'(s) \right| \, ds
\]

**Special case:**

\( D = (a, b), \psi \) decreasing, \( E = (\psi(b), \psi(a)) \)

\[
\int_{\psi(b)}^{\psi(a)} f(x) \, dx = \int_{a}^{b} \left[ f(\psi(s)) \right] \left[ -\psi'(s) \right] \, ds
\]
Single variable change of variables formula

Theorem:

Let $D, E \subseteq \mathbb{R}$ be open.
Assume $\psi : D \to E$ is smooth and bijective.
Assume $f : E \to \mathbb{R}$ is continuous.

Then

$$\int_E f(x) \, dx = \int_D \left[ f(\psi(s)) \right] \left| \psi'(s) \right| \, ds$$

Special case:

$D = (a, b)$, $\psi$ decreasing, $E = (\psi(b), \psi(a))$

$$\int_{\psi(b)}^{\psi(a)} f(x) \, dx = \int_a^b \left[ f(\psi(s)) \right] \left| \psi'(s) \right| \, ds$$

Note: This last formula is true even if $\psi$ is not bijective.
Single variable change of variables formula

Theorem:

Let \( D, E \subseteq \mathbb{R} \) be open.

Assume \( \psi : D \to E \) is smooth and bijective.

Assume \( f : E \to \mathbb{R} \) is continuous.

Then
\[
\int_{E} f(x) \, dx = \int_{D} [f(\psi(s))] \left| \psi'(s) \right| \, ds
\]

Goal: Find a similar formula of the form
\[
\int \int_{E} f(x, y) \, dx \, dy = \int \int_{D} [f(\psi(s, t))] [???] \, ds \, dt,
\]
where \( D, E \subseteq \mathbb{R}^2 \) are open and
where \( \psi : D \to E \) is smooth and bijective.

Next: def’n of multivariable integration
Definition of the Riemann integral (multivariable)

Recall the single-variable Riemann integral...

Graph of $g$ lives “over” $D$.

$$\int \int \limits_{D} g(s, t) \, ds \, dt = ?????
\[ D = (a, b) \]

\[ \int_D g(s) \, ds = ??? \]
Fix a small number $\delta > 0$. Cover $D$ by intervals of diameter $\leq \delta$.

(Not required, but often we use intervals of the same length.)

$$\int_D g(s) \, ds = ????
Fix a small number $\delta > 0$. Cover $D$ by intervals of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$.

(Not required, but often we take the midpoints.)

$$\int_D g(s) \, ds = \text{??????}$$
Fix a small number $\delta > 0$. Cover $D$ by intervals of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$. Call its point $s_j$. 

$$\int_D g(s) \, ds = ???$$
Fix a small number $\delta > 0$. Cover $D$ by intervals of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$. Call its point $s_j$.

$$\int_{D} g(s) \, ds = ???$$
Fix a small number $\delta > 0$.
Cover $D$ by intervals of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the $j$th, call it $D_j$.
Call its point $s_j$.

$$\int_D g(s) \, ds = ???$$
Fix a small number \( \delta > 0 \).
Cover \( D \) by intervals of diameter \( \leq \delta \).
Pick a point in each of the sets.
Focus on one set, the \( j \)th, call it \( D_j \).
Call its point \( s_j \).

\[
\int_D g(s) \, ds = ????
\]
Fix a small number $\delta > 0$. Cover $D$ by intervals of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$. Call its point $s_j$.

Compute $[g(s_j)][\text{Length}(D_j)]$.

$$\int_{D_j} g(s) \, ds \approx [g(s_j)][\text{Length}(D_j)]$$

Add over all $j$.

$$\int_D g(s) \, ds = ???$$
Next: replace $\delta$ by $\delta_k \to 0$

Fix a small number $\delta > 0$.

Cover $D$ by intervals of diameter $\leq \delta$.

Pick a point in each of the sets.

Focus on one set, the $j$th, call it $D_j$.

Call its point $s_j$.

Compute $[g(s_j)][\text{Length}(D_j)]$.

$$\int_{D_j} g(s) \, ds \approx [g(s_j)][\text{Length}(D_j)]$$

Add over all $j$.

$$\int_D g(s) \, ds \approx \sum_j [g(s_j)][\text{Length}(D_j)]$$
Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

e.g.: $\delta_1 = 1/2, \delta_2 = 1/3, \delta_3 = 1/4, \ldots$

$D_1^1, D_2^1, D_3^1, D_4^1$
all of diam. $\leq \delta_1 = 1/2$

$s_1^1, s_2^1, s_3^1$

$D_1^2, D_2^2, \ldots D_6^2$
all of diam. $\leq \delta_2 = 1/3$

$s_1^2, s_2^2, \ldots s_6^2$

$D_1^3, D_2^3, \ldots D_8^3$
all of diam. $\leq \delta_3 = 1/4$

\vdots
Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

$D = (a, b)$

Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.
Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

$\forall k$, cover $D$ by intervals of diameter $\leq \delta_k$.

$\forall k$, pick a point in each of the sets.

Focus on one $k$ and one set, the $j$th, $D_j^k$.

Call its point $s_j^k$.

Compute $[g(s_j^k)][\text{Length}(D_j^k)]$.

$$\int_{D_j^k} g(s) \, ds \approx [g(s_j^k)][\text{Length}(D_j^k)].$$

Add over all $j$, and let $k \to \infty$.

$$\int_D g(s) \, ds \approx \sum_j [g(s_j^k)][\text{Length}(D_j^k)]$$
Next: back to multivariable setting

Let \( \delta_1, \delta_2, \ldots \to 0 \) be pos. numbers.
\( \forall k \), cover \( D \) by intervals of diameter \( \leq \delta_k \).
\( \forall k \), pick a point in each of the sets.
Focus on one \( k \) and one set, the \( j \)th, \( D_j^k \).
Call its point \( s_j^k \).

Compute \( [g(s_j^k)][\text{Length}(D_j^k)] \).

\[
\int_{D_j^k} g(s) \, ds \approx [g(s_j^k)][\text{Length}(D_j^k)].
\]

Add over all \( j \), and let \( k \to \infty \).

\[
\int_D g(s) \, ds := \lim_{k \to \infty} \sum_j [g(s_j^k)][\text{Length}(D_j^k)]
\]
Fix a small number $\delta > 0$. Cover $D$ by sets of diameter $\leq \delta$.

(Not required, but helps if the areas of the sets are easily calculated, e.g., squares.)

$$\int \int_D g(s, t) \, ds \, dt := ??$$
Fix a small number $\delta > 0$. Cover $D$ by sets of diameter $\leq \delta$. Pick a point in each of the sets.

(Not required, but helps if the areas of the sets are easily calculated, e.g., squares.)

$$\int \int_D g(s, t) \, ds \, dt : = ???$$
Fix a small number $\delta > 0$. Cover $D$ by sets of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$.

(Not required, but often one takes the centers.)

$$\int \int_D g(s, t) \, ds \, dt : = ?????
Fix a small number \( \delta > 0 \).
Cover \( D \) by sets of diameter \( \leq \delta \).
Pick a point in each of the sets.
Focus on one set, the \( j \)th, call it \( D_j \).
Call its point \( (s_j, t_j) \).

\[
\int \int_\mathbb{R} g(s, t) \, ds \, dt := ??????
\]
Fix a small number $\delta > 0$. Cover $D$ by sets of diameter $\leq \delta$. Pick a point in each of the sets. Focus on one set, the $j$th, call it $D_j$. Call its point $(s_j, t_j)$.

Compute

$$\int \int_{D_j} g(s, t) \, ds \, dt \approx [g(s_j, t_j)][\text{Area}(D_j)].$$

Add over all $j$.

$$\int \int_{D} g(s, t) \, ds \, dt := ????$$

$0$, whenever $(s_j, t_j) \notin D$
Next: replace $\delta$ by $\delta_k \to 0$

Fix a small number $\delta > 0$.
Cover $D$ by sets of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the $j$th, call it $D_j$.
Call its point $(s_j, t_j)$.

Compute $\int \int_{D_j} g(s, t) \, ds \, dt \approx [g(s_j, t_j)][\text{Area}(D_j)]$.

Add over all $j$.

$\int \int_D g(s, t) \, ds \, dt \approx \sum_j [g(s_j, t_j)][\text{Area}(D_j)]$
Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

$\forall k$, cover $D$ by sets of diameter $\leq \delta_k$.

$\forall k$, pick a point in each of the sets.

Focus on one $k$ and one set, the $j$th, $D_{j}^{k}$.

Call its point $(s_{j}^{k}, t_{j}^{k})$.

Compute $[g(s_{j}^{k}, t_{j}^{k})][\text{Area}(D_{j}^{k})]$.

\[
\int \int_{D_{j}^{k}} g(s, t) \, ds \, dt \approx [g(s_{j}^{k}, t_{j}^{k})][\text{Area}(D_{j}^{k})]
\]

Add over all $j$, and let $k \to \infty$.

\[
\int \int_{D} g(s, t) \, ds \, dt \approx \sum_{j} [g(s_{j}^{k}, t_{j}^{k})][\text{Area}(D_{j}^{k})]
\]
Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

\forall k, \text{cover } D \text{ by sets of diameter } \leq \delta_k.

\forall k, \text{pick a point in each of the sets.}

Focus on one $k$ and one set, the $j$th, $D_j^k$.

Call its point $(s_j^k, t_j^k)$.

Compute $\left[g(s_j^k, t_j^k)\right][\text{Area}(D_j^k)]$.

$$\int \int_{D_j^k} g(s, t) \, ds \, dt \approx \left[g(s_j^k, t_j^k)\right][\text{Area}(D_j^k)]$$

Add over all $j$, and let $k \to \infty$.

$$\int \int_D g(s, t) \, ds \, dt := \lim_{k \to \infty} \sum_j \left[g(s_j^k, t_j^k)\right][\text{Area}(D_j^k)]$$
\[ D = (a, b) \times (c, d) \]

Fubini’s Theorem on rectangles:

\[
\int \int_D g(s, t) \, ds \, dt \quad \text{exists!}
\]

\[
\int_a^b \int_c^d g(s, t) \, dt \, ds \quad \text{and} \quad \int_c^d \int_a^b g(s, t) \, ds \, dt
\]

Proof: We omit existence proof. For the equalities, use rectangular partitions and “follow your nose”. QED
\[ \int \int_E f(x, y) \, dx \, dy \]

\[ = \int \int_D [f(\psi(s, t))] \, ds \, dt \]
\[ \int \int_E f(x, y) \, dx \, dy \]

\[ \approx \sum_j [f(x_j, y_j)] \text{[Area}(E_j)\text{]} \]

\[ \frac{?}{\det(\psi'(s, t))} \int \int_D [f(\psi(s, t))] \, ds \, dt \]
\[
\int \int_E f(x, y) \, dx \, dy 
\approx \sum_j [f(x_j, y_j)][\text{Area}(E_j)] 
\approx \sum_j [f(\psi(s_j, t_j))][|\text{det}(\psi'(s_j, t_j))|][\text{Area}(D_j)] 
\approx \int \int_D [f(\psi(s, t))][|\text{det}(\psi'(s, t))|] \, ds \, dt 
\]

Why close?
\[
E_j := \psi(D_j)
\]

\[
\text{ZOOM IN!!}
\]

Area\(E_j\) = \left| \det(\psi'(s_j, t_j)) \right| [\text{Area}(D_j)]
\[
\psi(s_j + h, t_j + k) \approx [\psi(s_j, t_j)] + L_{\psi'}(s_j, t_j)(h, k)
\]

\[
E_j \approx [\psi(s_j, t_j)] + L_{\psi'}(s_j, t_j)([-a, b] \times [-c, d])
\]

\[
\text{Area}(E_j) = \left| \det(\psi'(s_j, t_j)) \right| \text{Area}(D_j)
\]

\[
h \in [-a, b]
\]

\[
k \in [-c, d]
\]
\[
\psi(s_j + h, t_j + k)
\]

\[
\text{Area}(E_j) = \left| \det(\psi'(s_j, t_j)) \right| \cdot \text{Area}(D_j)
\]

\[
\text{Area}(E_j) \approx \text{Area}(L_{\psi'}(s_j, t_j)([-a, b] \times [-c, d]))
\]

\[
E_j \approx [\psi(s_j, t_j)] + L_{\psi'}(s_j, t_j)([-a, b] \times [-c, d])
\]
\[ \psi(s_j + h, t_j + k) \]

\[ \text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j, t_j)}([−a, b] \times [−c, d])) \]

\[ = \left| \det(\psi'(s_j, t_j)) \right| \text{Area}([−a, b] \times [−c, d]) \]
\[
\text{Area}(E_j) \approx \text{Area}(L_{\psi'}(s_j, t_j)([-a, b] \times [-c, d]))
\]
\[
= ||\text{det}(\psi'(s_j, t_j))||[(a + b)(c + d)]
\]
\[ \text{Area}(E_j) \approx \text{Area}(L_{\psi'}(s_j, t_j)([-a, b] \times [-c, d])) = |\det(\psi'(s_j, t_j))| \text{[Area}(D_j)] \]
\[\psi(s_j + h, t_j + k)\]

\[\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j, t_j)}([-a, b] \times [-c, d]))\]

\[= [| \det(\psi'(s_j, t_j))|][\text{Area}(D_j)]\]
\[ \int \int_{D_j} f(x, y) \, dx \, dy \]

\approx \sum_j \left[ f(x_j, y_j) \right] \left[ \text{Area}(E_j) \right]

\approx \sum_j \left[ f(\psi(s_j, t_j)) \right] \left[ | \det(\psi'(s_j, t_j)) | \right] \left[ \text{Area}(D_j) \right]

\approx \int \int_{D} f(\psi(s, t)) \left[ | \det(\psi'(s, t)) | \right] \, ds \, dt

\sum \text{err} \approx 0

Close!
\[
\int \int_E f(x, y) \, dx \, dy \\
\approx \int \int_D \left[ f(\psi(s, t)) \right][\left| \det(\psi'(s, t)) \right|] \, ds \, dt
\]

\[
\approx \int \int_D \left[ f(\psi(s, t)) \right][\left| \det(\psi'(s, t)) \right|] \, ds \, dt
\]
\[
\int \int_E f(x, y) \, dx \, dy \\
\approx \int \int_D [f(\psi(s, t))] |det(\psi'(s, t))| \, ds \, dt \\
\text{err} \to 0 \\
\text{Take limit as } k \to \infty.
\]
\[
\int \int_E f(x, y) \, dx \, dy = \int \int_D \left[ f(\psi(s, t)) \right] \left[ | \det(\psi'(s, t)) | \right] \, ds \, dt
\]

Change from \((s, t)\) to \((r, \theta)\).
\[\int_\mathbb{R} \int_E f(x, y) \, dx \, dy\]

\[= \int_\mathbb{R} \int_D [f(\psi(r, \theta))][| \det(\psi'(r, \theta))|] \, dr \, d\theta\]

Change from \((s, t)\) to \((r, \theta)\).
\[
\int \int_E f(x, y) \, dx \, dy = \int \int_D \left[ f(\psi(r, \theta)) \right] \left[ |\det(\psi'(r, \theta))| \right] \, dr \, d\theta
\]

**e.g.:**

\[D \coloneqq (0, \infty) \times (0, 2\pi)\]

\[E' \coloneqq [0, \infty) \times \{0\}\]

\[E \coloneqq \mathbb{R}^2 \setminus E'\]

\[\psi(r, \theta) = (r \cos \theta, r \sin \theta)\]

\[f(x, y) = e^{-\left(x^2+y^2\right)/2}\]
\[ f(\psi(r, \theta)) = e^{-\left(\frac{r^2 \cos^2 \theta + r^2 \sin^2 \theta}{2}\right)} = e^{-\frac{r^2}{2}} \]

Colloq.: As every \( x \) is repl. by \( r \cos \theta \), we write \( x = r \cos \theta \).

Similarly, \( y = r \sin \theta \).

\[ f(x, y) = e^{-\left(\frac{x^2 + y^2}{2}\right)} = e^{-\frac{r^2}{2}} \]

\[ \int \int_{E} f(x, y) \, dx \, dy = \int \int_{D} [f(\psi(r, \theta))] \left| \text{det}(\psi'(r, \theta)) \right| \, dr \, d\theta \]

e.g.: \( D := (0, \infty) \times (0, 2\pi) \)

\( E' := [0, \infty) \times \{0\} \)

\( E := \mathbb{R}^2 \setminus E' \)

\( \psi(r, \theta) = (r \cos \theta, r \sin \theta) \)

\( f(x, y) = e^{-\left(\frac{x^2 + y^2}{2}\right)} \)

\( r^2 \cos^2 \theta \quad r^2 \sin^2 \theta \)
\[ f(\psi(r, \theta)) = e^{-(r^2 \cos^2 \theta + r^2 \sin^2 \theta)/2} = e^{-r^2/2} \]

Colloq.: As every \( x \) is repl. by \( r \cos \theta \),
\[ x^2 + y^2 = r^2 \]
we write \( x = r \cos \theta \).
Similarly, \( y = r \sin \theta \).

\[ f(x, y) = e^{-(x^2+y^2)/2} = e^{-r^2/2} \]

\[
\int \int_E f(x, y) \, dx \, dy \\
= \int \int_D \left[ f(\psi(r, \theta)) \right] \left| \det(\psi'(r, \theta)) \right| \, dr \, d\theta
\]

Colloq.: As \( dx \, dy \) is repl. by \( \left| \det(\psi'(r, \theta)) \right| \, dr \, d\theta \),
we write \( dx \, dy = \left| \det(\psi'(r, \theta)) \right| \, dr \, d\theta \).

\[ \psi(r, \theta) = (r \cos \theta, r \sin \theta) \]
\[ \psi'(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \]
\[ dx\, dy = |\det(\psi'(r, \theta))| \, dr\, d\theta \]
\[ = |r \cos^2 \theta - (-r \sin^2 \theta)| \, dr\, d\theta = |r| \, dr\, d\theta \]

\[ \int \int_E f(x, y) \, dx\, dy = \int \int_D \left[ e^{-r^2/2} \right] |r| \, dr\, d\theta \]

\[ f(x, y) = e^{-(x^2+y^2)/2} = e^{-r^2/2} \]

\[ \int \int_E f(x, y) \, dx\, dy = \int \int_D [f(\psi(r, \theta))] |\det(\psi'(r, \theta))| \, dr\, d\theta \]

Colloq.: As \( dx\, dy \) is repl. by \( |\det(\psi'(r, \theta))| \, dr\, d\theta \), we write \( dx\, dy = |\det(\psi'(r, \theta))| \, dr\, d\theta \).

\[ \psi(r, \theta) = (r \cos \theta, r \sin \theta) \]
\[ \psi'(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \]
\[ \begin{align*}
\int \int_E f(x, y) \, dx \, dy &= \int \int_D [e^{-r^2/2}] \, dr \, d\theta \\
\int \int_E f(x, y) \, dx \, dy &= \int \int_D [f(\psi(r, \theta))][|\det(\psi'(r, \theta))|] \, dr \, d\theta \\
\text{Colloq.: As } dx \, dy \text{ is repl. by } |\det(\psi'(r, \theta))| \, dr \, d\theta, \\
\text{we write } dx \, dy &= |\det(\psi'(r, \theta))| \, dr \, d\theta. \\
\text{Colloq.: In “polar coordinates”, } x &= r \cos \theta, \\
x^2 + y^2 &= r^2 \\
y &= r \sin \theta, \\
\text{and } dx \, dy &= r \, dr \, d\theta.
\end{align*} \]
Prove:
\[ 0 < I := \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \]

Want:
\[ I^2 = 2\pi \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, e^{-\frac{y^2}{2}} \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} I \, dy \]

\[ = I \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \, dy = I^2 \]
Prove:
\[ I := \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

Want:
\[ I^2 = 2\pi \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy \]

\[ = I^2 \]
Prove:
\[ I := \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

Want:
\[ I^2 = 2\pi \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy = I^2 \]

\[ \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \, r \, dr \, d\theta = I^2 \]

\[ \int_{0}^{2\pi} \int_{0}^{\infty} e^{-s} \, ds \, d\theta \]

\[ x^2 + y^2 = r^2 \]

\[ ds \, d\theta \, dx \, dy = r \, dr \, d\theta \]

\[ s = r^2/2 \]

\[ ds = r \, dr \]
Prove: 

\[ I := \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

Want: \[ I^2 = 2\pi \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy = I^2 \]

\[ \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \, r \, dr \, d\theta \]

\[ \int_{0}^{2\pi} \int_{0}^{\infty} e^{-s} \, ds \, d\theta = \int_{0}^{2\pi} 1 \, d\theta = 2\pi \]

\[ \left[ -e^{-s} \right]_{s=0}^{s=\infty} = (-0) - (-1) = 1 \]

QED
Param. of $\frac{1}{2}$-ball in sph. coords... $\psi(r, \theta, \phi) =$

$$\begin{pmatrix}
r(\cos \phi)(\cos \theta), & r(\cos \phi)(\sin \theta), & r(\sin \phi)
\end{pmatrix}$$

$$\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & -\sin \phi \\
0 & 1 & 0 \\
\sin \phi & 0 & \cos \phi
\end{bmatrix}
\begin{bmatrix}
r \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
r(\cos \phi)(\cos \theta) \\
r(\cos \phi)(\sin \theta) \\
r(\sin \phi)
\end{bmatrix}$$

**SKILL:**
Use change of variables to compute the area/vol. of a well-parametrized set.