Financial Mathematics

Variations on Stokes' Theorem

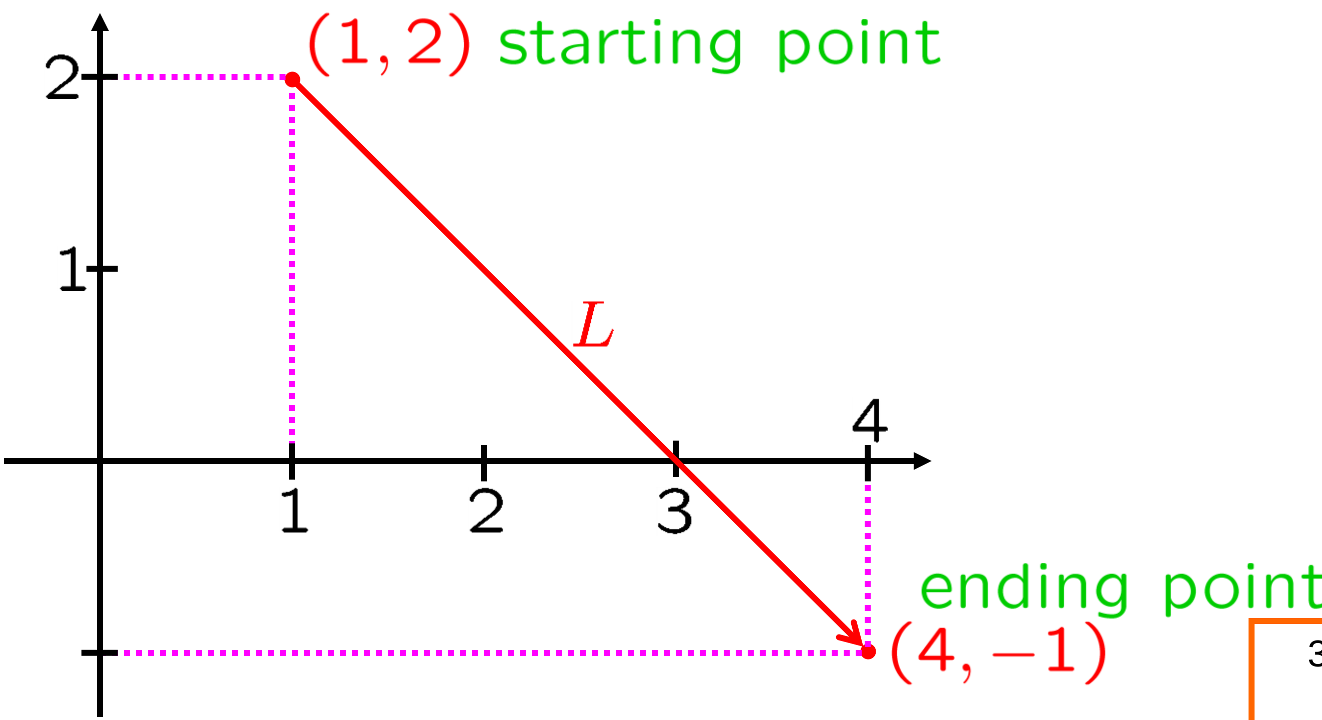
Green's Theorem
and
Cauchy's Theorem

Green's Theorem on rectangles

Definition:
A **directed line segment in** $\mathbb{R}^2$ is an ordered pair of points in $\mathbb{R}^2$, called the **starting point** and **ending point of** L.

E.g.: $L \quad := \quad (\quad (1,2) \quad , \quad (4,-1) \quad)$

Viz.:



Green's Theorem on rectangles

Definition:

A **directed line segment in** $\mathbb{R}^2$ is an ordered pair of points in $\mathbb{R}^2$, called the **starting point** and **ending point of** L.

Curves are assumed continuous.

Definition:

The **standard parametrization of** $L = (p, q)$ is the constant velocity curve $\phi : [0, 1] \to \mathbb{R}^2$ such that $\phi(0) = p$ and $\phi(1) = q$.

Definition:

Constant velocity means: ϕ' is constant, *i.e.*, that, $\forall s, t \in (0, 1)$,

$$\phi'(s) = \phi'(t)$$

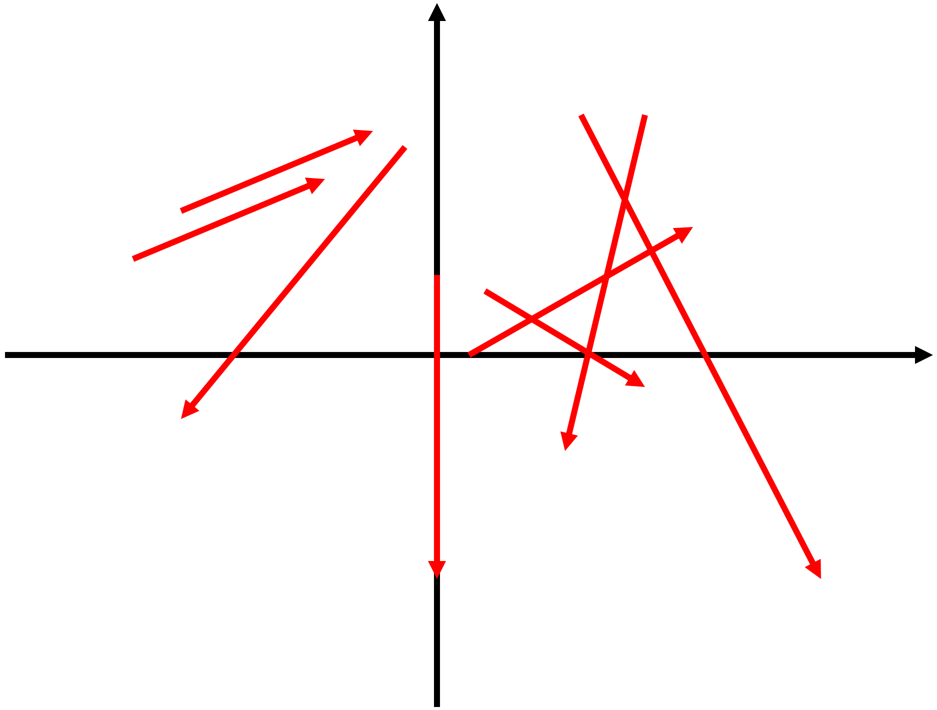
Green's Theorem on rectangles

Definition:

A **simple chain** is a finite set of directed line segments.

E.g.:

Viz.:

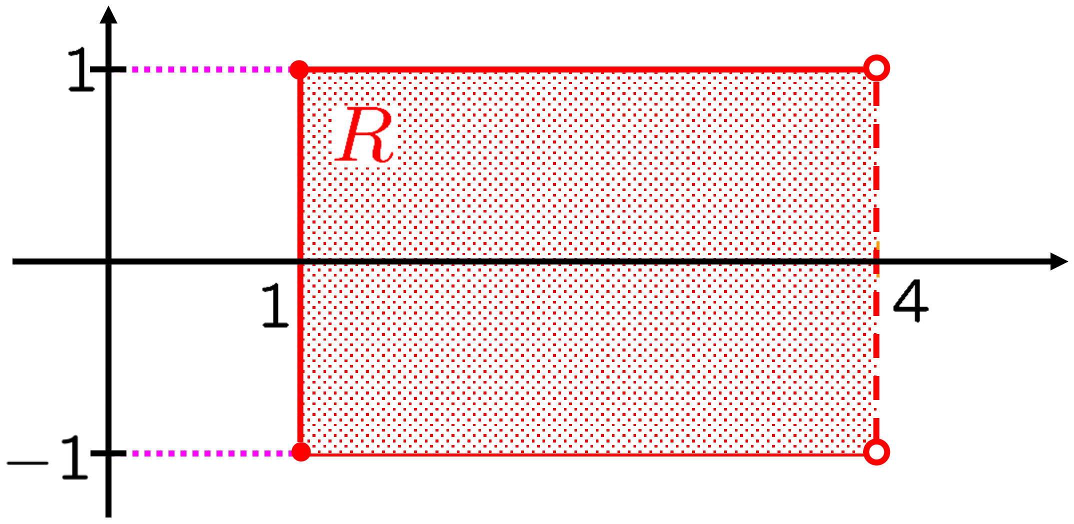


Green's Theorem on rectangles

Definition:
A **rectangle** is a subset of $\mathbb{R}^2$ of the form $I \times J$, where I and J are bounded intervals.

E.g.: $R \quad := \quad [1,4) \times [-1,1]$

Viz.:



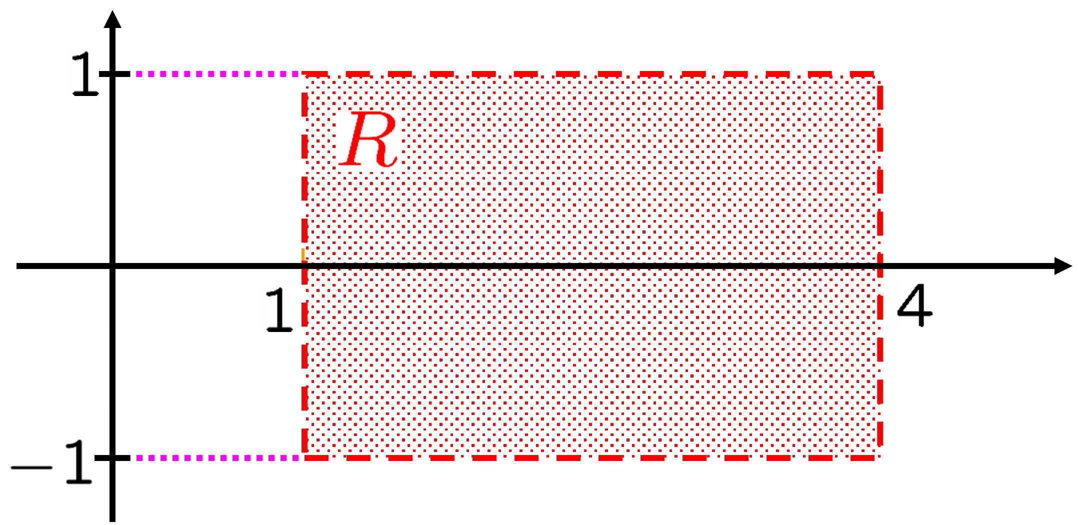
Green's Theorem on rectangles

Definition:
A **rectangle** is a subset of $\mathbb{R}^2$ of the form $I \times J$, where I and J are bounded intervals.

E.g.: $R := (1,4) \times (-1,1)$

is an open rectangle.

Viz.:



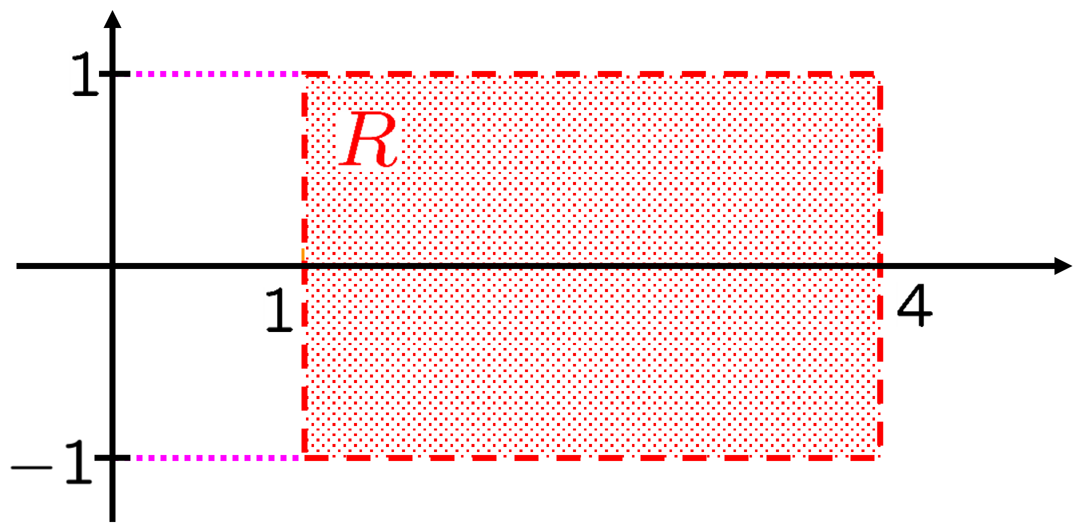
Green's Theorem on rectangles

Definition: Let R be an open rectangle.
The **counterclockwise boundary** $\boxed{\partial R}$ **of** R is the set of boundary line segments, directed counterclockwise.

E.g.: $R \quad := \quad (1,4) \times (-1,1)$

is an open rectangle.

Viz.:



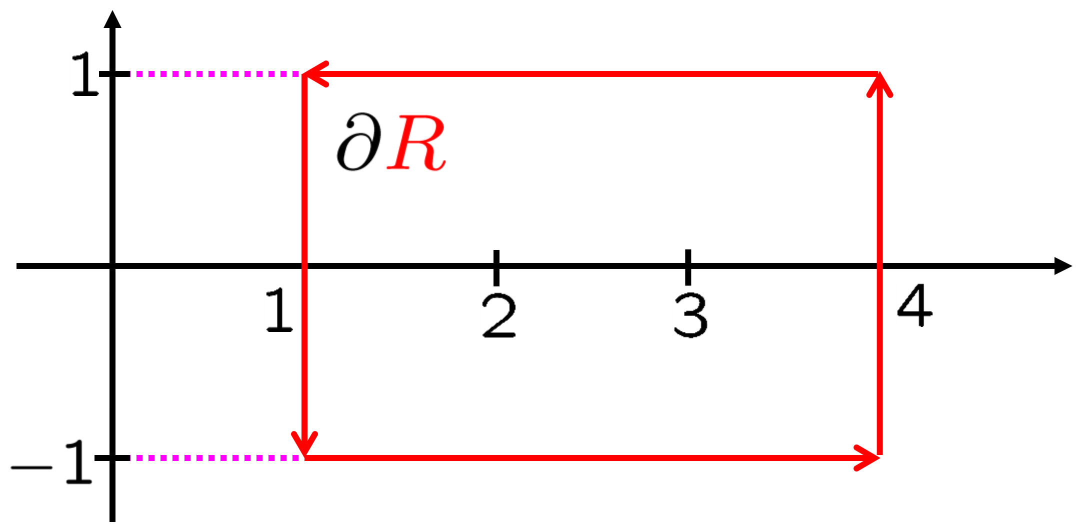
Green's Theorem on rectangles

Definition: Let R be an open rectangle.
The **counterclockwise boundary** $\boxed{\partial R}$ **of** R is the set of boundary line segments, directed counterclockwise.

E.g.: $R := (1,4) \times (-1,1)$

is an open rectangle.

Viz.:



Green's Theorem on rectangles

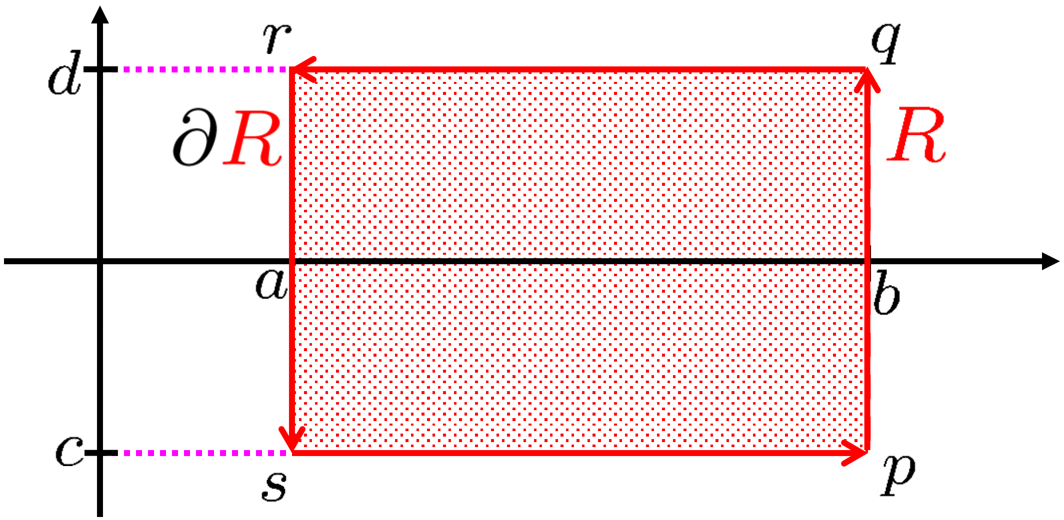
Definition: Let R be an open rectangle. The **counterclockwise boundary** $\boxed{\partial R}$ **of** R is the set of boundary line segments, directed counterclockwise.

Def'n: $R \quad := \quad (a, b) \times (c, d)$

$p := (b, c)$, $q := (b, d)$, $r := (a, d)$, $s := (a, c)$

implies $\quad \partial R = \{(p, q), (q, r), (r, s), (s, p)\}$

Viz.:



Green's Theorem on rectangles

Definition:

Let L be a directed line segment in $\mathbb{R}^2$.

Let $\phi = (\alpha, \beta) : [0, 1] \to \mathbb{R}^2$ be the standard parameterization of L.

Let $p, q : \phi([0, 1]) \to \mathbb{R}$ be continuous.

Then we define:

$$\boxed{\int_L p(x, y)\, dx + q(x, y)\, dy} :=$$

$$\int_0^1 [p(\phi(t))][\alpha'(t)] + [q(\phi(t))][\beta'(t)]\, dt.$$

Idea: Replace x by $\alpha(t)$, y by $\beta(t)$, and dx by $\alpha'(t)\, dt$, dy by $\beta'(t)\, dt$.

Green's Theorem on rectangles

Definition:

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain.
Let S be the union, over j, of the image of the standard parametrization of L_j.

Let $p, q : S \to \mathbb{R}$ be continuous.

Then we define:

$$\boxed{\int_C p(x,y)\,dx + q(x,y)\,dy} :=$$

$$\int_{L_1} p(x,y)\,dx + q(x,y)\,dy + \cdots$$

$$+ \int_{L_n} p(x,y)\,dx + q(x,y)\,dy.$$

Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{R}^2$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $p, q : \bar{R} \to \mathbb{R}$ be continuous, and smooth on R.

Let $P := p(x,y)$ and $Q := q(x,y)$.

Then:

$$\int_{\partial R} P\,dx + Q\,dy = \int\!\!\int_R \overbrace{[(\partial_x Q) - (\partial_y P)]}^{\det\begin{bmatrix} \partial_x & \partial_y \\ P & Q \end{bmatrix}} dx\,dy.$$

Definition: A **zero-form in x and y** is an expression in x and y.

Definition: An expression of the form $p(x, y)\, dx + q(x, y)\, dy$ is called a **one-form in x and y**.

Definition: The **exterior derivative of** $F = f(x, y)$, denoted $\boxed{dF}$, is the one-form $\partial_x F\, dx + \partial_y F\, dy$.

Note: Exterior differention carries zero-forms to one-forms.

$$\int_{\partial R} P\, dx + Q\, dy = \int \int_R [(\partial_x Q) - (\partial_y P)] dx\, dy.$$

Definition: A **zero-form in x and y** is an expression in x and y.

Definition: An expression of the form $p(x,y)\,dx + q(x,y)\,dy$ is called a **one-form in x and y**.

Definition: The **exterior derivative of** $F = f(x,y)$, denoted $\boxed{dF}$, is the one-form $\partial_x F\,dx + \partial_y F\,dy$.

SKILL: Compute the exterior derivative of a zero-form.

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

Definition: An expression of the form $p(x,y)\,dx \wedge dy$ is called a **two-form in x and y**.

Definition: An expression of the form $p(x,y)\,dx + q(x,y)\,dy$ is called a **one-form in x and y**.

Conventions:

f a 0-form in x, y

A, B, C forms in x,y

$$f \wedge A = fA = A \wedge f$$

$$(A+B) \wedge C = (A \wedge C) + (B \wedge C)$$

$$A \wedge (B+C) = (A \wedge B) + (A \wedge C)$$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

$$dx \wedge dx = dy \wedge dy = 0$$

$$dx \wedge dy = -dy \wedge dx$$

$$\int_{\partial R} P\,dx + Q\,dy = \int\!\!\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

Note: $f(A \wedge B) = (fA) \wedge B = A \wedge (fB)$

Proof:

$$f(A \wedge B) = f \wedge (A \wedge B) = (f \wedge A) \wedge B = (fA) \wedge B$$
$$= (A \wedge f) \wedge B = A \wedge (f \wedge B) = A \wedge (fB)$$

QED

Conventions: $f \wedge A = fA = A \wedge f$

f a 0-form in x, y
A, B, C forms in x,y

$$(A + B) \wedge C = (A \wedge C) + (B \wedge C)$$
$$A \wedge (B + C) = (A \wedge B) + (A \wedge C)$$
$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$
$$dx \wedge dx = dy \wedge dy = 0$$
$$dx \wedge dy = -dy \wedge dx$$

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

SKILL: Collect terms on a two-form.

e.g.: $[(4x^2 + 3xy)\,dx + (2\sin(xy))\,dy]$
$\wedge[(ye^x)\,dx + (5xy^3)\,dy] = [?????]\,dx \wedge dy$

$$????? = (4x^2 + 3xy)(5xy^3) - (2\sin(xy))(ye^x)$$

Conventions: $f \wedge A = fA = A \wedge f$

f a 0-form in x, y
A, B, C forms in x, y

$$(A + B) \wedge C = (A \wedge C) + (B \wedge C)$$
$$A \wedge (B + C) = (A \wedge B) + (A \wedge C)$$
$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$
$$dx \wedge dx = dy \wedge dy = 0$$
$$dx \wedge dy = -dy \wedge dx$$

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

SKILL: Compute exterior derivatives of zero-forms.

e.g.: $d[e^{x+y}\,\mathrm{sin}(x)] =$
$(\partial_x[e^{x+y}\,\mathrm{sin}(x)])\,dx + (\partial_y[e^{x+y}\,\mathrm{sin}(x)])\,dy = \cdots$

Definition: The **exterior derivative** **of** $F = f(x, y)$, denoted $\boxed{dF}$, is the one-form $\partial_x F\,dx + \partial_y F\,dy$.

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

Definition: An expression of the form $p(x, y)\,dx + q(x, y)\,dy$ is called a **one-form in x and y**.

Definition: An expression of the form $p(x, y)\,dx \wedge dy$ is called a **two-form in x and y**.

Definition: The **exterior derivative of** $F = P\,dx + Q\,dy$, denoted $\boxed{dF}$, is the two-form $dP \wedge dx + dQ \wedge dy$.

Note: Exterior differention carries one-forms to two-forms.

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

SKILL: Compute exterior derivatives of one-forms.

Definition: The **exterior derivative of** $F = P\,dx + Q\,dy$, denoted $\boxed{dF}$, is the two-form $dP \wedge dx + dQ \wedge dy$.

Definition: The **exterior derivative of** $F = P\,dx + Q\,dy$, denoted $\boxed{dF}$, is the two-form $dP \wedge dx + dQ \wedge dy$.

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)] dx\,dy.$$

SKILL: Compute exterior derivatives of one-forms.

e.g.: $d[(x\sin y)dx + (x^3y^2)dy] =$

$[d(x\sin y)] \wedge dx + [d(x^3y^2)] \wedge dy =$

$[(\partial_y(x\sin y))dy] \wedge dx + [(\partial_x(x^3y^2))dx] \wedge dy =$

$(\partial_y(x\sin y))(dy \wedge dx) + (\partial_x(x^3y^2))(dx \wedge dy) =$

$[-(\partial_y(x\sin y)) + (\partial_x(x^3y^2))][dx \wedge dy] = \cdots$

Definition: $\boxed{\int_R g(x,y)\, dx \wedge dy} := \int\int_R g(x,y)\, dx\, dy$

Definition: The **exterior derivative** **of** $F = P\,dx + Q\,dy$, denoted $\boxed{dF}$, is the two-form $dP \wedge dx + dQ \wedge dy$.

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

$$
\begin{aligned}
d(P\,dx + Q\,dy) &= \\
(\boxed{dP} \wedge dx) + (\boxed{dQ} \wedge dy) &= \\
[\boxed{((\partial_y P)dy)} \wedge dx] + [\boxed{((\partial_x Q)dx)} \wedge dy] &= \\
[(\partial_y P)(\boxed{dy \wedge dx})] + [(\partial_x Q)(\boxed{dx \wedge dy})] &= \\
[(-\partial_y P)(\boxed{dx \wedge dy})] + [(\partial_x Q)(\boxed{dx \wedge dy})] &= \\
[-(\partial_y P) + (\partial_x Q)][\boxed{dx \wedge dy}] &=
\end{aligned}
$$

Green's Theorem on rectangles:

$\forall$one-forms ω in x and y, $\forall$open rectangles R,

$$\int_{\partial R} \omega = \int_R d\omega.$$

ω continuous on $\bar{R}$ and smooth on R

$$\int_{\partial R} P\,dx + Q\,dy = \int\!\!\int_R [(\partial_x Q) - (\partial_y P)]dx\,dy.$$

Practice:

$$(x^2y\,dx + 3x\,dy + e^z\,dz)$$
$$\wedge(3\,dx + 2e^{xz}\,dy - 4y^3\,dz)$$
$$\wedge(dx + dy + dz) \;= [?????]\,dx \wedge dy \wedge dz$$

$$(-y\,dx + x^y\,dy + xyz\,dz)$$
$$\wedge(2ye^z\,dx - \;dy + 7x^{-1}\,dz)$$
$$\wedge(\cos(xy/z)\,dx + 3\,dy - z^5\,dz)$$
$$= [?????]\,dx \wedge dy \wedge dz$$

Solutions:

$$(x^2 y\, dx + 3x\, dy + e^z\, dz)$$
$$\wedge(3\, dx + 2e^{xz}\, dy - 4y^3\, dz)$$
$$\wedge(dx + dy + dz) = [?????]\, dx \wedge dy \wedge dz$$

$$????? =$$
$$(x^2 y)(2e^{xz})(1)$$
$$-(x^2 y)(-4y^3)(1)$$
$$-(3x)(3)(1)$$
$$+(3x)(-4y^3)(1)$$
$$+(e^z)(3)(1)$$
$$-(e^z)(2e^{xz})(1)$$

Solutions:

$$(-y\,dx + x^y\,dy + xyz\,dz)$$
$$\wedge(2ye^z\,dx - \;dy + 7x^{-1}\,dz)$$
$$\wedge(\cos(xy/z)\,dx + 3\,dy - z^5\,dz)$$
$$= [?????]\,dx \wedge dy \wedge dz$$

$$????? =$$
$$(-y)(-1)(-z^5)$$
$$-(-y)(7x^{-1})(3)$$
$$-(x^y)(2ye^z)(-z^5)$$
$$+(x^y)(7x^{-1})(\cos(xy/z))$$
$$+(xyz)(2ye^z)(3)$$
$$-(xyz)(-1)(\cos(xy/z))$$

Practice:

$$d(\sin(xye^z))$$

$$d([e^{-3xy}][\sin(z)])$$

$$d(x^2y\,dx + 3x\,dy + e^z\,dz)$$

$$d(-y\,dx + x^y\,dy + xyz\,dz)$$

Solutions:

$$d(\sin(xye^z))$$

$$= ([\cos(xye^z)][ye^z])\,dx$$
$$+([\cos(xye^z)][xe^z])\,dy$$
$$+([\cos(xye^z)][xye^z])\,dz$$

$$d([e^{-3xy}][\sin(z)])$$

$$= ([e^{-3xy}][-3y][\sin(z)])\,dx$$
$$+([e^{-3xy}][-3x][\sin(z)])\,dy$$
$$+([e^{-3xy}][\cos(z)])\,dz$$

Solutions:

$$
\begin{aligned}
d(x^2y\,dx + 3x\,dy + e^z\,dz) \\
&= ([3] - [x^2])\,dx \wedge dy \\
&\quad +([0] - [0])\,dx \wedge dz \\
&\quad +([0] - [0])\,dy \wedge dz
\end{aligned}
$$

$$
\begin{aligned}
d(-y\,dx + x^y\,dy + xyz\,dz) \\
&= ([yx^{y-1}] - [-1])\,dx \wedge dy \\
&\quad +([yz] - [0])\,dx \wedge dz \\
&\quad +([xz] - [0])\,dy \wedge dz
\end{aligned}
$$

Practice:

$\boxed{x=-1+t}$ $\boxed{dx=dt}$

$$\int_{(-1,2)}^{(2,5)} e^{x+y}\,dx + xy^3\,dy$$

$\boxed{y=2+t}$ $\boxed{dy=dt}$

$\boxed{t\in[0,3]}$

$$=\int_0^3 e^{(-1+t)+(2+t)}\,dt + (-1+t)(2+t)^3\,dt$$

$$=\int_0^3 e^{1+2t} + (-1+t)(2^3+3\cdot 2^2 t + 3\cdot 2t^2 + t^3)\,dt$$

$$=\cdots$$

Compute $\displaystyle\int_{(3,2,1)}^{(5,6,7)} x(\sin y)\,dx + z^2e^x\,dy$

$$x = 3 + 2t$$
$$y = 2 + 4t$$
$$z = 1 + 6t$$

$$\int_{(3,2,1)}^{(5,6,7)} x(\sin y)\,dx + z^2e^x\,dy$$

$$= \int_0^1 (3+2t)(\sin(2+4t))\,2\,dt$$
$$+ \int_0^1 (1+6t)^2e^{3+2t}\,4\,dt$$

$$\int_0^1 t(\sin(2+4t))\,dt = \int_0^4 \frac{t}{4}(\sin(2+t))\,\frac{dt}{4}$$

$$\int_2^6 t(\sin t)\,dt = \int_2^6 \left(\frac{t-2}{4}\right)(\sin t)\,\frac{dt}{4}$$

Let $R := (1,2) \times (3,4)$.

Compute $\displaystyle\int_R [e^{2x+3y}]\, dy \wedge dx$.

$$\int_R [e^{2x+3y}]\, dy \wedge dx = - \int \int_R [e^{2x+3y}]\, dx\, dy$$

$$= - \int_3^4 \int_1^2 [e^{2x+3y}]\, dx\, dy = - \int_3^4 \left[\frac{e^{2x+3y}}{2}\right]_{x:\to 1}^{x:\to 2} dy$$

$$= - \int_3^4 \left[\frac{e^{4+3y}}{2}\right] - \left[\frac{e^{2+3y}}{2}\right] dy$$

$$= - \left[\left[\frac{e^{4+3y}}{2 \cdot 3}\right]_{y:\to 3}^{y:\to 4}\right] + \left[\left[\frac{e^{2+3y}}{2 \cdot 3}\right]_{y:\to 3}^{y:\to 4}\right] = \cdots$$

(Real) Green's Theorem on rectangles, SETUP

Definition:

Let L be a directed line segment in $\mathbb{R}^2$.

Let $\phi = (\alpha, \beta) : [0,1] \to \mathbb{R}^2$ be the standard parameterization of L.

Let $p, q : \phi([0,1]) \to \mathbb{R}$ be continuous.

Then we define:

$$\boxed{\int_L p(x,y)\,dx + q(x,y)\,dy} :=$$

$$\int_0^1 [p(\phi(t))][\alpha'(t)] + [q(\phi(t))][\beta'(t)]\,dt.$$

Idea: Replace x by $\alpha(t)$, y by $\beta(t)$,

dx by $\alpha'(t)\,dt$, dy by $\beta'(t)\,dt$.

Complex Green's Theorem on rectangles, SETUP

Definition:

Let L be a directed line segment in $\mathbb{C}$.

Let $\phi : [0,1] \to \mathbb{C}$ be the standard parameterization of L.

Let $p : \phi([0,1]) \to \mathbb{C}$ be continuous.

Then we define:

$$\boxed{\int_L p(z)\,dz} := \int_0^1 [p(\phi(t))][\phi'(t)]\,dt.$$

Idea: Replace z by $\phi(t)$,

dz by $\phi'(t)\,dt$.

(Real) Green's Theorem on rectangles, SETUP

Definition:

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain.
Let S be the union, over j, of the image of the standard parametrization of L_j.

Let $p, q : S \to \mathbb{R}$ be continuous.

Then we define:

$$\boxed{\int_C p(x,y)\,dx + q(x,y)\,dy} :=$$

$$\int_{L_1} p(x,y)\,dx + q(x,y)\,dy + \cdots$$

$$+ \int_{L_n} p(x,y)\,dx + q(x,y)\,dy.$$

Definition:

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain in $\mathbb{C}$.

Let S be the union, over j, of the image of the standard parametrization of L_j.

Let $\phi : S \to \mathbb{C}$ be continuous.

Then we define:

$$\boxed{\int_C \phi(z)\, dz} := \int_{L_1} \phi(z)\, dz + \cdots + \int_{L_n} \phi(z)\, dz.$$

(Real) Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{R}^2$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $p, q : \bar{R} \to \mathbb{R}$ be continuous, and smooth on R.

Let $\omega := p(x,y)\,dx + q(x,y)\,dy$.

Then:

$$\int_{\partial R} \omega = \int_R d\omega.$$

Complex Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{C}$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and smooth on R.

Let $\omega := \phi(z)\, dz$.

One variable!

Then:

$$\int_{\partial R} \omega = \int_R d\omega.$$

Want:
Complex exterior differentiation.

Exercise: Compute $d[(\text{sin } x)\, dx]$,

the exterior derivative of

$$(\text{sin } x)\, dx$$

with respect to x.

Solution:

$$\begin{aligned} d[(\text{sin } x)\, dx] &= [d(\text{sin } x)] \wedge dx \\ &= [\partial_x(\text{sin } x)\, dx] \wedge dx \\ &= 0 \end{aligned}$$

Exercise: Compute $d[e^x\,dx]$

the exterior derivative of

$$e^x\,dx$$

with respect to x.

Solution:

$$\begin{aligned} d[e^x\,dx] &= [d(e^x)] \wedge dx \\ &= [\partial_x(e^x)\,dx] \wedge dx \\ &= 0 \end{aligned}$$

Fact:

$d[\phi(x)\,dx] = 0$, for any smooth ϕ.

Proof:

$$
\begin{aligned}
d[\phi(x)\,dx] &= [d(\phi(x))] \wedge dx \\
&= [\partial_x(\phi(x))\,dx] \wedge dx \\
&= 0
\end{aligned}
$$

QED

Complex Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{C}$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and smooth on R.

Need: Complex differentiable

Let $\omega := \phi(z)\, dz$.

Then:

$\boxed{\text{Expect: } 0}$

sometimes!

$$\int_{\partial R} \omega = \int_R d\omega.$$

Complex Green's Theorem on rectangles, SETUP

Definition:

Let R be an open set in $\mathbb{C}$.

Let $\phi : R \to \mathbb{C}$ be smooth. Let $z \in R$.

If $L := \lim_{h \to 0} \dfrac{[f(z+h)] - [f(z)]}{h}$ exists,

then we say that f is

complex differentiable at z,

and we define $\boxed{f'(z)} = L$.

Exercise: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{3z}$.

Show, for all $z \in \mathbb{C}$,

that f is complex differentiable at z,

and that $f'(z) = 3e^{3z}$.

Complex Green's Theorem on rectangles, SETUP

If $L := \lim_{h \to 0} \frac{[f(z+h)] - [f(z)]}{h}$ exists,

then we say that f is

If $L := \lim_{h \to 0} \frac{[f(z+h)] - [f(z)]}{h}$ exists,

then we say that f is

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

complex differentiable at z,

and we define $\boxed{f'(z)} = L$.

Complex Green's Theorem on rectangles, SETUP

If $L := \lim_{h \to 0} \frac{[f(z+h)] - [f(z)]}{h}$ exists,

then we say that f is

complex differentiable at z,

and we define $\boxed{f'(z)} = L$.

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim_{j \to \infty} \frac{[f(1 + h_j^R)] - [f(1)]}{h_j^R}$,

$i := \sqrt{-1}$

$L^I := \lim_{j \to \infty} \frac{[f(1 + h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{\substack{j \to \infty \\ h \to 0}} \frac{[(1+\overset{h}{\tfrac{1}{j}})^2 + 0^2] - [1^2 + 0^2]}{h \quad \frac{1}{j}}$$

$$L^I = \lim_{\substack{j \to \infty \\ h \to 0}} \frac{[1^2 + (\overset{h}{\tfrac{1}{j}})^2] - [1^2 + 0^2]}{ih \quad \frac{i}{j}}$$

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim_{j \to \infty} \dfrac{[f(1+h_j^R)] - [f(1)]}{h_j^R}$, $\qquad i := \sqrt{-1}$

$L^I := \lim_{j \to \infty} \dfrac{[f(1+h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

h

$$L^R = \lim_{h\to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h\,h} = \left[\frac{d}{dx}(x^2 + \overset{y}{0^2})\right]_{x=1}$$

$h \to 0$ $\qquad$ $y = 0$

h

$$L^I = \lim_{h\to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih\;ih}$$

$h \to 0$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim_{j\to\infty} \frac{[f(1 + h_j^R)] - [f(1)]}{h_j^R}$, $\qquad i := \sqrt{-1}$

$L^I := \lim_{j\to\infty} \frac{[f(1 + h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{d}{dx}([x^2 + \overset{y}{y^2}]_{y=0})\right]_{x=1}$$

$y = 0$

$$L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih}$$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim\limits_{j \to \infty} \dfrac{[f(1 + h_j^R)] - [f(1)]}{h_j^R}$,

$L^I := \lim\limits_{j \to \infty} \dfrac{[f(1 + h_j^I)] - [f(1)]}{h_j^I}$.

$i := \sqrt{-1}$

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{d}{dx}([x^2+y^2]_{y=0})\right]_{x=1}$$

$$= \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih}$$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

$$\text{Let } L^R := \lim_{j \to \infty} \frac{[f(1+h_j^R)] - [f(1)]}{h_j^R},$$

$$i := \sqrt{-1}$$

$$L^I := \lim_{j \to \infty} \frac{[f(1+h_j^I)] - [f(1)]}{h_j^I}.$$

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{\boxed{i}h} = \left[\frac{\partial}{\partial x}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim\limits_{j \to \infty} \dfrac{[f(1 + h_j^R)] - [f(1)]}{h_j^R}$, $\qquad i := \sqrt{-1}$

$L^I := \lim\limits_{j \to \infty} \dfrac{[f(1 + h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h\to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \lim_{h\to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{\boxed{i}h} = \frac{1}{\boxed{i}} \lim_{h\to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h}$$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \displaystyle\lim_{j\to\infty} \frac{[f(1+h_j^R)] - [f(1)]}{h_j^R}$,

$i := \sqrt{-1}$

$L^I := \displaystyle\lim_{j\to\infty} \frac{[f(1+h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h\to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \frac{1}{i}\lim_{h\to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} = \frac{1}{i}\left[\frac{d}{dy}(1^2+y^2)\right]_{y=0}$$

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim\limits_{j\to\infty} \dfrac{[f(1+h_j^R)] - [f(1)]}{h_j^R}$,

$L^I := \lim\limits_{j\to\infty} \dfrac{[f(1+h_j^I)] - [f(1)]}{h_j^I}$.

$i := \sqrt{-1}$

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h\to 0} \frac{[(1+h)^2+0^2]-[1^2+0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \frac{1}{i}\lim_{h\to 0} \frac{[1^2+h^2]-[1^2+0^2]}{h} = \frac{1}{i}\left[\frac{d}{dy}(\overset{x}{1}{}^2+y^2)\right]_{y=0}$$

$x = 1$

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \displaystyle\lim_{j\to\infty} \frac{[f(1+h_j^R)]-[f(1)]}{h_j^R}$,

$i := \sqrt{-1}$

$L^I := \displaystyle\lim_{j\to\infty} \frac{[f(1+h_j^I)]-[f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i}\left[\frac{d}{dy}([\overset{x}{x^2} + y^2]_{x=1})\right]_{y=0}$$

$x = 1$

---

non-e.g.: Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

f is not complex diff. at 1:

$\forall$integers $j > 0$, let $h_j^R := 1/j$ and $h_j^I := i/j$.

Let $L^R := \lim_{j \to \infty} \frac{[f(1 + h_j^R)] - [f(1)]}{h_j^R}$,

$i := \sqrt{-1}$

$L^I := \lim_{j \to \infty} \frac{[f(1 + h_j^I)] - [f(1)]}{h_j^I}$.

Want: $L^R \neq L^I$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h\to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$L^I = \frac{1}{i} \lim_{h\to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i}\left[\frac{d}{dy}([x^2+y^2]_{x=1})\right]_{y=0}$$

$$= \frac{1}{i}\left[\frac{\partial}{\partial y}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

$$\nparallel$$

$$L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i}\left[\frac{\partial}{\partial y}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

$$= \frac{1}{i}\left[\frac{\partial}{\partial y}(x^2 + y^2)\right]_{(x,y)=(1,0)}$$

Complex Green's Theorem on rectangles, SETUP

$\forall x, y \in \mathbb{R}, f(x+iy) = x^2 + y^2.$

$$L^R = \lim_{h \to 0} \frac{[(1+h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[\frac{\partial}{\partial x}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

$$\nparallel$$

$$L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i}\left[\frac{\partial}{\partial y}(x^2+y^2)\right]_{(x,y)=(1,0)}$$

For f to be cx-diff $\boxed{\text{at } a+bi}$, we need

$$\left[\frac{\partial}{\partial x}(f(x+iy))\right]_{\boxed{(x,y)=(a,b)}} = \frac{1}{i}\left[\frac{\partial}{\partial y}(f(x+iy))\right]_{\boxed{(x,y)=(a,b)}}$$

$\boxed{\text{complex differentiable}}$

Complex Green's Theorem on rectangles, SETUP

For f to be $\overbrace{\text{cx-diff}}^{\text{analytic}}$, we need

complex differentiable

$$\frac{\partial}{\partial x}(f(x+iy)) =$$

$$\frac{1}{i}\left[\frac{\partial}{\partial y}(f(x+iy))\right]$$

For f to be $\overbrace{\text{cx-diff}}^{\text{analytic}}$ $\boxed{\text{at } a+bi}$, we need

complex differentiable

$$\left[\frac{\partial}{\partial x}(f(x+iy))\right]_{\boxed{(x,y)=(a,b)}} =$$

$$\frac{1}{i}\left[\frac{\partial}{\partial y}(f(x+iy))\right]_{\boxed{(x,y)=(a,b)}}$$

Complex Green's Theorem on rectangles, SETUP

For f to be $\overbrace{\text{cx-diff}}^{\text{analytic}}$, we need

$\boxed{\text{complex differentiable}}$

$$\frac{\partial}{\partial x}(f(x+iy)) = \frac{1}{i}\left[\frac{\partial}{\partial y}(f(x+iy))\right]$$

Say $f(x+iy) = [u(x,y)] + i[v(x,y)]$.

Complex Green's Theorem on rectangles, SETUP

For f to be $\overbrace{\text{cx-diff}}^{\text{analytic}}$, we need

$$\frac{\partial}{\partial x}(\overbrace{f(x+iy)}^{U+iV}) = \frac{1}{i}\left[\frac{\partial}{\partial y}(\underbrace{f(x+iy)}_{U+iV})\right]$$

complex differentiable

Say $f(x+iy) = [u(x,y)] + i[v(x,y)]$.

$U := u(x,y), \qquad V := v(x,y)$

$$\partial_x(U+iV) = [1/i][\partial_y(U+iV)]$$

$$\boxed{\partial_x U} + i\boxed{(\partial_x V)} = -i\boxed{(\partial_y U)} + \boxed{\partial_y V}$$

$$\boxed{\partial_x U = \partial_y V \ \ \& \ \ \partial_x V = -\partial_y U}$$ Cauchy-Riemann equations

Problem:

Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{z^2/2}$.

Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x+iy) = [u(x,y)] + [v(x,y)]i.$$

Let $U := u(x,y)$ and $V := v(x,y)$.

Find U and V.

$$f(x+iy) = e^{(x+iy)^2/2} = e^{(x^2-y^2+2ixy)/2}$$

$$= e^{[(x^2-y^2)/2]+i[xy]} = e^{[(x^2-y^2)/2]}e^{i[xy]}$$

$$= e^{(x^2-y^2)/2}\,[(\cos(xy)) + i\,(\sin(xy))]$$

$$= \underbrace{[e^{(x^2-y^2)/2}]\,[(\cos(xy))]}_{U} + i\underbrace{[e^{(x^2-y^2)/2}]\,[(\sin(xy))]}_{V}$$

Exercise: Check $\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$

Complex Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{C}$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and smooth on R.

Let $\omega := \phi(z)\, dz$.

Then:

$$\int_{\partial R} \omega = \int_R d\omega.$$

Complex Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{C}$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and analytic on R.

Let $\omega := \phi(z)\, dz$.

Then:

$$\int_{\partial R} \omega = \int_R \overbrace{d\omega}^{0}.$$

Cauchy's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{C}$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and analytic on R.

Let $\omega := \phi(z)\, dz$.

Next: Proof of Green's Th'm for rectangles in $\mathbb{R}^2$

Then:

$$\int_{\partial R} \omega = 0.$$

Green's Theorem on rectangles

Theorem:

Let R be an open rectangle in $\mathbb{R}^2$.

Let $\bar{R}$ be the union of R and the boundary of R.

Let $p, q : \bar{R} \to \mathbb{R}$ be continuous, and smooth on R.

Let $P := p(x, y)$ and $Q := q(x, y)$.

Then: $$\int_{\partial R} P\,dx + Q\,dy = \int\int_R [(\partial_x Q) - (\partial_y P)] dx\,dy.$$

Proof:

Want: $$\int_{\partial R} P\,dx = -\int\int_R \partial_y P\,dx\,dy$$

Exercise: $$\int_{\partial R} Q\,dy = \int\int_R \partial_x Q\,dx\,dy$$

ADD

Want: $\displaystyle\int_{\partial R} P\,dx = -\int\int_R \partial_y P\,dx\,dy$

Write $R = (a,b) \times (c,d)$.

Want: $\displaystyle\int_{\partial R} P\,dx = -\int\int_R \partial_y P dx\,dy$

Want: $\int_{\partial R} P\,dx = -\int\int_R \partial_y P\,dx\,dy$

Write $R = (a,b) \times (c,d)$.

$$\int\int_R \partial_y P\,dx\,dy = \int_a^b \int_c^d \partial_y P\,dy\,dx$$

$$/\!/$$

$$\int_c^d \int_a^b \partial_y P\,dx\,dy$$

$\boxed{\text{Fubini's Theorem}}$

Want: $\int_{\partial R} P\,dx = -\int\int_R \partial_y P\,dx\,dy$

Write $R = (a,b)\times(c,d)$.

Fund. thm of calc.

$$\int\int_R \partial_y P\,dx\,dy = \int_a^b \boxed{\int_c^d \partial_y P\,dy}\,dx$$

$\boxed{P = p(x,y)}$

$$= \int_a^b [P]_{y=c}^{y=d}\,dx$$

$$= \int_a^b [p(x,y)]_{y=c}^{y=d}\,dx$$

$$= \int_a^b [p(x,d)] - [p(x,c)]\,dx$$

Want: $\int_{\partial R} P\,dx = -\int_a^b [p(x,d)] - [p(x,c)]\,dx$

Want: $$\int_{\partial R} P\,dx = -\int\int_R \partial_y P\,dx\,dy$$

Write $R = (a,b) \times (c,d)$.

$$\int\int_R \partial_y P\,dx\,dy = \int_a^b \int_c^d \partial_y P\,dy\,dx$$

$$= \int_a^b [P]_{y=c}^{y=d}\,dx \qquad \boxed{P = p(x,y)}$$

$$= \int_a^b [p(x,y)]_{y=c}^{y=d}\,dx$$

$$= \int_a^b [p(x,d)] - [p(x,c)]\,dx$$

Want: $$\int_{\partial R} P\,dx = \int_a^b [p(x,c)] - [p(x,d)]\,dx$$

$R = (a,b) \times (c,d)$

Want: $\displaystyle\int_{\partial R} P\,dx = \int_a^b [p(x,c)] - [p(x,d)]\,dx$

$R = (a,b) \times (c,d)$

$q := (b,c)$, $r := (b,d)$, $s := (a,d)$, $t := (a,c)$

Want: $\displaystyle\int_{\partial R} P\,dx = \int_a^b [p(x,c)] - [p(x,d)]\,dx$

$R = (a,b) \times (c,d)$

Want: $\displaystyle\int_{\partial R} P\,dx = \int_a^b [p(x,c)] - [p(x,d)]\,dx$

$\|$

$$\underbrace{\left[\cancel{\int_{(q,r)} P\,dx}\right]}_{I_R} + \underbrace{\left[\int_{(r,s)} P\,dx\right]}_{I_U} + \underbrace{\left[\cancel{\int_{(s,t)} P\,dx}\right]}_{I_L} + \underbrace{\left[\int_{(t,q)} P\,dx\right]}_{I_D}$$

$q := (b,c),\ r := (b,d),\ s := (a,d),\ t := (a,c)$

implies $\partial R = \{\underbrace{(q,r)}_{\text{right}},\ \underbrace{(r,s)}_{\text{up}},\ \underbrace{(s,t)}_{\text{left}}\ \underbrace{(t,q)}_{\text{down}}\}$

Want: $I_R = 0 \quad \& \quad I_U = -\displaystyle\int_a^b p(x,d)\,dx$

Exercise: $I_L = 0 \quad \& \quad I_D = \displaystyle\int_a^b p(x,c)\,dx$

Want:

$$\int_{(q,r)} P\,dx = 0 \qquad \& \qquad \int_{(r,s)} P\,dx = -\int_a^b p(x,d)\,dx$$

$$\underbrace{\left[\int_{(q,r)} P\,dx\right]}_{I_R} + \underbrace{\left[\int_{(r,s)} P\,dx\right]}_{I_U} + \underbrace{\left[\int_{(s,t)} P\,dx\right]}_{I_L} + \underbrace{\left[\int_{(t,q)} P\,dx\right]}_{I_D}$$

$$q := (b,c),\ r := (b,d),\ s := (a,d),\ t := (a,c)$$

implies $\quad \partial R = \{\underbrace{(q,r)}_{\text{right}},\ \underbrace{(r,s)}_{\text{up}},\ \underbrace{(s,t)}_{\text{left}}\ \underbrace{(t,q)}_{\text{down}}\}$

Want: $\quad I_R = 0 \qquad \& \qquad I_U = -\int_a^b p(x,d)\,dx$

Exercise: $\quad I_L = 0 \qquad \& \qquad I_D = \int_a^b p(x,c)\,dx$

Want:

$$\underbrace{\int_{(q,r)} P\,dx}_{\substack{\alpha_R(t) = b \\ \beta_R(t) = (1-t)c + td \\ \phi_R = (\alpha_R, \beta_R) : [0,1] \to \mathbb{R}^2}} = 0 \qquad \& \qquad \underbrace{\int_{(r,s)} P\,dx}_{\substack{\alpha_U(t) = (1-t)b + ta \\ \beta_U(t) = d \\ \phi_U = (\alpha_U, \beta_U) : [0,1] \to \mathbb{R}^2}} = -\int_a^b p(x,d)\,dx$$

$$q := (b,c),\; r := (b,d),\; s := (a,d),\; t := (a,c)$$

implies $\quad \partial R = \{\underbrace{(q,r)}_{\text{right}},\ \underbrace{(r,s)}_{\text{up}},\ \underbrace{(s,t)}_{\text{left}}\ \underbrace{(t,q)}_{\text{down}}\}$

Want: $\quad I_R = 0 \quad \& \quad I_U = -\int_a^b p(x,d)\,dx$

Exercise: $\quad I_L = 0 \quad \& \quad I_D = \int_a^b p(x,c)\,dx$

Want:

$$\int_{(q,r)} P\,dx = 0 \qquad \& \qquad \int_{(r,s)} \boxed{P}\,\boxed{dx} = \boxed{-\int_a^b p(x,d)\,dx}$$

$\alpha_R(t) = b$

$\beta_R(t) = (1-t)c + td$

$\phi_R = (\alpha_R, \beta_R) : [0,1] \to \mathbb{R}^2$

$\alpha_U(t) = (1-t)b + ta$

$\beta_U(t) = d$

$\phi_U = (\alpha_U, \beta_U) : [0,1] \to \mathbb{R}^2$

$P = p(x,y)$

Repl. x by $\alpha_R(t)$

Repl. y by $\beta_R(t)$

Repl. dx by $\alpha'_R(t)\,dt = 0\,dt$

Repl. dy by $\beta'_R(t)\,dt$

$P = p(x,y)$

Repl. x by $\alpha_U(t)$

Repl. y by $\beta_U(t)$

Repl. dx by $\alpha'_U(t)\,dt$

Repl. dy by $\beta'_U(t)\,dt$

Want: $$\int_0^1 \boxed{[p(\alpha_U(t), \beta_U(t))]}\,\boxed{[\alpha'_U(t)]\,dt} = \boxed{-\int_a^b p(x,d)\,dx}$$

Want: $\int_0^1 [p(\alpha_U(t), d)]\,[\alpha_U'(t)]\,dt \stackrel{\checkmark}{=}$

$$-\int_1^0 [p(\alpha_U(t), d)][\alpha_U'(t)]\,dt$$

STOP

$\alpha_U(t) = (1-t)b + ta$

$\beta_U(t) = d$

$\phi_U = (\alpha_U, \beta_U) : [0,1] \to \mathbb{R}^2$

Change variables

$x = \alpha_U(t)$

$dx = \alpha_U'(t)\,dt$

$a = \alpha_U(1)$

$b = \alpha_U(0)$

Want: $\int_0^1 [p(\alpha_U(t), \overbrace{\beta_U(t)}^{d})]\,[\alpha_U'(t)]\,dt = \overbrace{-\int_a^b p(x,d)\,dx}$