Financial Mathematics
Variations on Stokes’ Theorem
Green’s Theorem
and
Cauchy’s Theorem
Green’s Theorem on rectangles

Definition:
A **directed line segment** in $\mathbb{R}^2$ is an ordered pair of points in $\mathbb{R}^2$, called the **starting point** and **ending point** of $L$.

*E.g.*: $L := ( (1, 2), (4, -1) )$

*Viz.*:

![Diagram showing a directed line segment with endpoints (1, 2) and (4, -1).]
Green’s Theorem on rectangles

Definition:
A **directed line segment** in \( \mathbb{R}^2 \) is an ordered pair of points in \( \mathbb{R}^2 \), called the **starting point** and **ending point** of \( L \).

Definition:
Curves are assumed continuous.

The **standard parametrization** of \( L = (p, q) \) is the constant velocity curve \( \phi : [0, 1] \rightarrow \mathbb{R}^2 \) such that \( \phi(0) = p \) and \( \phi(1) = q \).

Definition:
**Constant velocity** means: \( \phi' \) is constant, i.e., that, \( \forall s, t \in (0, 1) \),
\[
\phi'(s) = \phi'(t)
\]
Green’s Theorem on rectangles

Definition:
A **simple chain** is a finite set of directed line segments.

*E.g.:*

*Viz.:*
Green’s Theorem on rectangles

Definition:
A **rectangle** is a subset of \( \mathbb{R}^2 \) of the form \( I \times J \), where \( I \) and \( J \) are bounded intervals.

*E.g.:* \( R := [1, 4) \times [-1, 1] \)

**Viz.:**

![Diagram of a rectangle with vertices at (1,-1), (4,-1), (4,1), and (1,1)]
Green’s Theorem on rectangles

Definition:
A **rectangle** is a subset of $\mathbb{R}^2$ of the form $I \times J$, where $I$ and $J$ are bounded intervals.

*E.g.*: $R := (1, 4) \times (-1, 1)$

is an open rectangle.

[Viz.]

\[\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
-1
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
4
\end{array}
\end{array}\]
Green’s Theorem on rectangles

Definition: Let $R$ be an open rectangle. The \textbf{counterclockwise boundary} $\partial R$ of $R$ is the set of boundary line segments, directed counterclockwise.

E.g.: $R := (1, 4) \times (-1, 1)$ is an open rectangle.

Viz.:

\[\begin{array}{c}
\begin{tikzpicture}
    \draw[->] (-1.5,0) -- (5,0);
    \draw[->] (0,-1.5) -- (0,5);
    \draw[dashed, red] (1,1) -- (1,4) -- (4,4) -- (4,-1) -- cycle;
    \draw[red, fill=red!30] (1,1) rectangle (4,4);
    \node at (2.5, 2.5) {\textcolor{red}{$R$}};
\end{tikzpicture}
\end{array}\]
Green’s Theorem on rectangles

**Definition:** Let $R$ be an open rectangle. The **counterclockwise boundary** $\partial R$ of $R$ is the set of boundary line segments, directed counterclockwise.

**E.g.:** $R := (1, 4) \times (-1, 1)$ is an open rectangle.

**Viz.:**
Green’s Theorem on rectangles

Definition: Let \( R \) be an open rectangle. The **counterclockwise boundary** \( \partial R \) of \( R \) is the set of boundary line segments, directed counterclockwise.

Def’n: \( R := (a, b) \times (c, d) \)
\( p := (b, c), \ q := (b, d), \ r := (a, d), \ s := (a, c) \)
implies \( \partial R = \{(p, q), (q, r), (r, s), (s, p)\} \)

Viz.:
Green’s Theorem on rectangles

Definition:
Let $L$ be a directed line segment in $\mathbb{R}^2$.
Let $\phi = (\alpha, \beta) : [0, 1] \to \mathbb{R}^2$ be the standard parameterization of $L$.
Let $p, q : \phi([0, 1]) \to \mathbb{R}$ be continuous.

Then we define:

$$\int_L p(x, y) \, dx + q(x, y) \, dy := \int_0^1 [p(\phi(t))] [\alpha'(t)] + [q(\phi(t))] [\beta'(t)] \, dt.$$

Idea: Replace $x$ by $\alpha(t)$, $y$ by $\beta(t)$, and $dx$ by $\alpha'(t) \, dt$, $dy$ by $\beta'(t) \, dt$. 
Green’s Theorem on rectangles

Definition:

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain.
Let $S$ be the union, over $j$, of the image of the standard parametrization of $L_j$.
Let $p, q : S \to \mathbb{R}$ be continuous.

Then we define:

$$\int_C p(x, y) \, dx + q(x, y) \, dy := $$

$$\int_{L_1} p(x, y) \, dx + q(x, y) \, dy + \cdots$$

$$+ \int_{L_n} p(x, y) \, dx + q(x, y) \, dy.$$
Green's Theorem on rectangles

Theorem:

Let $R$ be an open rectangle in $\mathbb{R}^2$.
Let $\bar{R}$ be the union of $R$ and the boundary of $R$.
Let $p, q : \bar{R} \to \mathbb{R}$ be continuous, and smooth on $R$.
Let $P := p(x, y)$ and $Q := q(x, y)$.

Then:

$$\int_{\partial R} P \, dx + Q \, dy = \iint_R \left[ \left( \frac{\partial}{\partial x} Q \right) - \left( \frac{\partial}{\partial y} P \right) \right] \, dx \, dy.$$
Definition: A **zero-form in** $x$ and $y$ is an expression in $x$ and $y$.

Definition: An expression of the form

$p(x, y) \, dx + q(x, y) \, dy$ is called a **one-form in** $x$ and $y$.

Definition: The **exterior derivative** of $F = f(x, y)$, denoted $\text{d}F$, is the one-form $\partial_x F \, dx + \partial_y F \, dy$.

Note: Exterior differentiation carries zero-forms to one-forms.

$$\int_{\partial R} P \, dx + Q \, dy = \int \int_R \left[ (\partial_x Q) - (\partial_y P) \right] dx \, dy.$$
Definition: A zero-form in \( x \) and \( y \) is an expression in \( x \) and \( y \).

Definition: An expression of the form \( p(x, y) \, dx + q(x, y) \, dy \) is called a one-form in \( x \) and \( y \).

Definition: The exterior derivative of \( F = f(x, y) \), denoted \( dF \), is the one-form \( \partial_x F \, dx + \partial_y F \, dy \).

SKILL: Compute the exterior derivative of a zero-form.

\[
\int_{\partial R} P \, dx + Q \, dy = \int \int_R [(\partial_x Q) - (\partial_y P)] \, dx \, dy.
\]
Definition: An expression of the form \( p(x, y) \, dx \wedge dy \) is called a **two-form** in \( x \) and \( y \).

Definition: An expression of the form \( p(x, y) \, dx + q(x, y) \, dy \) is called a **one-form** in \( x \) and \( y \).

Conventions:

\[
\begin{align*}
\inta A &= \inta f = f A = A \wedge f \\
(A + B) \wedge C &= (A \wedge C) + (B \wedge C) \\
A \wedge (B + C) &= (A \wedge B) + (A \wedge C) \\
A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\
dx \wedge dx &= dy \wedge dy = 0 \\
dx \wedge dy &= -dy \wedge dx
\end{align*}
\]

\[
\int_{\partial R} P \, dx + Q \, dy = \int \int_R \left[ (\partial_x Q) - (\partial_y P) \right] \, dx \, dy.
\]
Note: \( f(A \land B) = (fA) \land B = A \land (fB) \)

Proof:
\[
\begin{align*}
f(A \land B) &= f \land (A \land B) = (f \land A) \land B = (fA) \land B \\
&= (A \land f) \land B = A \land (f \land B) = A \land (fB)
\end{align*}
\]

Conventions:
\[
\begin{align*}
f \land A &= fA = A \land f \\
(A + B) \land C &= (A \land C) + (B \land C) \\
A \land (B + C) &= (A \land B) + (A \land C) \\
A \land (B \land C) &= (A \land B) \land C \\
dx \land dx &= dy \land dy = 0 \\
dx \land dy &= -dy \land dx
\end{align*}
\]

\[
\int_{\partial R} P \, dx + Q \, dy = \int \int_R \left[ (\partial_x Q) - (\partial_y P) \right] dx \, dy.
\]
SKILL: Collect terms on a two-form.

\[ \begin{align*}
\text{e.g.: } & [(4x^2 + 3xy) \, dx + (2 \sin(xy)) \, dy] \\
& \wedge [(ye^x) \, dx + (5xy^3) \, dy] = [??????] \, dx \wedge dy \\
?????? &= (4x^2 + 3xy)(5xy^3) \\
& \quad - (2 \sin(xy))(ye^x)
\end{align*} \]

Conventions:

\[ f \wedge A = fA = A \wedge f \]

\( f \) a 0-form in \( x, y \)

\( A + B \wedge C = (A \wedge C) + (B \wedge C) \)

\( A \wedge (B + C) = (A \wedge B) + (A \wedge C) \)

\( A \wedge (B \wedge C) = (A \wedge B) \wedge C \)

\( \int \int_{\partial R} P \, dx + Q \, dy = \int \int_{R} [(\partial_x Q) - (\partial_y P)] \, dx \, dy. \)
SKILL: Compute exterior derivatives of zero-forms.

e.g.: \[ d[e^x + y \sin(x)] = \\
(\partial_x [e^x + y \sin(x)]) \, dx + (\partial_y [e^x + y \sin(x)]) \, dy = \ldots \]

Definition: The exterior derivative of \( F = f(x, y) \), denoted \( dF \), is the one-form \( \partial_x F \, dx + \partial_y F \, dy \).

\[ \int_{\partial R} P \, dx + Q \, dy = \int \int_R \left[ (\partial_x Q) - (\partial_y P) \right] dx \, dy. \]
Definition: An expression of the form \( p(x, y) \, dx + q(x, y) \, dy \) is called a **one-form** in \( x \) and \( y \).

Definition: An expression of the form \( p(x, y) \, dx \wedge dy \) is called a **two-form** in \( x \) and \( y \).

Definition: The **exterior derivative** of \( F = P \, dx + Q \, dy \), denoted \( dF \), is the two-form \( dP \wedge dx + dQ \wedge dy \).

Note: Exterior differentiation carries one-forms to two-forms.

\[
\int_{\partial R} P \, dx + Q \, dy = \int \int_R \left[ (\partial_x Q) - (\partial_y P) \right] \, dx \, dy.
\]
SKILL: Compute exterior derivatives of one-forms.

Definition: The **exterior derivative** of \( F = P \, dx + Q \, dy \), denoted \( dF \), is the two-form \( dP \wedge dx + dQ \wedge dy \).

\[
\int_{\partial R} P \, dx + Q \, dy = \iint_{R} \left[ (\partial_x Q) - (\partial_y P) \right] dx \, dy.
\]
SKILL: Compute exterior derivatives of one-forms.

\[ d[(x \sin y) \, dx + (x^3 y^2) \, dy] = \]
\[ [d(x \sin y)] \wedge dx + [d(x^3 y^2)] \wedge dy = \]
\[ [\partial_y(x \sin y) \, dy] \wedge dx + [\partial_x(x^3 y^2) \, dx] \wedge dy = \]
\[ (\partial_y(x \sin y))(dy \wedge dx) + (\partial_x(x^3 y^2))(dx \wedge dy) = \]
\[ -(\partial_y(x \sin y)) + (\partial_x(x^3 y^2))] [dx \wedge dy] = \cdots \]

Definition: \[
\int_R g(x, y) \, dx \wedge dy := \int \int_R g(x, y) \, dx \, dy \]

Definition: The exterior derivative of \( F = P \, dx + Q \, dy \), denoted \( dF \), is the two-form \( dP \wedge dx + dQ \wedge dy \).

\[
\int_{\partial R} P \, dx + Q \, dy = \int \int_R [\partial_x Q - \partial_y P] \, dx \, dy. \]
\[ d(P \, dx + Q \, dy) = \]
\[ (dP \wedge dx) + (dQ \wedge dy) = \]
\[ ((\partial_y P)dy \wedge dx) + ((\partial_x Q)dx \wedge dy) = \]
\[ ((\partial_y P)(dy \wedge dx)) + ((\partial_x Q)(dx \wedge dy)) = \]
\[ ((-\partial_y P)(dx \wedge dy)) + ((\partial_x Q)(dx \wedge dy)) = \]
\[ -(\partial_y P) + (\partial_x Q)[dx \wedge dy] = \]

Green's Theorem on rectangles:

\( \forall \) one-forms \( \omega \) in \( x \) and \( y \), \( \forall \) open rectangles \( R \),

\[ \int_{\partial R} \omega = \int_{R} d\omega. \]

\( \omega \) continuous on \( \bar{R} \) and smooth on \( R \)

\[ \int_{\partial R} P \, dx + Q \, dy = \int_{R} \int_{R} [(\partial_x Q) - (\partial_y P)] \, dx \, dy. \]
Practice:

\[(x^2 y \, dx + 3x \, dy + e^z \, dz)\]
\[\wedge (3 \, dx + 2e^{xz} \, dy - 4y^3 \, dz)\]
\[\wedge (dx + dy + dz) = [?????] \, dx \wedge dy \wedge dz\]

\[(-y \, dx + x^y \, dy + xyz \, dz)\]
\[\wedge (2ye^z \, dx - dy + 7x^{-1} \, dz)\]
\[\wedge (\cos(xy/z) \, dx + 3 \, dy - z^5 \, dz)\]
\[= [?????] \, dx \wedge dy \wedge dz\]
Solutions:

\[(x^2y \, dx + 3x \, dy + e^z \, dz)\]
\[\wedge (3 \, dx + 2e^{xz} \, dy - 4y^3 \, dz)\]
\[\wedge (dx + dy + dz) = [??????] \, dx \wedge dy \wedge dz\]

\[?????? = \]
\[\left(x^2y\right)(2e^{xz})(1)\]
\[- (x^2y)(-4y^3)(1)\]
\[- (3x)(3)(1)\]
\[+ (3x)(-4y^3)(1)\]
\[+ (e^z)(3)(1)\]
\[- (e^z)(2e^{xz})(1)\]
Solutions:

\(-y \, dx + x^y \, dy + xyz \, dz\)
\(\wedge (2ye^{-z} \, dx - \, dy + 7x^{-1} \, dz)\)
\(\wedge (\cos(xy/z) \, dx + 3 \, dy - z^5 \, dz)\)

\[= [?????] \, dx \wedge dy \wedge dz\]

?????? =

\((-y)(-1)(-z^5)\)
\(-(-y)(7x^{-1})(3)\)
\(-(xy)(2ye^{-z})(-z^5)\)
\((xy)(7x^{-1})(\cos(xy/z))\)
\((xyz)(2ye^{-z})(3)\)
\(-(xyz)(-1)(\cos(xy/z))\)
Practice:

\[ d(\sin(xye^z)) \]
\[ d([e^{-3xy}][\sin(z)]) \]

\[ d(x^2y\,dx + 3x\,dy + e^z\,dz) \]
\[ d(-y\,dx + xy\,dy + xyz\,dz) \]
Solutions:

\[ d(\sin(xye^z)) = (\cos(xye^z) \cdot ye^z) \, dx + (\cos(xye^z) \cdot xe^z) \, dy + (\cos(xye^z) \cdot yye^z) \, dz \]

\[ d([e^{-3xy}][\sin(z)]) = (e^{-3xy} \cdot [-3y][\sin(z)]) \, dx + (e^{-3xy} \cdot [-3x][\sin(z)]) \, dy + (e^{-3xy} \cdot \cos(z)) \, dz \]
Solutions:

\[ d(x^2 y \, dx + 3x \, dy + e^z \, dz) \]
\[ = ([3] - [x^2]) \, dx \wedge dy \]
\[ + ([0] - [0]) \, dx \wedge dz \]
\[ + ([0] - [0]) \, dy \wedge dz \]

\[ d(-y \, dx + xy \, dy + xyz \, dz) \]
\[ = ([yxx^{-1}] - [-1]) \, dx \wedge dy \]
\[ + ([yz] - [0]) \, dx \wedge dz \]
\[ + ([xz] - [0]) \, dy \wedge dz \]
Practice:

\[
\int_{(2,5)}^{(-1,2)} e^x y \, dx + xy^3 \, dy
\]

\[
t \in [0, 3]
\]

\[
= \int_{0}^{3} e^{(-1+t)+(2+t)} \, dt + (-1 + t)(2 + t)^3 \, dt
\]

\[
= \int_{0}^{3} e^{1+2t} + (-1 + t)(2^3 + 3 \cdot 2^2 t + 3 \cdot 2t^2 + t^3) \, dt
\]

\[
= \ldots
\]
Compute \[ \int_{(3,2,1)}^{(5,6,7)} x (\sin y) \, dx + z^2 e^x \, dy \]

\[ x = 3 + 2t \\
y = 2 + 4t \\
z = 1 + 6t \]

\[ = \int_0^1 (3 + 2t)(\sin(2 + 4t)) \cdot 2 \, dt \]

\[ + \int_0^1 \left(1 + 6t\right)^2 e^{3+2t} \cdot 4 \, dt \]

\[ \int_0^1 t (\sin(2 + 4t)) \, dt = \int_0^4 \frac{t}{4} (\sin(2 + t)) \, \frac{dt}{4} \]

\[ \int_2^6 t (\sin t) \, dt = \int_2^6 \left(\frac{t - 2}{4}\right) (\sin t) \, \frac{dt}{4} \]
Let \( R := (1, 2) \times (3, 4) \).

Compute \( \int_R [e^{2x+3y}] \, dy \wedge dx \).

\[
\int_R [e^{2x+3y}] \, dy \wedge dx = \int \int_R [e^{2x+3y}] \, dx \, dy
\]

\[
= -\int_3^4 \int_1^2 [e^{2x+3y}] \, dx \, dy = -\int_3^4 \left[ \frac{e^{2x+3y}}{2} \right]_{x: \to 2}^{x: \to 1} \, dy
\]

\[
= -\int_3^4 \left[ \frac{e^{4+3y}}{2} \right] - \left[ \frac{e^{2+3y}}{2} \right] \, dy
\]

\[
= -\left[ \left[ \frac{e^{4+3y}}{2 \cdot 3} \right]_{y: \to 4} - \left[ \frac{e^{2+3y}}{2 \cdot 3} \right]_{y: \to 3} \right] + \left[ \frac{e^{2+3y}}{2 \cdot 3} \right]_{y: \to 4} - \left[ \frac{e^{2+3y}}{2 \cdot 3} \right]_{y: \to 3} = \ldots
\]
(Real) Green’s Theorem on rectangles, SETUP

**Definition:**

Let $L$ be a directed line segment in $\mathbb{R}^2$. Let $\phi = (\alpha, \beta) : [0, 1] \rightarrow \mathbb{R}^2$ be the standard parameterization of $L$. Let $p, q : \phi([0, 1]) \rightarrow \mathbb{R}$ be continuous.

Then we define:

$$
\int_L p(x, y) \, dx + q(x, y) \, dy := \int_0^1 [p(\phi(t))] [\alpha'(t)] + [q(\phi(t))] [\beta'(t)] \, dt.
$$

**Idea:** Replace $x$ by $\alpha(t)$, $y$ by $\beta(t)$, $dx$ by $\alpha'(t) \, dt$, $dy$ by $\beta'(t) \, dt$. 


Complex Green’s Theorem on rectangles, Definition:

Let \( L \) be a directed line segment in \( \mathbb{C} \).
Let \( \phi : [0, 1] \rightarrow \mathbb{C} \) be the standard parameterization of \( L \).
Let \( p : \phi([0, 1]) \rightarrow \mathbb{C} \) be continuous.

Then we define:

\[
\int_L p(z) \, dz := \int_0^1 [p(\phi(t))] \phi'(t) \, dt.
\]

Idea: Replace \( z \) by \( \phi(t) \),
\( \, dz \) by \( \phi'(t) \, dt \).
(Real) Green’s Theorem on rectangles, SETUP

Definition:

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain.
Let $S$ be the union, over $j$, of the image of the standard parametrization of $L_j$.

Let $p, q : S \to \mathbb{R}$ be continuous.

Then we define:

$$\int_C p(x, y) \, dx + q(x, y) \, dy := \int_{L_1} p(x, y) \, dx + q(x, y) \, dy + \cdots$$

$$+ \int_{L_n} p(x, y) \, dx + q(x, y) \, dy.$$
Complex Green’s Theorem on rectangles,

**Definition:**

Let $C = \{L_1, \ldots, L_n\}$ be a simple chain in $\mathbb{C}$.

Let $S$ be the union, over $j$, of the image of the standard parametrization of $L_j$.

Let $\phi : S \to \mathbb{C}$ be continuous.

Then we define:

$$\int_C \phi(z) \, dz := \int_{L_1} \phi(z) \, dz + \cdots + \int_{L_n} \phi(z) \, dz.$$
(Real) Green’s Theorem on rectangles

**Theorem:**

Let \( R \) be an open rectangle in \( \mathbb{R}^2 \).

Let \( \bar{R} \) be the union of \( R \) and the boundary of \( R \).

Let \( p, q : \bar{R} \to \mathbb{R} \) be continuous, and smooth on \( R \).

Let \( \omega := p(x, y) \, dx + q(x, y) \, dy \).

Then:

\[
\int_{\partial R} \omega = \int_{R} d\omega.
\]
Complex Green’s Theorem on rectangles

Theorem:

Let $R$ be an open rectangle in $\mathbb{C}$.
Let $\bar{R}$ be the union of $R$ and the boundary of $R$.
Let $\phi : \bar{R} \to \mathbb{C}$ be continuous, and smooth on $R$.
Let $\omega := \phi(z) \, dz$.

Then:

One variable!

$$\int_{\partial R} \omega = \int_{R} \, d\omega.$$
Exercise: Compute $d[(\sin x) \, dx]$, the exterior derivative of $(\sin x) \, dx$ with respect to $x$.

Solution:

\[
    d[(\sin x) \, dx] = [d(\sin x)] \wedge dx \\
    = [\partial_x (\sin x) \, dx] \wedge dx \\
    = 0
\]
Exercise: Compute $d[e^x \, dx]$
the exterior derivative of $e^x \, dx$
with respect to $x$.

Solution:

$$d[e^x \, dx] = [d(e^x)] \wedge dx$$
$$= [\partial_x (e^x) \, dx] \wedge dx$$
$$= 0$$
Fact:
\[ d[\phi(x) \, dx] = 0, \text{ for any smooth } \phi. \]

Proof:
\[ d[\phi(x) \, dx] = [d(\phi(x))] \wedge dx \]
\[ = [\partial_x(\phi(x)) \, dx] \wedge dx \]
\[ = 0 \]
\[ \text{QED} \]
Complex Green’s Theorem on rectangles

Theorem:

Let \( R \) be an open rectangle in \( \mathbb{C} \).

Let \( \bar{R} \) be the union of \( R \) and the boundary of \( R \).

Let \( \phi : \bar{R} \to \mathbb{C} \) be continuous, and smooth on \( R \).

Let \( \omega := \phi(z) \, dz \).

Then: \[
\int_{\partial R} \omega = \int_{R} d\omega.
\]

Expect: 0 sometimes! 42
Complex Green’s Theorem on rectangles, \textbf{SETUP}

\textbf{Definition:}

Let $R$ be an open set in $\mathbb{C}$. Let $\phi : R \rightarrow \mathbb{C}$ be smooth. Let $z \in R$. If

$$L := \lim_{h \to 0} \frac{[f(z + h)] - [f(z)]}{h}$$

exists, then we say that $f$ is complex differentiable at $z$, and we define $[f'(z)] = L$.

\textbf{Exercise:} Define $f : \mathbb{C} \rightarrow \mathbb{C}$ by $f(z) = e^{3z}$. Show, for all $z \in \mathbb{C}$, that $f$ is complex differentiable at $z$, and that $f'(z) = 3e^{3z}$. 
Complex Green’s Theorem on rectangles, \( \text{SETUP} \)

If \( L := \lim_{h \to 0} \frac{[f(z + h)] - [f(z)]]}{h} \) exists,

then we say that \( f \) is \( \text{non-e.g. we define} \) \( f'(z) = L \).

Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \).

and we define \( f'(z) = L \).
Complex Green's Theorem on rectangles, \textbf{SETUP}

If $L := \lim_{h \to 0} \frac{[f(z + h)] - [f(z)]}{h}$ exists,

then we say that $f$ is \textbf{complex differentiable} at $z$,

and we define $f'(z) = L$.

\textbf{non-e.g.:} Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

$f$ is not complex diff. at 1:

\forall integers $j > 0$, let $h^R_j := 1/j$ and $h^I_j := i/j$.

Let $L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j}$, $i := \sqrt{-1}$

$L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j}$.

Want: $L^R \neq L^I$. 

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Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2 \).

\[
L^R = \lim_{j \to \infty} \lim_{h \to 0} \frac{h}{(1 + \frac{1}{j})^2 + 0^2} - \frac{h}{1^2 + 0^2}
\]

\[
L^I = \lim_{j \to \infty} \lim_{h \to 0} \frac{ih}{1^2 + (\frac{1}{j})^2} - \frac{ih}{1^2 + 0^2}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \).

\( f \) is not complex diff. at 1:

\( \forall \) integers \( j > 0 \), let \( h^R_j := 1/j \) and \( h^I_j := i/j \).

Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j} \), \( i := \sqrt{-1} \)

\( L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j} \).

Want: \( L^R \neq L^I \).
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{{h \to 0}} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{hh} = \left[ \frac{d}{dx}(x^2 + 0^2) \right]_{x=1}^{y=0}
\]

\[
L^I = \lim_{{h \to 0}} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih \cdot ih}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2. \) \( f \) is not complex diff. at 1:

\( \forall \) integers \( j > 0, \) let \( h^R_j := 1/j \) and \( h^I_j := i/j. \)

Let

\[
L^R := \lim_{{j \to \infty}} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j}
\]

\[
L^I := \lim_{{j \to \infty}} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j}
\]

Want:

\[ L^R \neq L^I \]
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{d}{dx}([x^2 + y^2]_{y=0}) \right]_{x=1}
\]

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L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \).

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Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j} \),

\[
L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j}
\]

Want: \( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{d}{dx}([x^2 + y^2]_{y=0}) \right]_{x=1} \\
L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih} = \left[ \frac{\partial}{\partial x}(x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

**non-e.g.**: Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \).

\( f \) is not complex diff. at 1:

\( \forall \) integers \( j > 0 \), let \( h^R_j := 1/j \) and \( h^I_j := i/j \).

Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j} \), \( i := \sqrt{-1} \)

\[
L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j}
\]

Want: \( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2 \).

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\[
L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih} = \left[ \frac{\partial}{\partial x}(x^2 + y^2) \right]_{(x,y)=(1,0)}
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\( L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j} \).

**Want:** \( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2 \).

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x} (x^2 + y^2) \right]_{(x,y) = (1,0)}
\]

\[
L^I = \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{ih} = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \). *f* is not complex diff. at 1:

\( \forall \) integers \( j > 0 \), let \( h^R_j := 1/j \) and \( h^I_j := i/j \).

Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j} \), \( i := \sqrt{-1} \)

\( L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j} \).

Want:
\( L^R \neq L^I \).
Complex Green’s Theorem on rectangles, \[ \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \]

**SETUP**

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x}(x^2 + y^2) \right]_{(x,y) = (1,0)}
\]

\[
L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} = \frac{1}{i} \left[ \frac{d}{dy}(1^2 + y^2) \right]_{y = 0}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \). \( f \) is not complex diff. at 1:

\( \forall \) integers \( j > 0 \), let \( h_j^R := 1/j \) and \( h_j^I := i/j \).

Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h_j^R)] - [f(1)]}{h_j^R} \), \( i := \sqrt{-1} \)

\( L^I := \lim_{j \to \infty} \frac{[f(1 + h_j^I)] - [f(1)]}{h_j^I} \).

Want: \( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x} (x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

\[
L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} \left[ \frac{d}{dy} (1^2 + y^2) \right]_{x=1, y=0}
\]

**non-e.g.:** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2. \)

\( f \) is not complex diff. at \( 1: \)

\( \forall \) integers \( j > 0, \) let \( h^R_j := 1/j \) and \( h^I_j := i/j. \)

Let \( L^R := \lim_{j \to \infty} \frac{[f(1 + h^R_j)] - [f(1)]}{h^R_j}, \)

\( L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j}. \)

Want: \( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x} (x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

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L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} \left[ \frac{d}{dy} ([x^2 + y^2]_{x=1}) \right]_{y=0}
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non-e.g.: Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = |z|^2 \).
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\( i := \sqrt{-1} \)

Let \( L^I := \lim_{j \to \infty} \frac{[f(1 + h^I_j)] - [f(1)]}{h^I_j} \).

Want:\( L^R \neq L^I \)
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2 \).

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L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x}(x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

\[
L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} \left[ \frac{d}{dy}([x^2 + y^2]_{x=1}) \right]_{y=0}
\]

\[
= \frac{1}{i} \left[ \frac{\partial}{\partial y}(x^2 + y^2) \right]_{(x,y)=(1,0)}
\]
Complex Green’s Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2. \)

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x} (x^2 + y^2) \right]_{(x,y) = (1,0)}
\]

\[
L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} \left[ \frac{\partial}{\partial y} (x^2 + y^2) \right]_{(x,y) = (1,0)} = \frac{1}{i} \left[ \frac{\partial}{\partial y} (x^2 + y^2) \right]_{(x,y) = (1,0)}
\]
Complex Green's Theorem on rectangles, \( \forall x, y \in \mathbb{R}, f(x + iy) = x^2 + y^2 \).

\[
L^R = \lim_{h \to 0} \frac{[(1 + h)^2 + 0^2] - [1^2 + 0^2]}{h} = \left[ \frac{\partial}{\partial x}(x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

\[
L^I = \frac{1}{i} \lim_{h \to 0} \frac{[1^2 + h^2] - [1^2 + 0^2]}{h} = \frac{1}{i} \left[ \frac{\partial}{\partial y}(x^2 + y^2) \right]_{(x,y)=(1,0)}
\]

For \( f \) to be complex differentiable at \( a + bi \), we need

\[
\left[ \frac{\partial}{\partial x}(f(x + iy)) \right]_{(x,y)=(a,b)} = \frac{1}{i} \left[ \frac{\partial}{\partial y}(f(x + iy)) \right]_{(x,y)=(a,b)}
\]
Complex Green’s Theorem on rectangles, SETUP

For \( f \) to be \( \text{cx-diff} \), we need

\[
\frac{\partial}{\partial x} (f(x + iy)) = \frac{1}{i} \left[ \frac{\partial}{\partial y} (f(x + iy)) \right]
\]

For \( f \) to be \( \text{cx-diff at } a + bi \), we need

\[
\left[ \frac{\partial}{\partial x} (f(x + iy)) \right]_{(x,y)=(a,b)} = \frac{1}{i} \left[ \frac{\partial}{\partial y} (f(x + iy)) \right]_{(x,y)=(a,b)}
\]
Complex Green’s Theorem on rectangles, SETUP

For \( f \) to be \( \text{cx-diff} \), we need

\[
\frac{\partial}{\partial x}(f(x + iy)) = \frac{1}{i} \left[ \frac{\partial}{\partial y}(f(x + iy)) \right]
\]

Say \( f(x + iy) = [u(x, y)] + i[v(x, y)] \).
Complex Green's Theorem on rectangles, SETUP

For $f$ to be complex differentiable, we need

$$\frac{\partial}{\partial x} (f(x + iy)) = \left[ \frac{\partial}{\partial y} (f(x + iy)) \right]_U + i \left[ \frac{\partial}{\partial y} (f(x + iy)) \right]_V$$

Say $f(x + iy) = [u(x, y)] + i[v(x, y)]$.

$$U := u(x, y), \quad V := v(x, y)$$

$$\partial_x (U + iV) = \left[ \frac{1}{i} \right] \left[ \partial_y (U + iV) \right]$$

$$\partial_x U + i(\partial_x V) = -i(\partial_y U) + \partial_y V$$

$\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$ Cauchy-Riemann equations
Problem:

Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = e^{z^2/2} \).

Define \( u, v : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x + iy) = [u(x, y)] + [v(x, y)]i.
\]

Let \( U := u(x, y) \) and \( V := v(x, y) \).

Find \( U \) and \( V \).

\[
f(x + iy) = e^{(x+iy)^2/2} = e^{(x^2 - y^2 + 2ixy)/2}
\]

\[
= e^{[(x^2 - y^2)/2] + i[xy]} = e^{[(x^2 - y^2)/2]} e^{i[xy]}
\]

\[
= e^{(x^2 - y^2)/2} \left[ (\cos (xy)) + i (\sin (xy)) \right]
\]

\[
= [e^{(x^2 - y^2)/2}] \left[ (\cos (xy)) \right] + i [e^{(x^2 - y^2)/2}] \left[ (\sin (xy)) \right]
\]

\[
\underbrace{U}_{\text{ }} \underbrace{V}_{\text{ }}
\]

Exercise: Check \( \partial_x U = \partial_y V \) & \( \partial_x V = -\partial_y U \)
Complex Green’s Theorem on rectangles

Theorem:

Let $R$ be an open rectangle in $\mathbb{C}$.

Let $\overline{R}$ be the union of $R$ and the boundary of $R$.

Let $\phi : \overline{R} \to \mathbb{C}$ be continuous, and smooth on $R$.

Let $\omega := \phi(z) \, dz$.

Then:

$$\int_{\partial R} \omega = \int_{R} \, d\omega.$$
Complex Green’s Theorem on rectangles

**Theorem:**

Let \( R \) be an open rectangle in \( \mathbb{C} \).

Let \( \bar{R} \) be the union of \( R \) and the boundary of \( R \).

Let \( \phi : \bar{R} \to \mathbb{C} \) be continuous, and analytic on \( R \).

Let \( \omega := \phi(z) \, dz \).

Then:

\[
\int_{\partial R} \omega = \int_{R} d\omega.
\]
Cauchy’s Theorem on rectangles:

Theorem:

Let \( R \) be an open rectangle in \( \mathbb{C} \).
Let \( \bar{R} \) be the union of \( R \) and the boundary of \( R \).
Let \( \phi : \bar{R} \to \mathbb{C} \) be continuous, and analytic on \( R \).
Let \( \omega := \phi(z) \, dz \).

Then:

\[
\int_{\partial R} \omega = 0.
\]

Next: Proof of Green’s Th’m for rectangles in \( \mathbb{R}^2 \).
Green’s Theorem on rectangles

Theorem:
Let $R$ be an open rectangle in $\mathbb{R}^2$.
Let $\overline{R}$ be the union of $R$ and the boundary of $R$.
Let $p, q : \overline{R} \to \mathbb{R}$ be continuous, and smooth on $R$.
Let $P := p(x, y)$ and $Q := q(x, y)$.

Then: \[ \int_{\partial R} P \, dx + Q \, dy = \iint_R [(\partial_x Q) - (\partial_y P)] \, dx \, dy. \]

Proof:
Want: \[ \int_{\partial R} P \, dx = - \iint_R \partial_y P \, dx \, dy \]
Exercise: \[ \int_{\partial R} Q \, dy = \iint_R \partial_x Q \, dx \, dy \]
Want: \( \int_{\partial R} P \, dx = - \int \int_R \partial_y P \, dx \, dy \)

Write \( R = (a, b) \times (c, d) \).
Want: \[ \int_{\partial R} P \, dx = -\int \int_{R} \partial_y P \, dx \, dy \]

Write \( R = (a, b) \times (c, d) \).

\[ \int \int_{R} \partial_y P \, dx \, dy = \int_{a}^{b} \int_{c}^{d} \partial_y P \, dy \, dx \]

\[ \int_{c}^{d} \int_{a}^{b} \partial_y P \, dx \, dy \]

Fubini’s Theorem
Want: \[ \int_{\partial R} P \, dx = - \int \int_{R} \partial_y P \, dx \, dy \]

Write \( R = (a, b) \times (c, d) \).

Fund. thm of calc.

\[
\int \int_{R} \partial_y P \, dx \, dy = \int_{a}^{b} \int_{c}^{d} \partial_y P \, dy \, dx
\]

\[
= \int_{a}^{b} [P]_{y=c}^{d} \, dx
\]

\[
= \int_{a}^{b} [p(x, y)]_{y=c}^{d} \, dx
\]

\[
= \int_{a}^{b} [p(x, d)] - [p(x, c)] \, dx
\]

Want: \[ \int_{\partial R} P \, dx = - \int_{a}^{b} [p(x, d)] - [p(x, c)] \, dx \]
Want: \[ \int_{\partial R} P \, dx = - \int \int_R \partial_y P \, dx \, dy \]

Write \( R = (a, b) \times (c, d) \).

\[ \int \int_R \partial_y P \, dx \, dy = \int_a^b \int_c^d \partial_y P \, dy \, dx \]

\[ = \int_a^b [P]_{y=c}^{y=d} \, dx \]

\[ = \int_a^b [p(x, y)]_{y=c}^{y=d} \, dx \]

\[ = \int_a^b [p(x, d)] - [p(x, c)] \, dx \]

Want: \[ \int_{\partial R} P \, dx = \int_a^b [p(x, c)] - [p(x, d)] \, dx \]
Want: \[ \int_{\partial R} P \, dx = \int_a^b [p(x, c)] - [p(x, d)] \, dx \]
\[ R = (a, b) \times (c, d) \]

\[ q := (b, c), \ r := (b, d), \ s := (a, d), \ t := (a, c) \]

Want: \[ \int_{\partial R} P \, dx = \int_a^b [p(x, c)] - [p(x, d)] \, dx \]
Want: \[ \int_{\partial R} P \, dx = \int_a^b [p(x, c)] - [p(x, d)] \, dx \]

\[ \begin{align*}
I_R &= \left[ \int_{(q,r)} P \, dx \right] \\
I_U &= \left[ \int_{(r,s)} P \, dx \right] \\
I_L &= \left[ \int_{(s,t)} P \, dx \right] \\
I_D &= \left[ \int_{(t,q)} P \, dx \right]
\end{align*} \]

where: 
- \( q := (b, c) \), \( r := (b, d) \), \( s := (a, d) \), \( t := (a, c) \)

\( \partial R = \{(q,r), (r,s), (s,t), (t,q)\} \) implies right \( I_R \), up \( I_U \), left \( I_L \), down \( I_D \)

Want: \( I_R = 0 \) \& \( I_U = -\int_a^b p(x, d) \, dx \)

Exercise: \( I_L = 0 \) \& \( I_D = \int_a^b p(x, c) \, dx \)
Want: \[ \int_{(q,r)} P \, dx = 0 \quad \& \quad \int_{(r,s)} P \, dx = -\int_{a}^{b} p(x, d) \, dx \]

\[
\begin{align*}
[\int_{(q,r)} P \, dx] &+ [\int_{(r,s)} P \, dx] + [\int_{(s,t)} P \, dx] + [\int_{(t,q)} P \, dx] \\
= & \quad I_R + I_U + I_L + I_D
\end{align*}
\]

\[q := (b, c), \quad r := (b, d), \quad s := (a, d), \quad t := (a, c)\]

implies \[\partial R = \{(q, r), (r, s), (s, t), (t, q)\}\]

\[\text{right up left down}\]

Want: \[I_R = 0 \quad \& \quad I_U = -\int_{a}^{b} p(x, d) \, dx\]

Exercise: \[I_L = 0 \quad \& \quad I_D = \int_{a}^{b} p(x, c) \, dx\]
Want:
\[
\int_{(q,r)} P \, dx = 0 \quad \& \quad \int_{(r,s)} P \, dx = -\int_{a}^{b} p(x, d) \, dx
\]

\[\begin{align*}
\alpha_R(t) &= b \\
\beta_R(t) &= (1 - t)c + td \\
\phi_R &= (\alpha_R, \beta_R) : [0, 1] \to \mathbb{R}^2
\end{align*}\]

\[\begin{align*}
\alpha_U(t) &= (1 - t)b + ta \\
\beta_U(t) &= d \\
\phi_U &= (\alpha_U, \beta_U) : [0, 1] \to \mathbb{R}^2
\end{align*}\]

\[q := (b, c), \quad r := (b, d), \quad s := (a, d), \quad t := (a, c)\]

implies 
\[
\partial R = \{(q, r), (r, s), (s, t), (t, q)\}
\]

right \quad up \quad left \quad down

Want:
\[I_R = 0 \quad \& \quad I_U = -\int_{a}^{b} p(x, d) \, dx\]

Exercise: 
\[I_L = 0 \quad \& \quad I_D = \int_{a}^{b} p(x, c) \, dx\]
Want: \[
\int_{(q,r)} P \, dx = 0 \quad \& \quad \int_{(r,s)} P \, dx = -\int_{a}^{b} p(x, d) \, dx
\]

\[\alpha_R(t) = b\]
\[\beta_R(t) = (1-t)c + td\]
\[\phi_R = (\alpha_R, \beta_R) : [0, 1] \to \mathbb{R}^2\]

\[\alpha_U(t) = (1-t)b + tc\]
\[\beta_U(t) = d\]
\[\phi_U = (\alpha_U, \beta_U) : [0, 1] \to \mathbb{R}^2\]

\[P = p(x, y)\]
Repl. \(x\) by \(\alpha_R(t)\)
Repl. \(y\) by \(\beta_R(t)\)
Repl. \(dx\) by \(\alpha_R'(t) \, dt = 0 \, dt\)
Repl. \(dy\) by \(\beta_R'(t) \, dt\)

Want: \[
\int_{0}^{1} \left[ p(\alpha_U(t), \beta_U(t)) \right] [\alpha_U'(t)] \, dt = -\int_{a}^{b} p(x, d) \, dx
\]
Want: \[ \int_0^1 [p(\alpha_U(t), \beta_U(t))] [\alpha'_U(t)] \, dt = - \int_1^0 [p(\alpha_U(t), d)][\alpha'_U(t)] \, dt \]

\[ \alpha_U(t) = (1 - t)b + ta \]
\[ \beta_U(t) = d \]
\[ \phi_U = (\alpha_U, \beta_U) : [0, 1] \to \mathbb{R}^2 \]

Change variables
\[ x = \alpha_U(t) \]
\[ dx = \alpha'_U(t) \, dt \]
\[ a = \alpha_U(1) \]
\[ b = \alpha_U(0) \]