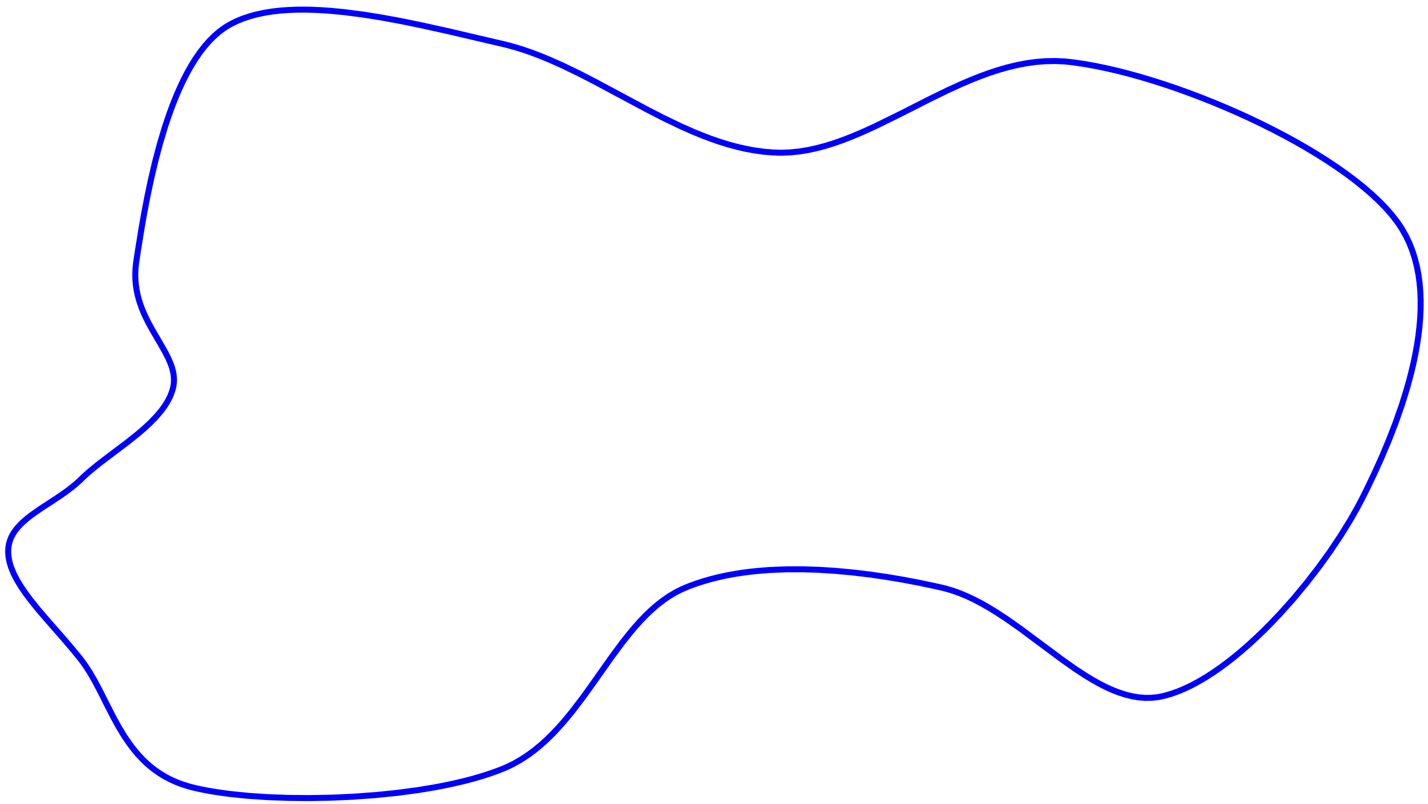
Financial Mathematics

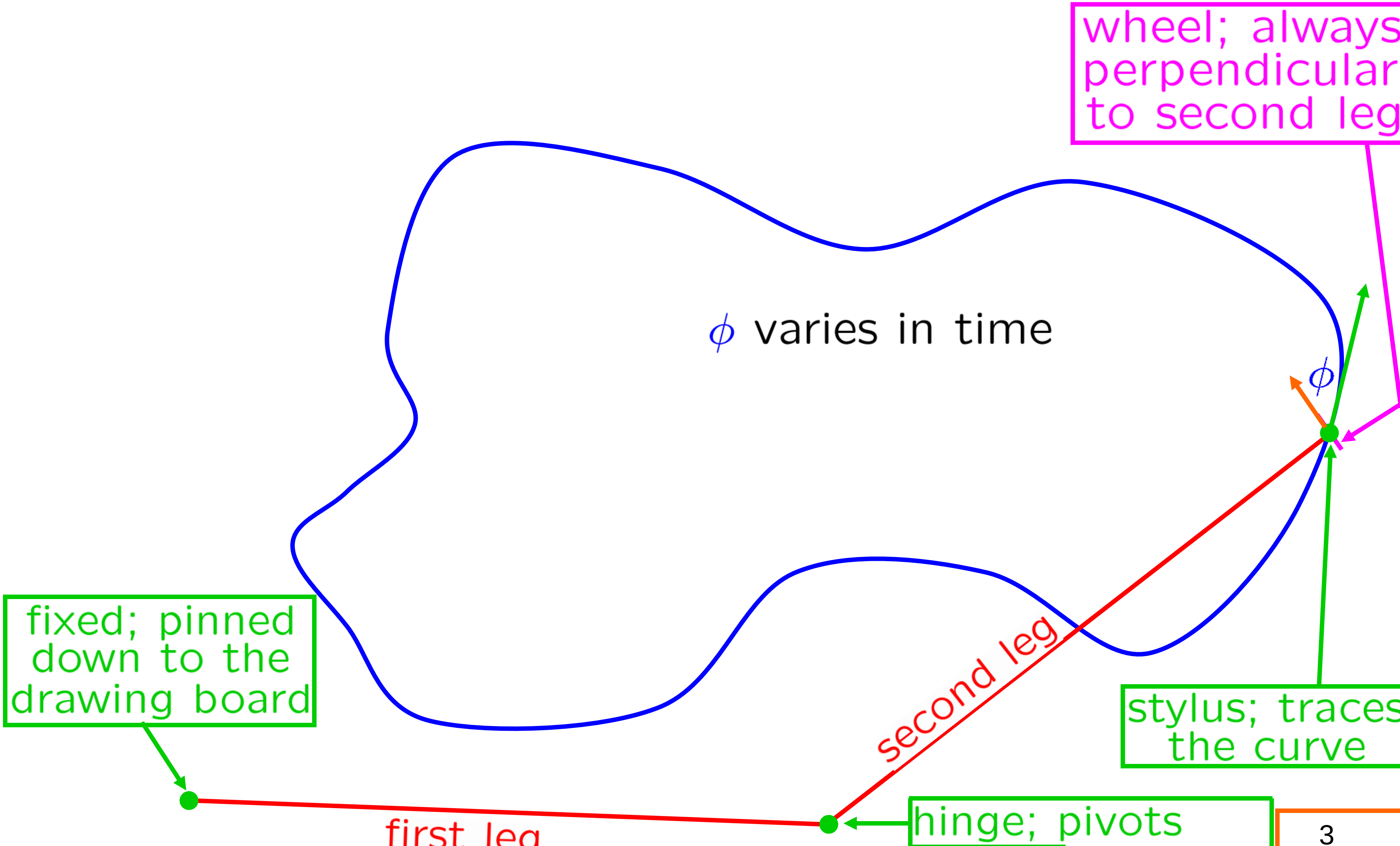
Planimeters

TOP VIEW OF A DRAWING BOARD



How can we find the area enclosed?

TOP VIEW OF A DRAWING BOARD



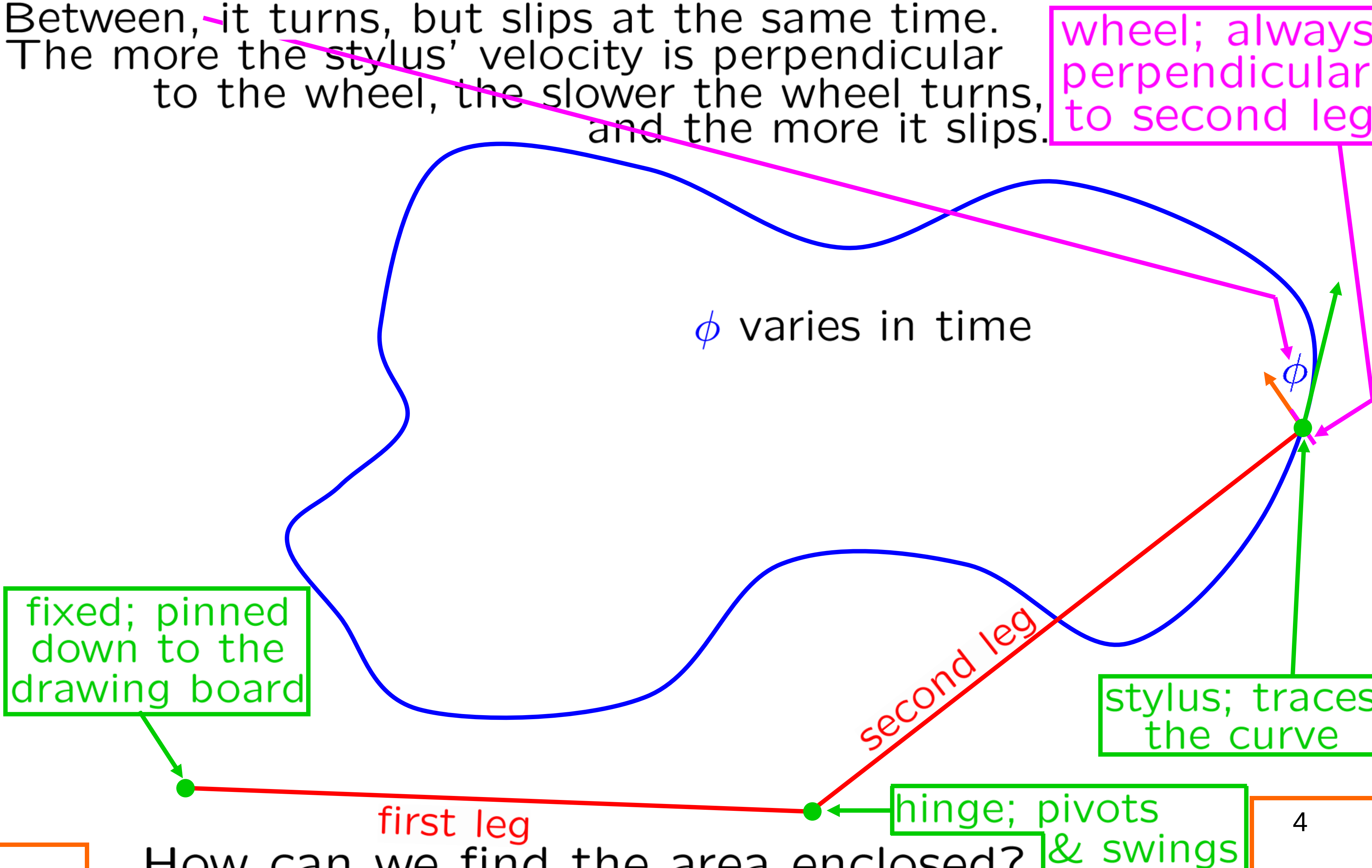
How can we find the area enclosed?

When stylus moves perpendicular to the wheel, *i.e.*, when $\phi = \frac{\pi}{2}$, the wheel slips and doesn't turn at all.

When stylus moves in direction of the wheel, *i.e.*, when $\phi = 0$, the wheel turns as fast as possible.

Between, it turns, but slips at the same time.

The more the stylus' velocity is perpendicular to the wheel, the slower the wheel turns, and the more it slips.

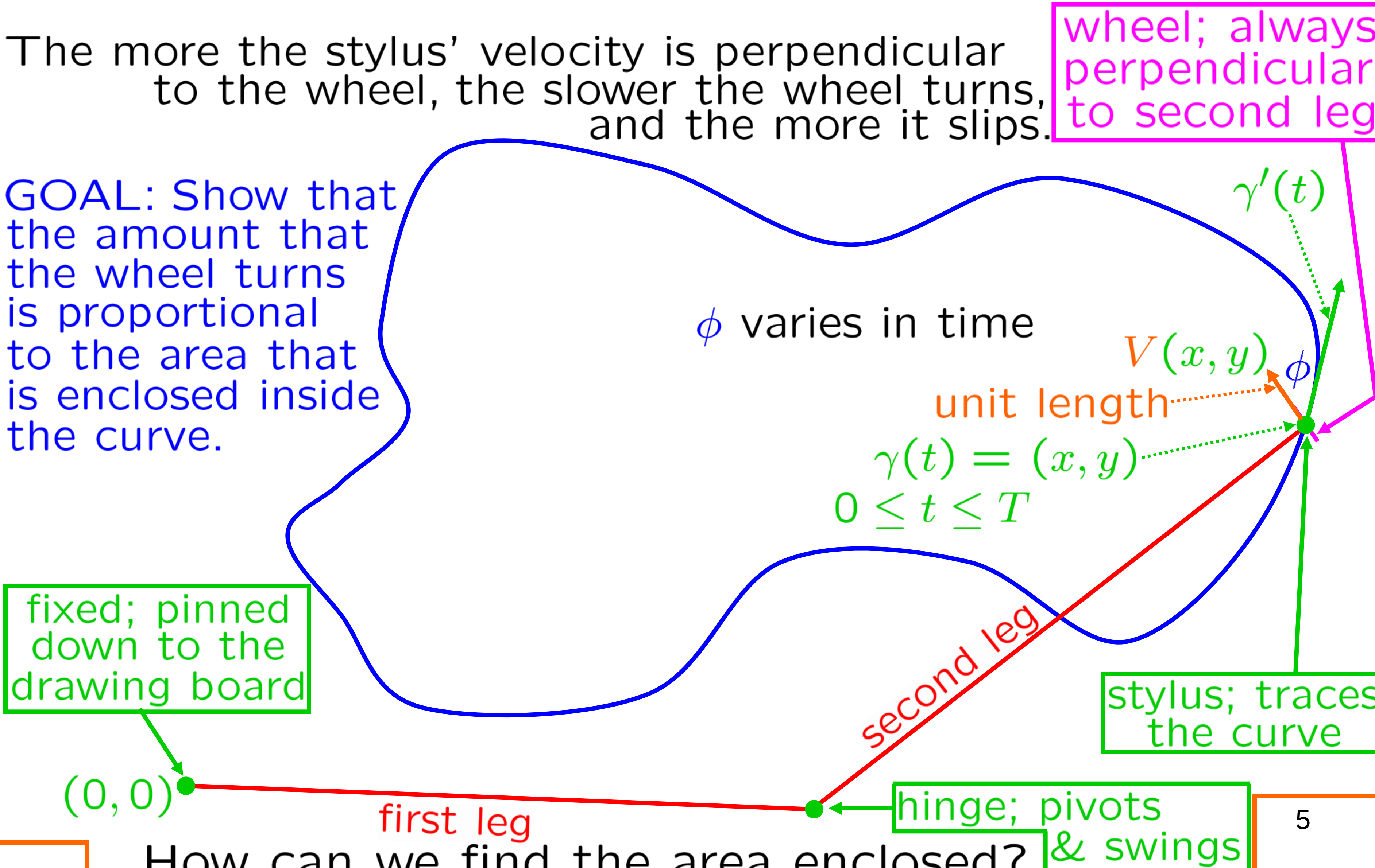


How can we find the area enclosed?

The faster we push the stylus,
the faster the wheel turns.

The more the stylus' velocity is perpendicular to the wheel, the slower the wheel turns, and the more it slips.

GOAL: Show that the amount that the wheel turns is proportional to the area that is enclosed inside the curve.



How can we find the area enclosed?

The faster we push the stylus,
the faster the wheel turns.

Rate of turning of wheel:
$[V(\gamma(t))] \cdot [\gamma'(t)]$

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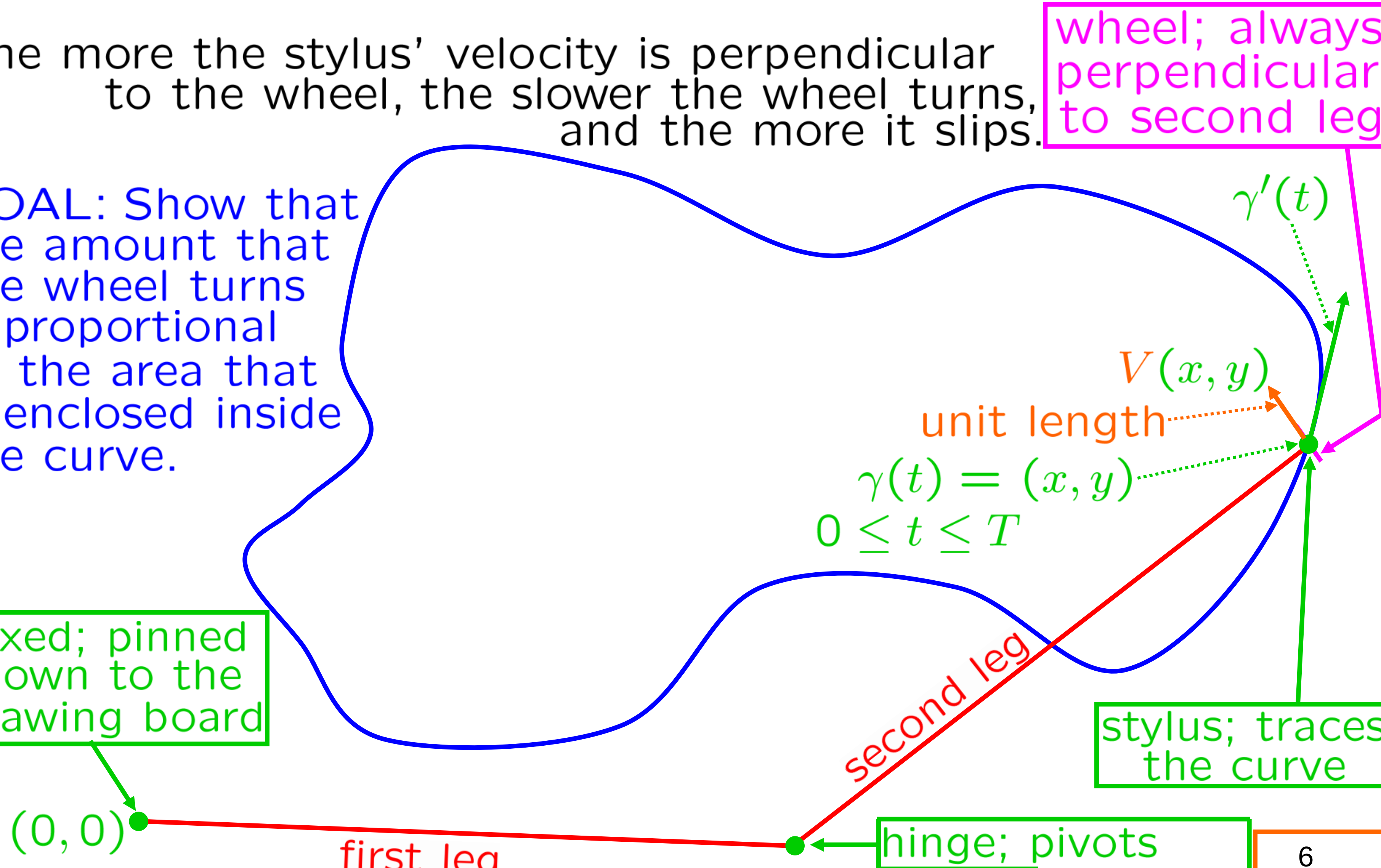


How can we find the area enclosed?

The faster we push the stylus, the faster the wheel turns.

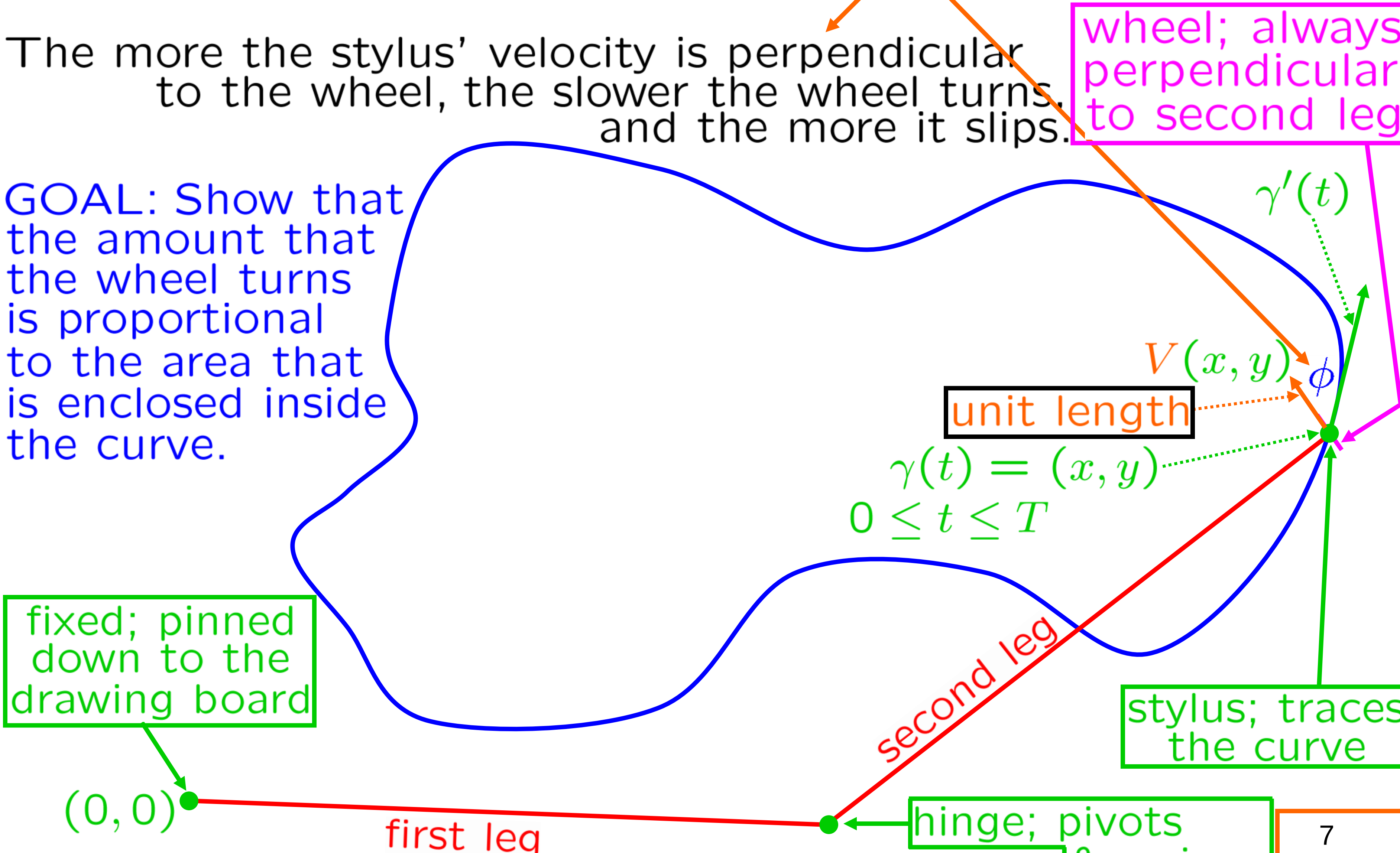
Why?

Rate of turning of wheel:

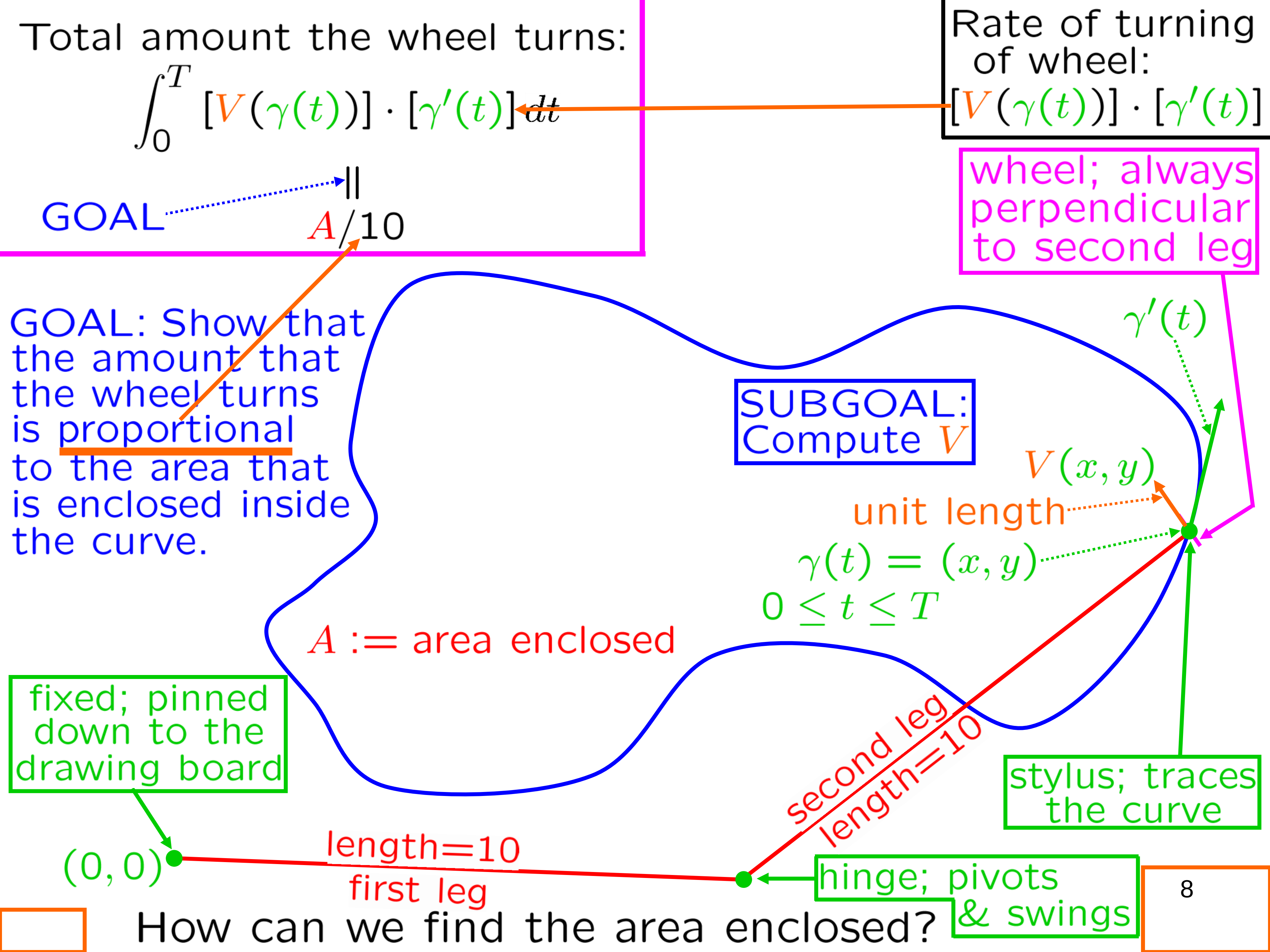
$$[|V(\gamma(t))|]\ [|\gamma'(t)|]\ [\cos\phi] = [V(\gamma(t))]\cdot[\gamma'(t)]$$

The more the stylus' velocity is perpendicular to the wheel, the slower the wheel turns, and the more it slips.

GOAL: Show that the amount that the wheel turns is proportional to the area that is enclosed inside the curve.



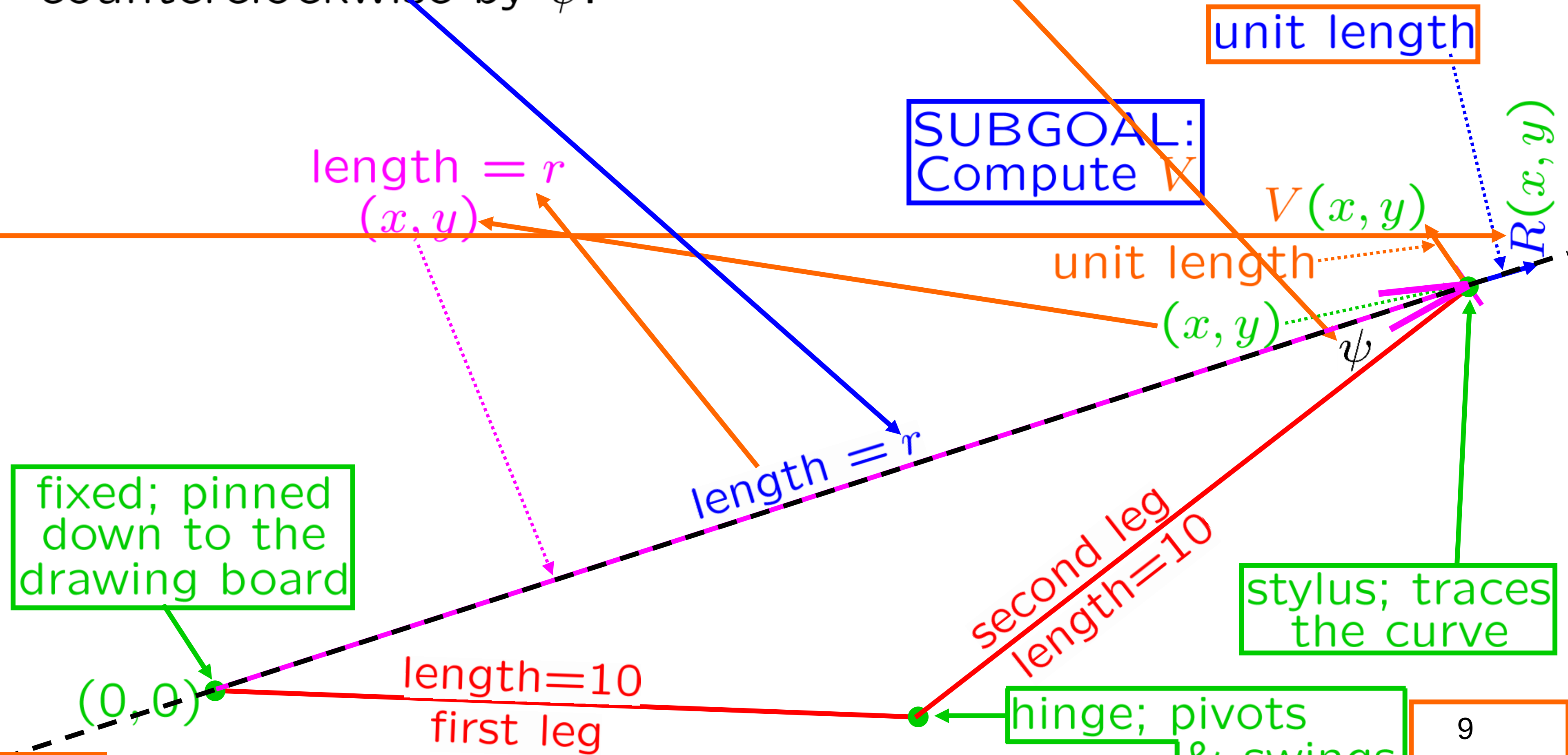
How can we find the area enclosed?


Total amount the wheel turns:
$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$
||
GOAL
$A/10$
Rate of turning of wheel:
$[V(\gamma(t))] \cdot [\gamma'(t)]$
wheel; always perpendicular to second leg
GOAL: Show that the amount that the wheel turns is proportional to the area that is enclosed inside the curve.
SUBGOAL: Compute V
$\gamma'(t)$
$V(x,y)$
unit length
$\gamma(t) = (x,y)$
$0 \leq t \leq T$
$A :=$ area enclosed
second leg
length=10
stylus; traces the curve
fixed; pinned down to the drawing board
(0,0)
length=10
first leg
hinge; pivots & swings
How can we find the area enclosed?

$r := \sqrt{x^2 + y^2}$

$R(x,y) = (x,y)/r$ $\qquad \begin{bmatrix} x \\ y \end{bmatrix} \frac{1}{r}$

Let $W(x,y)$ be the vector obtained by rotating $R(x,y)$ counterclockwise by ψ.



How can we find the area enclosed?

$r := \sqrt{x^2 + y^2}$

$R(x, y) = (x, y)/r$

$$\begin{bmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \frac{1}{r}$$

0   −1   1   0

Let $W(x, y)$ be the vector obtained by rotating $R(x, y)$ counterclockwise by ψ.

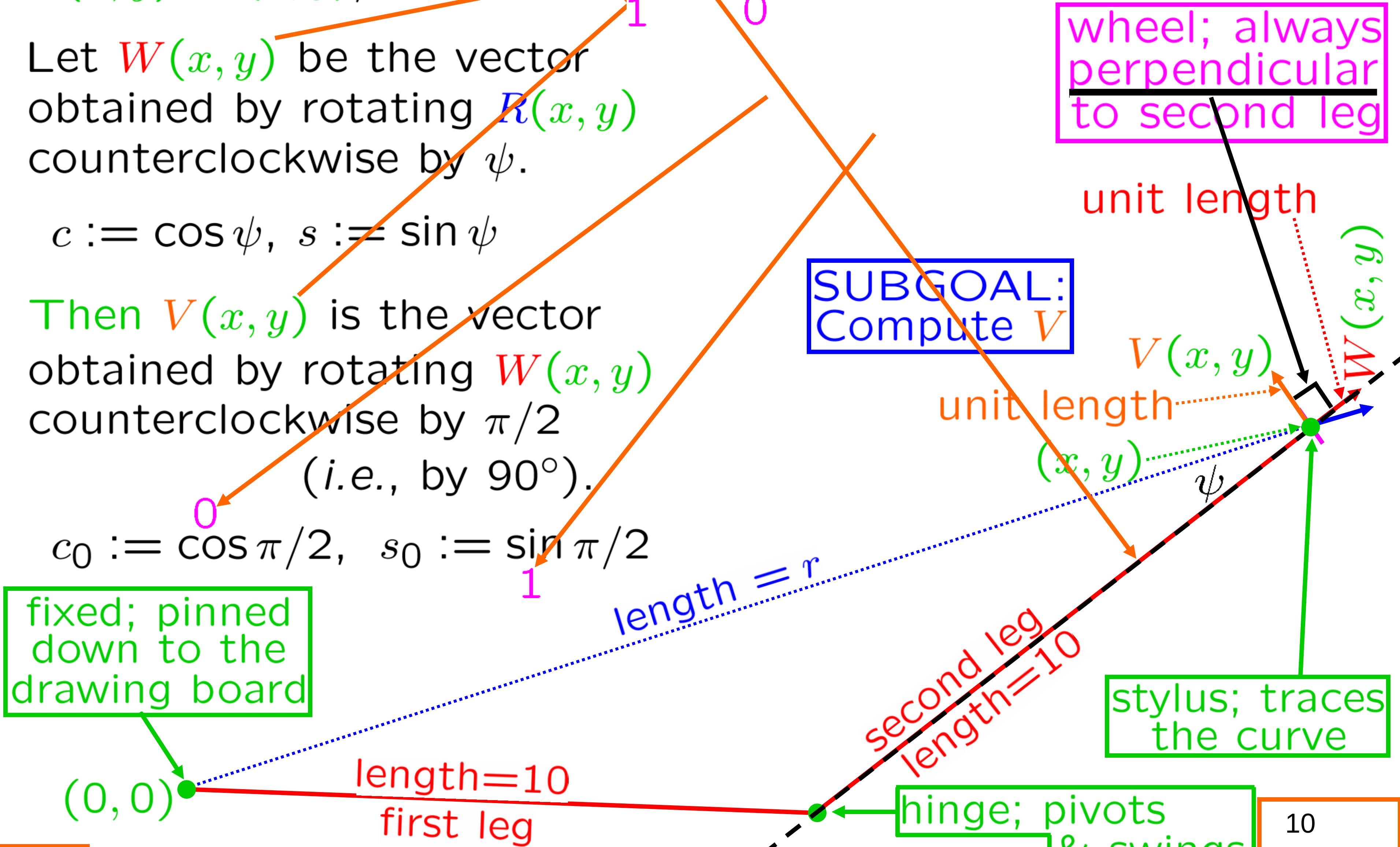
$c := \cos\psi$, $s := \sin\psi$

Then $V(x, y)$ is the vector obtained by rotating $W(x, y)$ counterclockwise by $\pi/2$ (*i.e.*, by 90°).

$c_0 := \cos\pi/2$, $s_0 := \sin\pi/2$

0   1

SUBGOAL: Compute V



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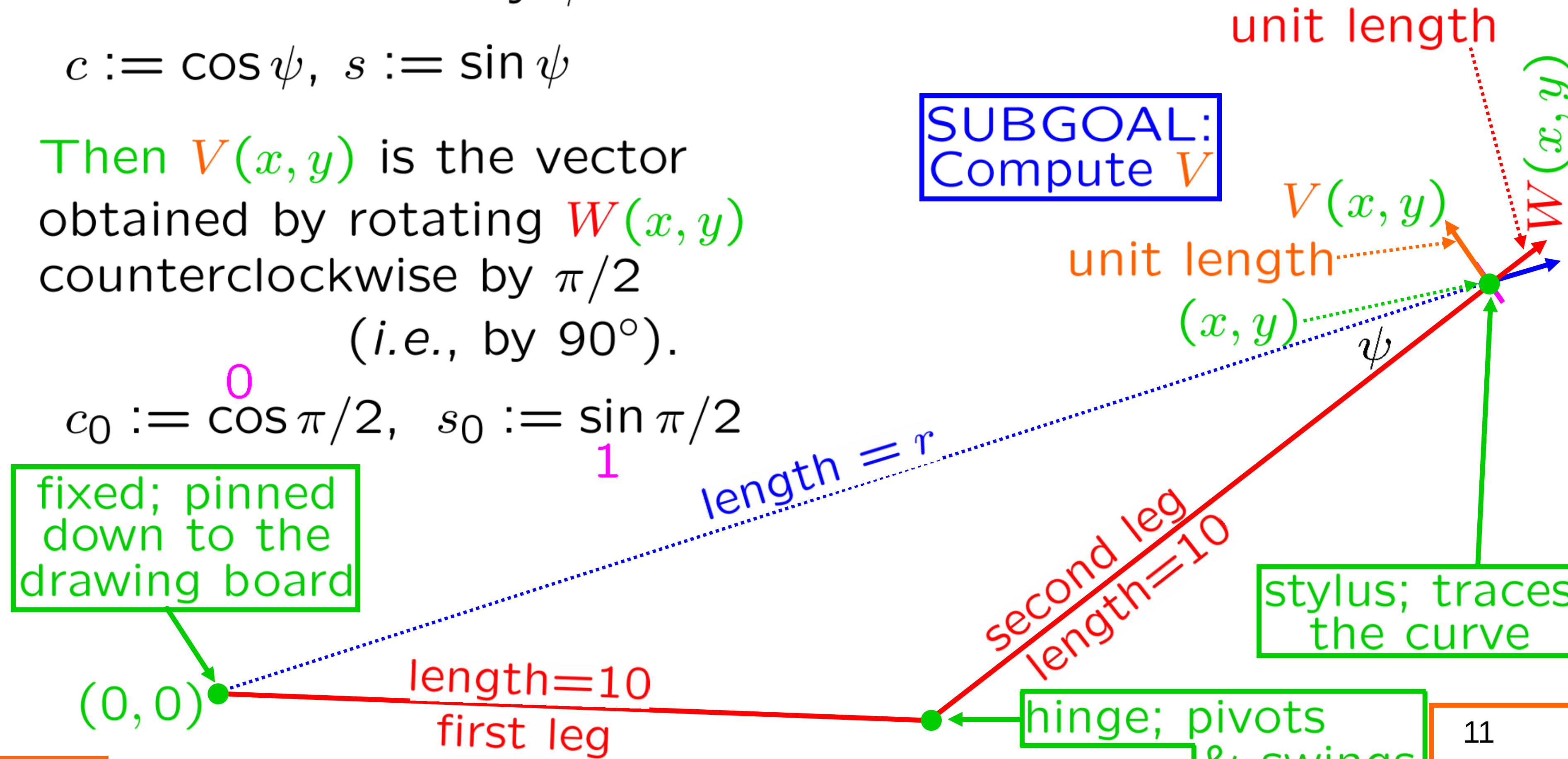
0 1

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \frac{1}{r} =$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cx - sy \\ sx + cy \end{bmatrix} \ldots$$

wheel; always perpendicular to second leg

SUBGOAL: Compute V



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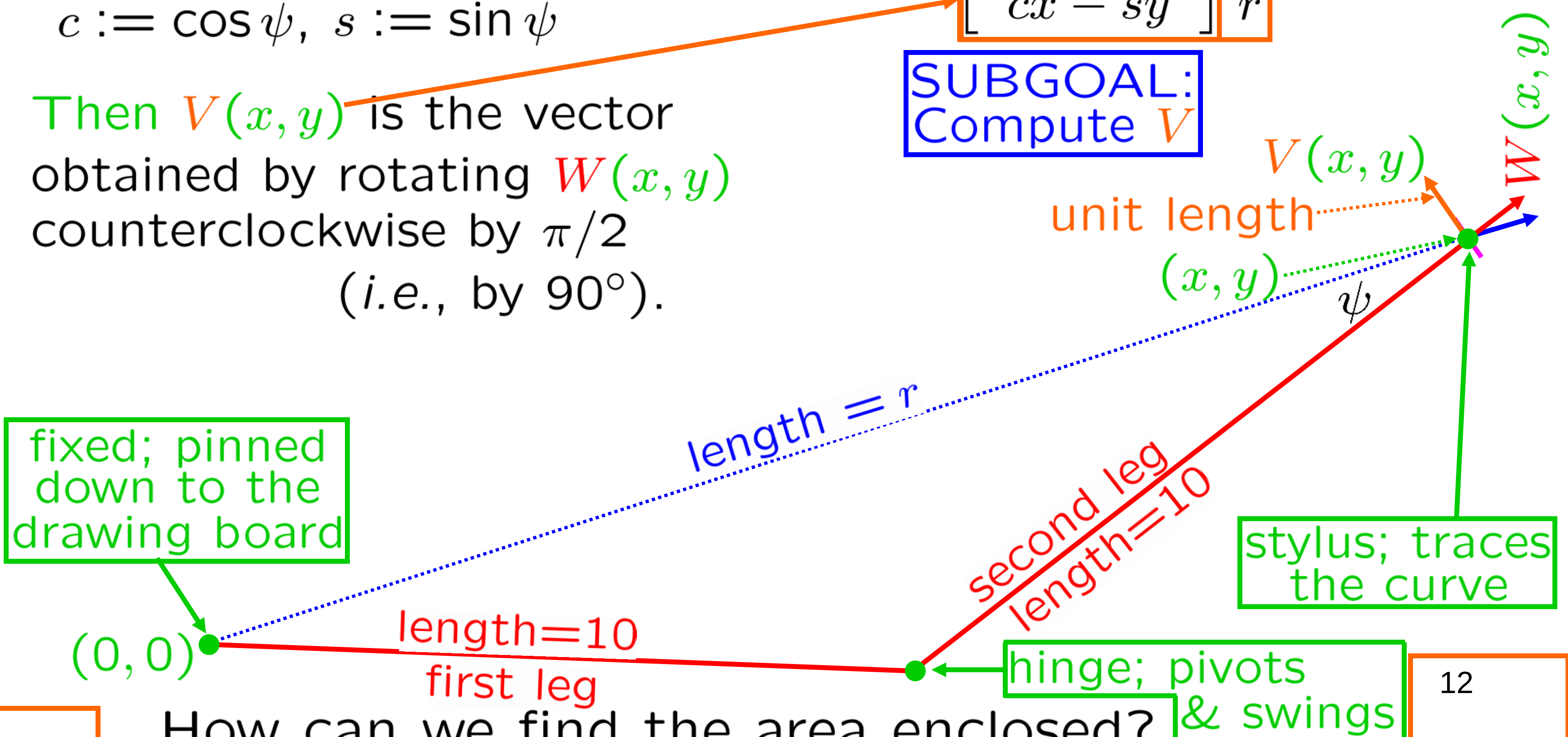
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cx - sy \\ sx + cy \end{bmatrix} \frac{1}{r} =$$

$$\begin{bmatrix} -sx - cy \\ cx - sy \end{bmatrix} \frac{1}{r}$$

SUBGOAL: Compute V



How can we find the area enclosed?

$$r := \sqrt{x^2 + y^2}$$

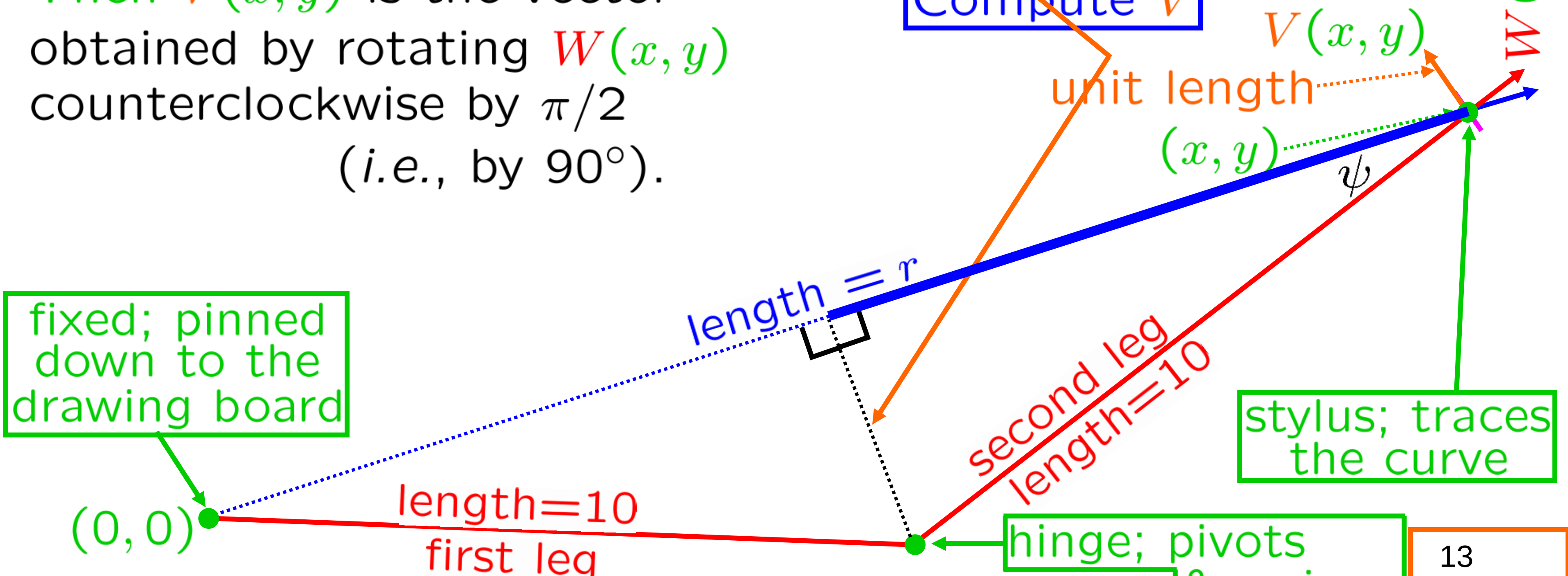
$$c := \cos\psi, \quad s := \sin\psi$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\frac{1}{r} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} cx - sy \\ sx + cy \end{bmatrix}\frac{1}{r} = \begin{bmatrix} -sx - cy \\ cx - sy \end{bmatrix}\frac{1}{r}$$

$$c := V(x,y) = (-sx - cy, cx - sy)/r$$

Then $V(x,y)$ is the vector obtained by rotating $W(x,y)$ counterclockwise by $\pi/2$ (*i.e.*, by 90°).

SUBGOAL: Compute V



How can we find the area enclosed?

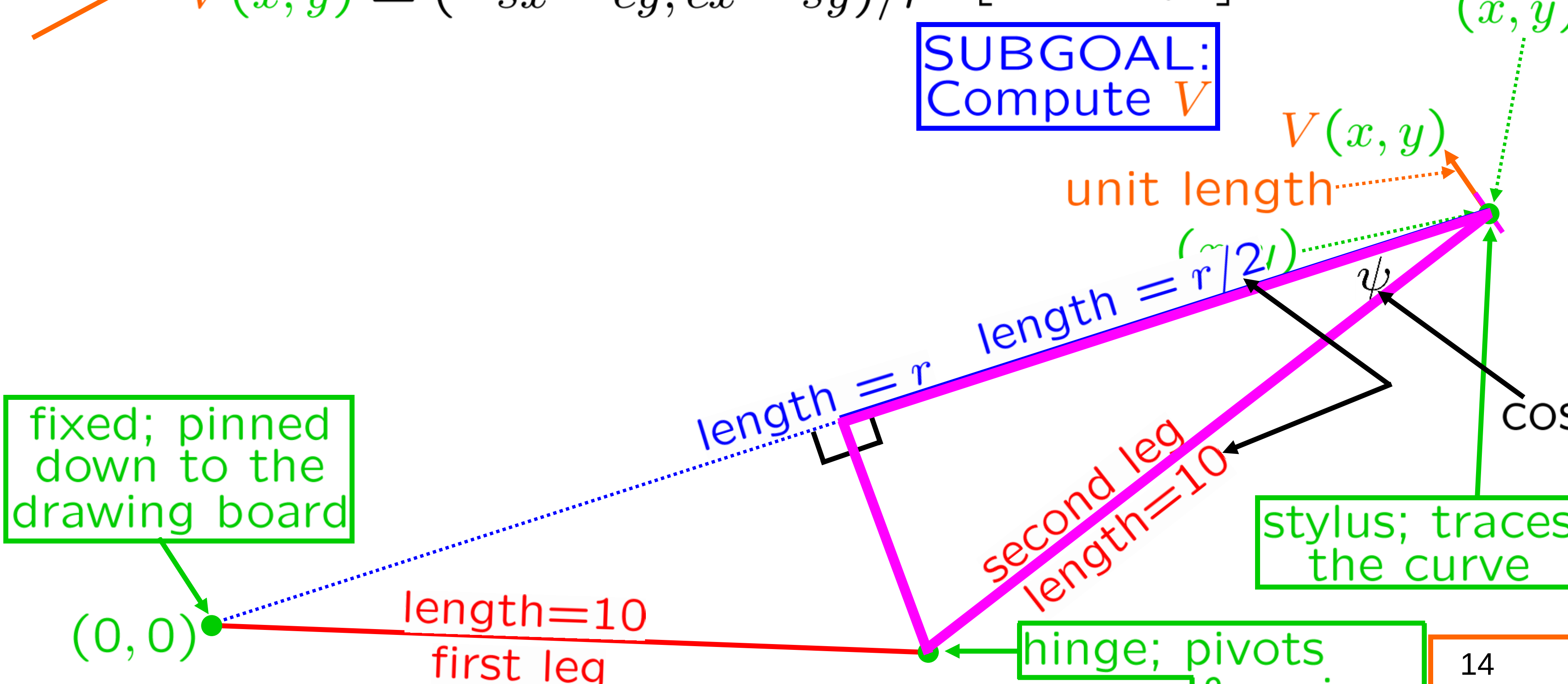
$$r := \sqrt{x^2 + y^2}$$

$$c := \boxed{\cos\psi}, \quad s := \boxed{\sin\psi}$$

$$\cos\psi = \frac{r/2}{10}, \qquad \sin\psi = \pm\sqrt{1 - c^2}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \frac{1}{r} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cx - sy \\ sx + cy \end{bmatrix} \frac{1}{r} = \begin{bmatrix} -sx - cy \\ cx - sy \end{bmatrix} \frac{1}{r}$$

$$V(x, y) = (-sx - cy, cx - sy)/r$$



How can we find the area enclosed?

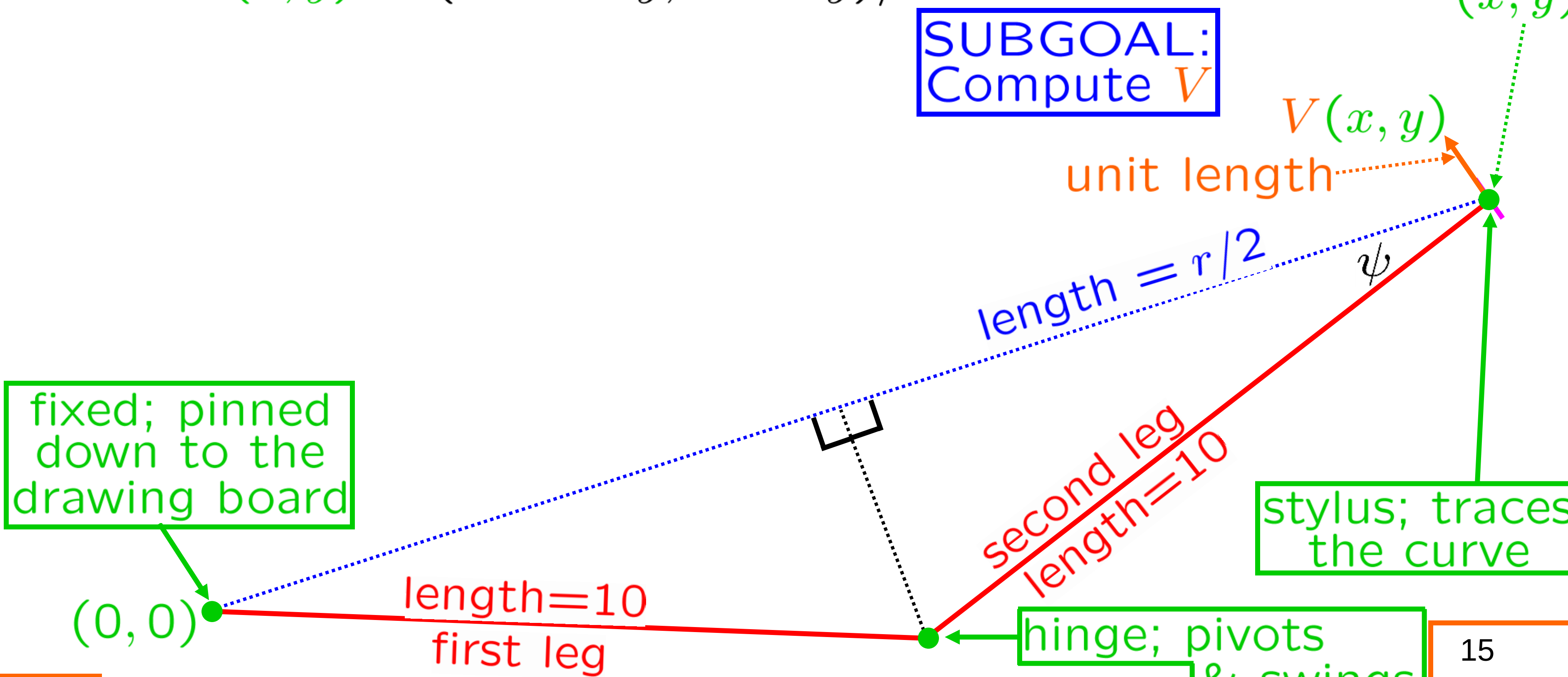
$r := \sqrt{x^2 + y^2}$

$c := \cos\psi, \; s := \sin\psi$

$c = \dfrac{r/2}{10}$ → $c = r/20,$

$s = \sqrt{1 - c^2}$ → $s = \sqrt{1 - c^2}$

$$V(x, y) = (-sx - cy, cx - sy)/r$$



How can we find the area enclosed?

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20, \quad s = \sqrt{1 - c^2}$$

$$V(x, y) = (-sx - cy, cx - sy)/r$$

$$r := \sqrt{x^2 + y^2}$$

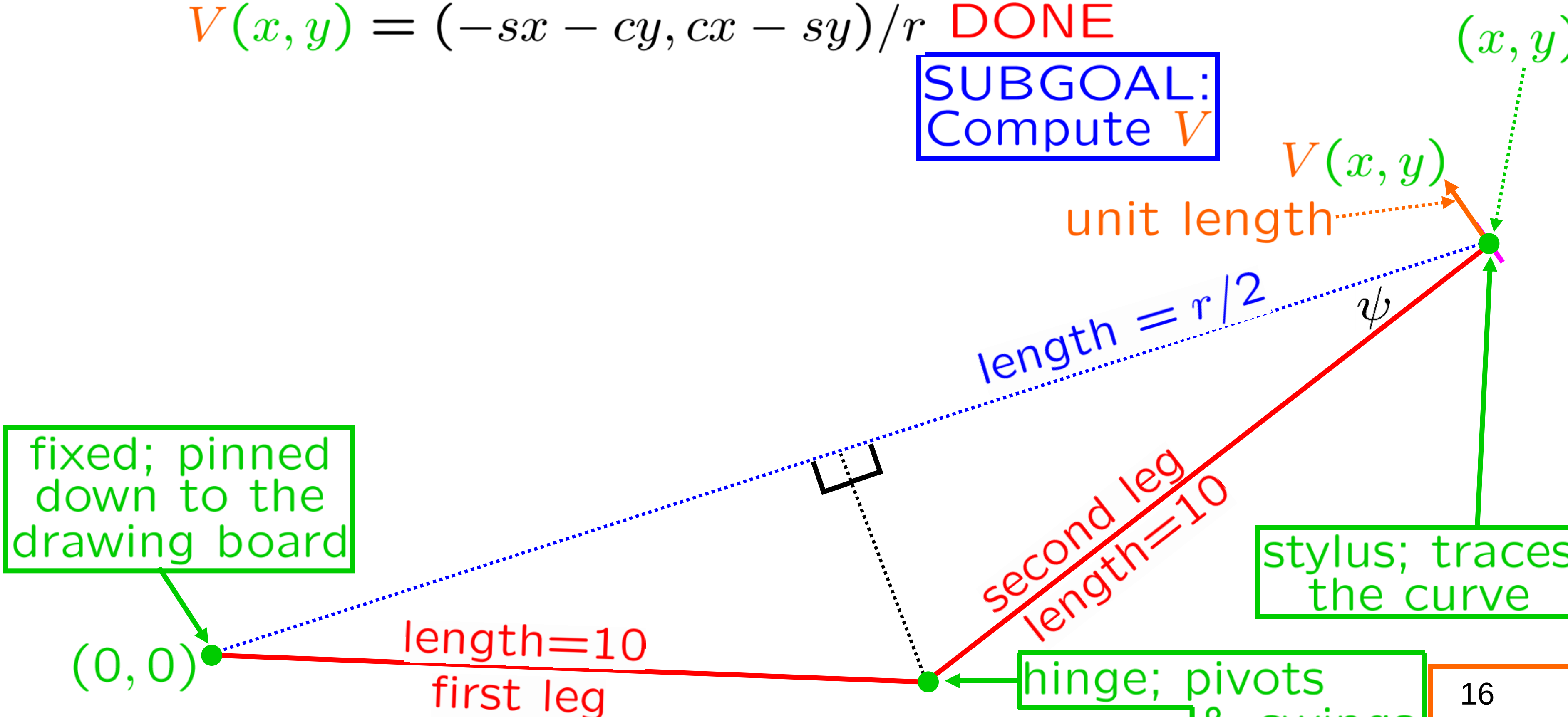
$$c = r/20 = [\sqrt{x^2 + y^2}]/[20]$$

$$s = \sqrt{1 - c^2}$$

$$= \sqrt{1 - [(x^2 + y^2)/(400)]}$$

$$V(x, y) = (-sx - cy, cx - sy)/r$$ DONE

SUBGOAL: Compute V



How can we find the area enclosed?

Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

GOAL $= A/10$

$$V(x,y) = (-sx - cy, cx - sy)/r$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

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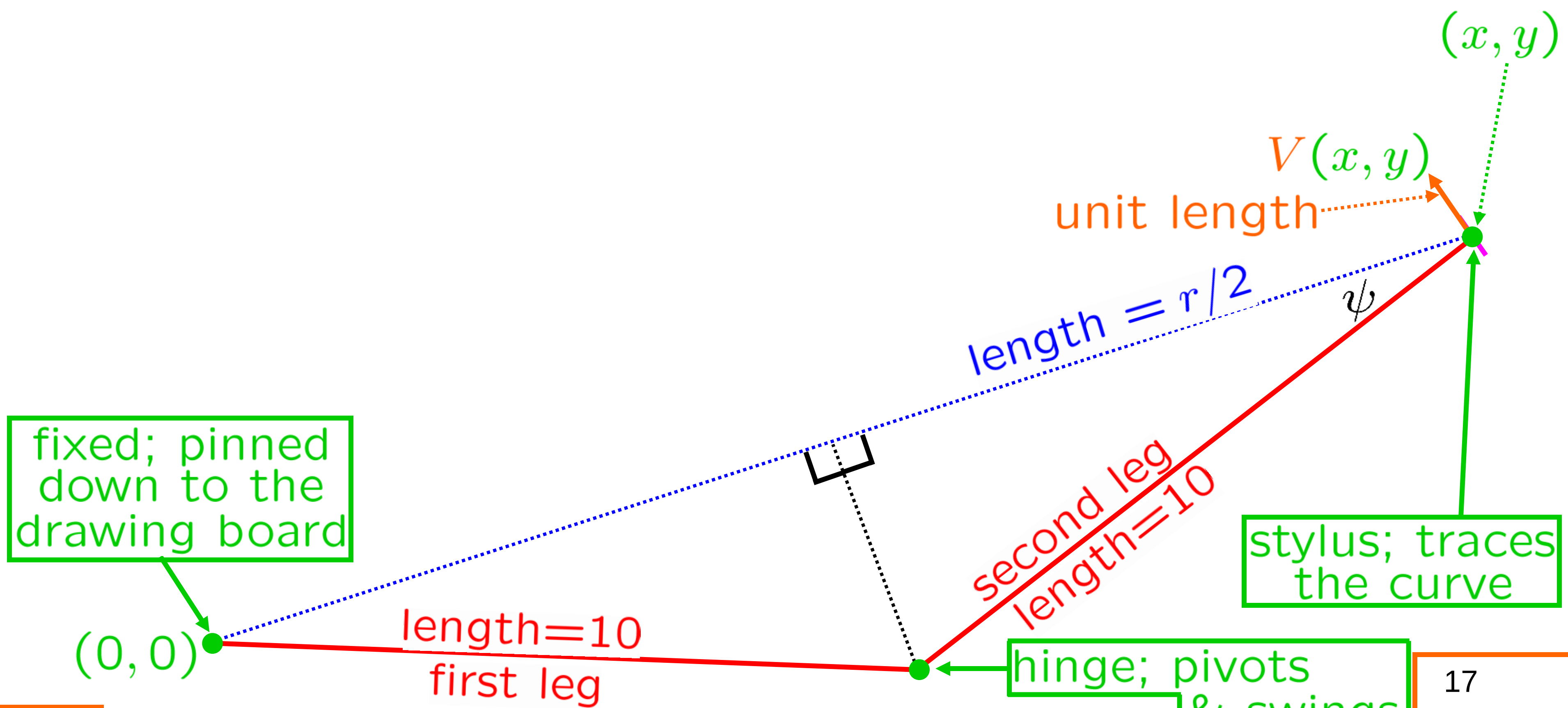

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GOAL $\cdots\to$ $\|$

$A/10$

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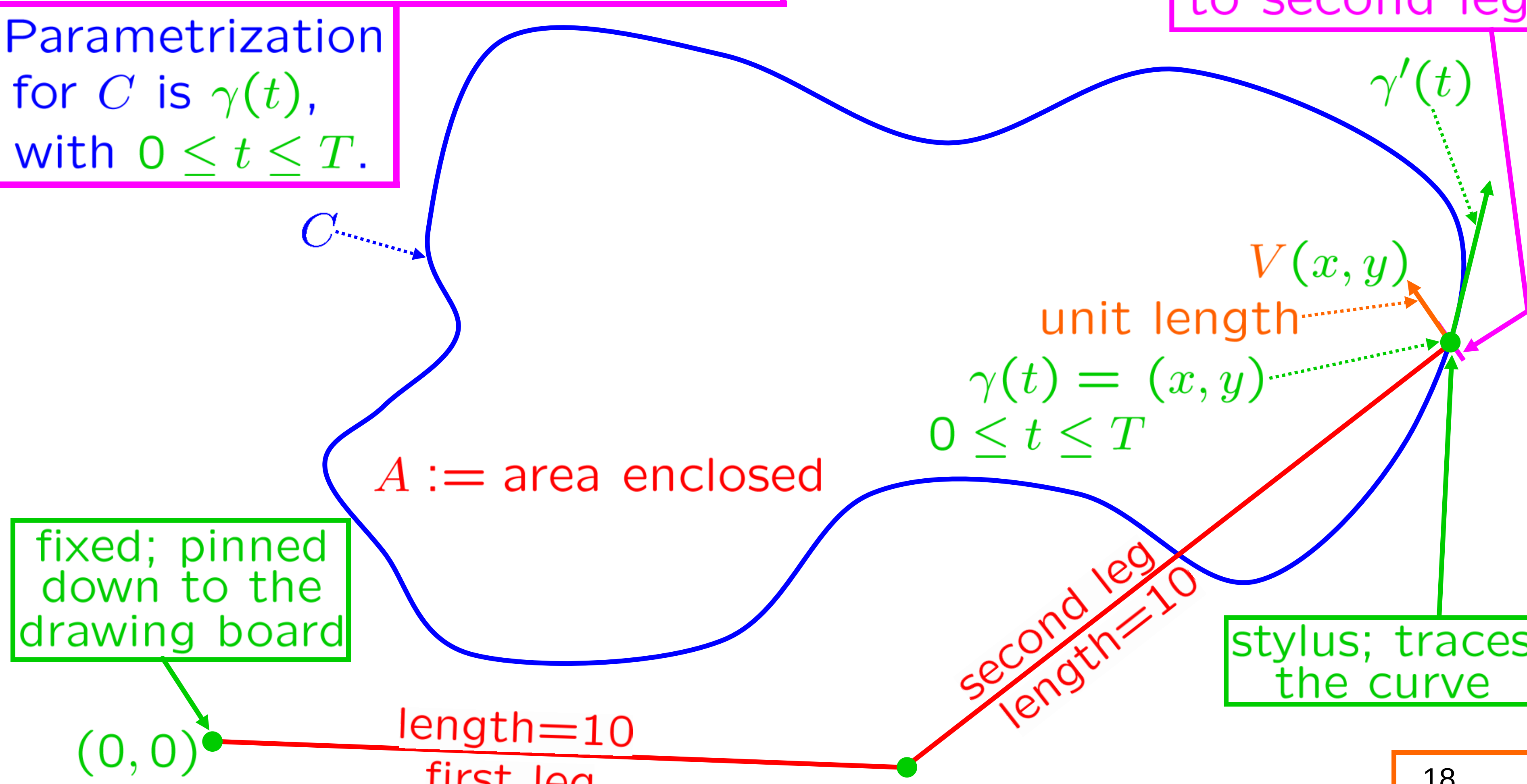

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GOAL $\Rightarrow$ $\|$

$A/10$

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Parametrization for C is $\gamma(t)$, with $0 \leq t \leq T$.

Choose $\alpha, \beta : [0,T] \to \mathbb{R}$ s.t. $\gamma(t) = (\alpha(t), \beta(t))$.

$$p(x,y) := \frac{-sx - cy}{r}$$

$$q(x,y) := \frac{cx - sy}{r}$$

Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

GOAL $\;\|\; A/10$

$V(x,y) = (\ p(x,y)\ ,\ q(x,y)\ )$

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Parametrization for C is $\gamma(t)$, with $0 \le t \le T$.

Choose $\alpha, \beta : [0,T] \to \mathbb{R}$ s.t. $\gamma(t) = (\alpha(t), \beta(t))$.

$$\omega := (p(x,y))\, dx + (q(x,y))\, dy$$

$$p(x,y) := \frac{-sx - cy}{r}$$

$$q(x,y) := \frac{cx - sy}{r}$$

equal?

$$\int_C \omega = \int_C (p(x,y))\, dx + (q(x,y))\, dy$$

$$= \int_C (p(x,y))\, dx + \int_C (q(x,y))\, dy$$

$x = \alpha(t)$

$dx = (\alpha'(t))\, dt$

$y = \beta(t)$

$dy = (\beta'(t))\, dt$

$$= \int_0^T (p(\alpha(t), \beta(t)))\, (\alpha'(t))\, dt + \int_0^T (q(\alpha(t), \beta(t)))\, (\beta'(t))\, dt$$

Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

GOAL → $\| A/10$

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$$\int_C \omega = \int_0^T (p(\alpha(t),\beta(t)))\,(\alpha'(t))\, dt + \int_0^T (q(\alpha(t),\beta(t)))\,(\beta'(t))\, dt$$

$$p(x,y) := \frac{-sx - cy}{r}$$

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Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

GOAL $\rightarrow$ $\|$

$A/10$

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$$\int_C \omega = \int_0^T (p(\alpha(t), \beta(t)))\, (\alpha'(t))\, dt + \int_0^T (q(\alpha(t), \beta(t)))\, (\beta'(t))\, dt$$

$$(\ p(\alpha(t), \beta(t))\ ,\ q(\alpha(t), \beta(t))\ ) \cdot (\ \alpha'(t)\ ,\ \beta'(t)\ )$$

Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

$$\|$$

GOAL $\to$ $A/10$

$V(x,y) = (\ p(x,y)\ ,\ q(x,y)\ )$

$$r := \sqrt{x^2+y^2}$$
$$c = r/20$$
$$s = \sqrt{1-c^2}$$

Parametrization for C is $\gamma(t)$, with $0 \leq t \leq T$.

Choose $\alpha, \beta : [0,T] \to \mathbb{R}$ s.t. $\gamma(t) = (\alpha(t), \beta(t))$.

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$$p(x,y) := \frac{-sx-cy}{r}$$

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$$\int_C \omega = \int_0^T (p(\alpha(t),\beta(t)))\,(\alpha'(t))\, dt + \int_0^T (q(\alpha(t),\beta(t)))\,(\beta'(t))\, dt$$

$$= \int_0^T (\ p(\alpha(t),\beta(t))\ ,\ q(\alpha(t),\beta(t))\ ) \cdot (\ \alpha'(t)\ ,\ \beta'(t)\ )\, dt$$

Total amount the wheel turns:

$$\int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

$$\|$$

GOAL $A/10$

$$V(x,y) = (\ p(x,y)\ ,\ q(x,y)\ )$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

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Choose $\alpha, \beta : [0, T] \to \mathbb{R}$ s.t. $\gamma(t) = (\alpha(t), \beta(t))$.

$$\omega := (p(x,y))\, dx + (q(x,y))\, dy$$

$$p(x,y) := \frac{-sx - cy}{r}$$

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$$\int_C \omega = \int_0^T (p(\alpha(t), \beta(t)))\, (\alpha'(t))\, dt + \int_0^T (q(\alpha(t), \beta(t)))\, (\beta'(t))\, dt$$

$$= \int_0^T (\ V(\ \alpha(t)\ ,\ \beta(t)\ )\ ) \cdot (\ \alpha'(t)\ ,\ \beta'(t)\ )\ dt$$

$$= \int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\, dt$$

Total amount the wheel turns:

$$\int_C \omega$$

$\|$

GOAL $A/10$

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$$\int_C \omega = \int_0^T (p(\alpha(t), \beta(t)))\,(\alpha'(t))\,dt + \int_0^T (q(\alpha(t), \beta(t)))\,(\beta'(t))\,dt$$

$$= \int_0^T (\ V\ (\ \alpha(t)\ ,\ \beta(t)\ )\ ) \cdot (\ \alpha'(t)\ ,\ \beta'(t)\ )\ dt$$

$$= \int_0^T [V(\gamma(t))] \cdot [\gamma'(t)]\,dt$$

Total amount the wheel turns:

$$\int_C \omega$$

GOAL → $\|$

$$A/10$$

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$$s = \sqrt{1 - c^2}$$

Parametrization for C is $\gamma(t)$, with $0 \leq t \leq T$.

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$\omega := (p(x,y))\, dx + (q(x,y))\, dy$$

$$p(x,y) := \frac{-sx - cy}{r}$$

$$q(x,y) := \frac{cx - sy}{r}$$

Total amount the wheel turns:

$$\int_C \omega = \int_R d\omega$$

$$\int_C \omega = A/10$$

GOAL

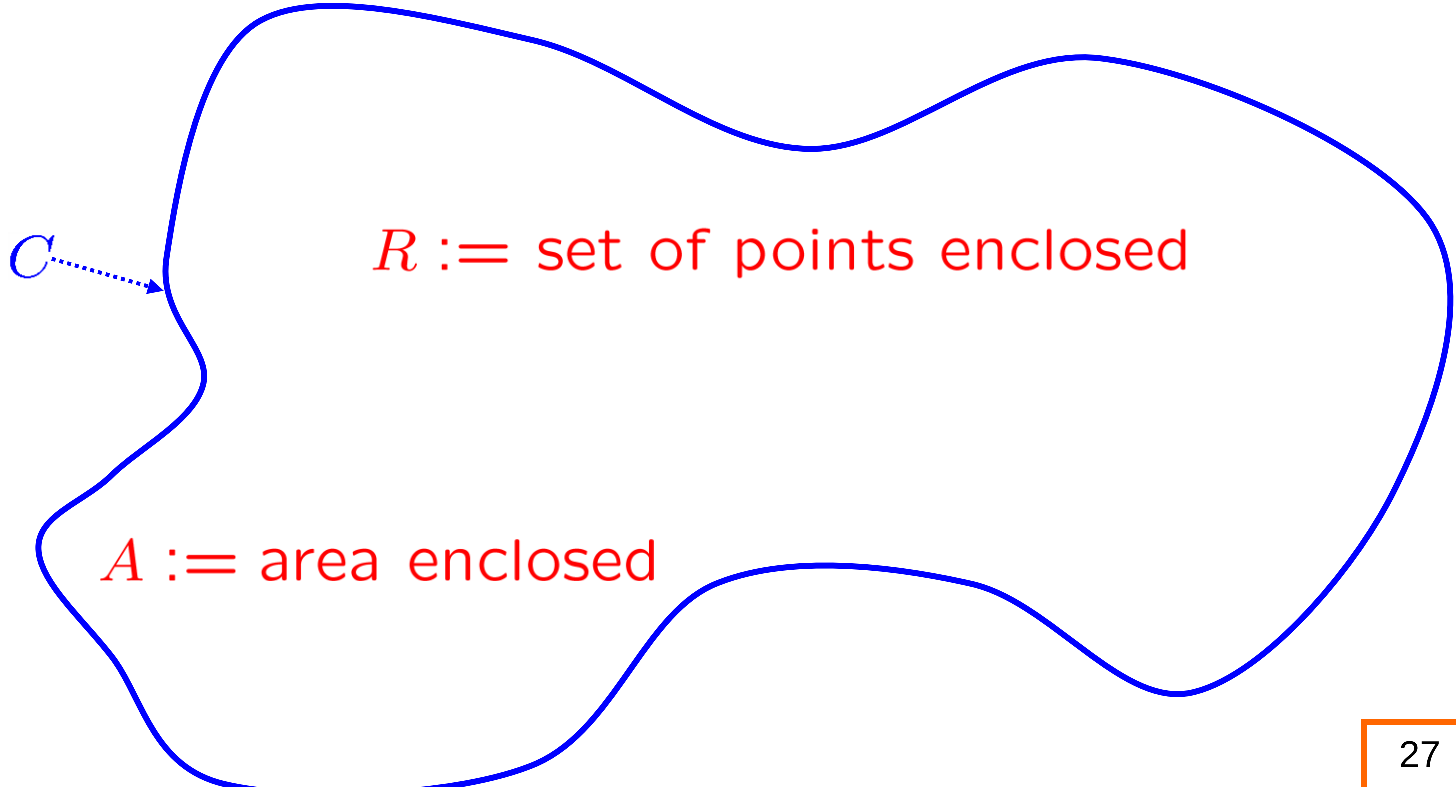
STOKES' THEOREM

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$



Total amount the wheel turns:

$$\int_R d\omega \qquad \int_R d\omega$$

$$\| $$

GOAL → $A/10$

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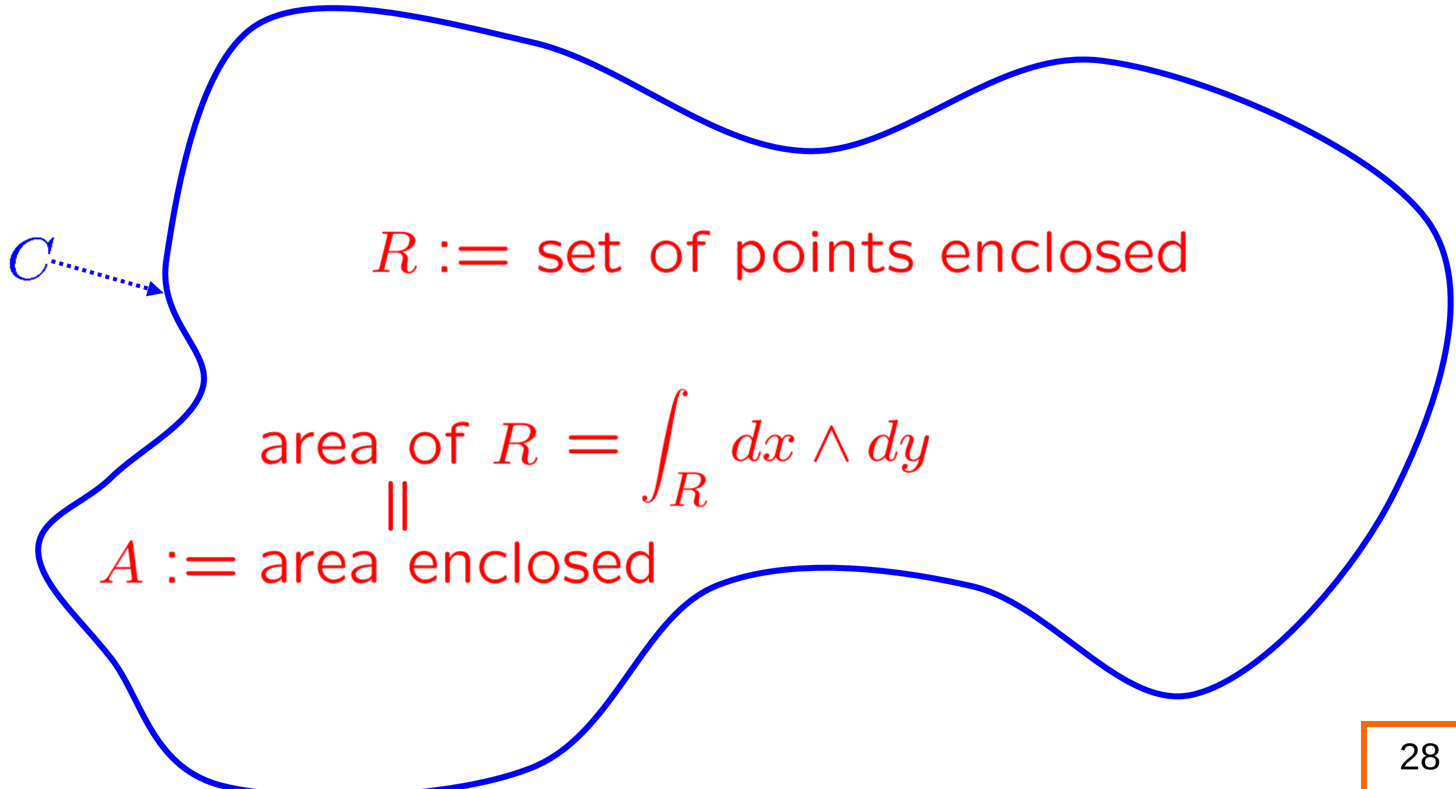
$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$



Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL → $A/10$

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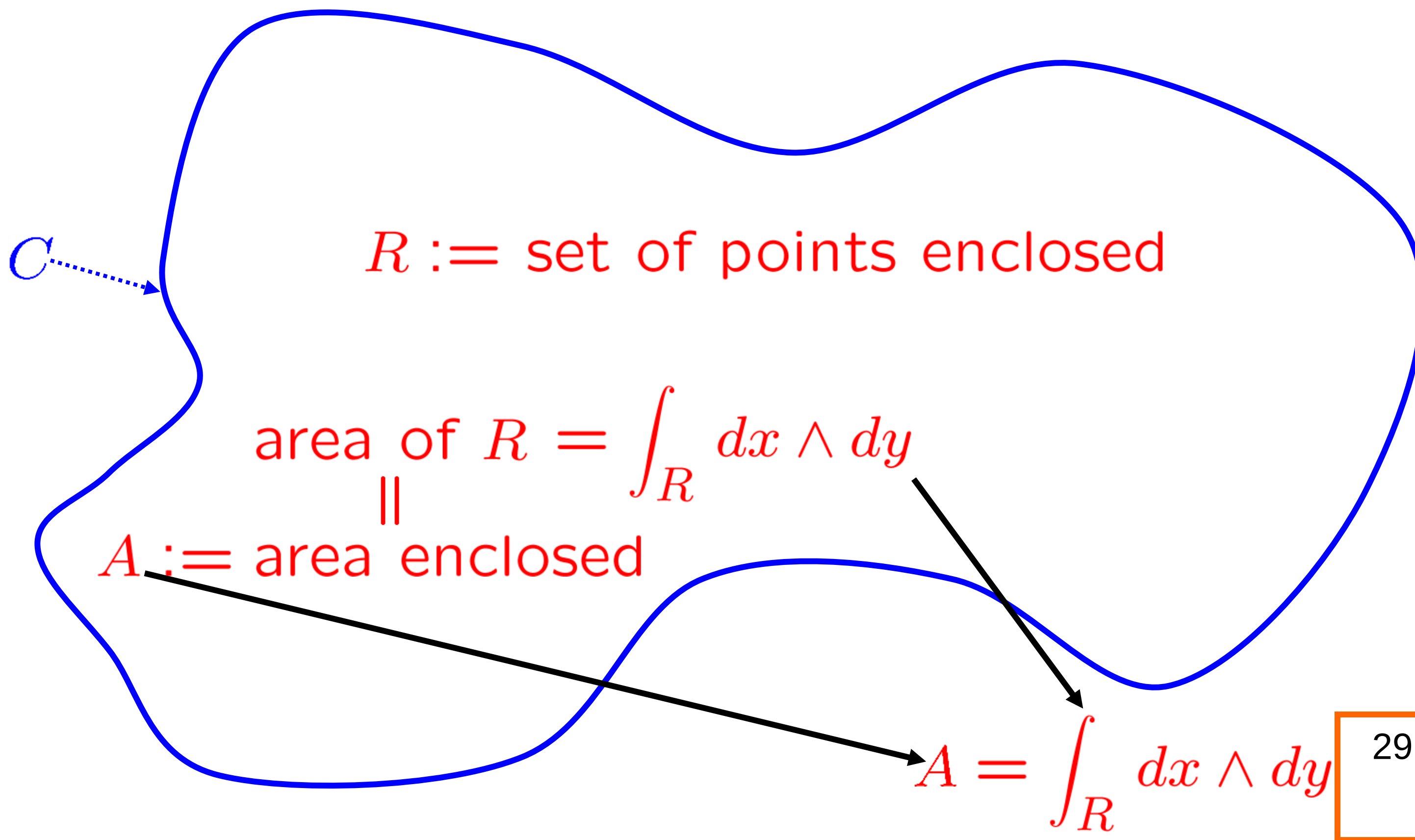

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left[\frac{\partial}{\partial y}\left(\frac{-sx - cy}{r}\right)\right] [dy \wedge dx] + \left[\frac{\partial}{\partial x}\left(\frac{cx - sy}{r}\right)\right] [dx \wedge dy]$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx + cy}{r}\right)\right] + \left[\frac{\partial}{\partial x}\left(\frac{cx - sy}{r}\right)\right]\right) (dx \wedge dy)$$

$$\frac{c}{r} = \frac{1}{20}$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r} + \frac{y}{20}\right)\right] + \left[\frac{\partial}{\partial x}\left(\frac{x}{20} - \frac{sy}{r}\right)\right]\right) (dx \wedge dy)$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] + \frac{1}{20} + \frac{1}{20} - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right]\right) (dx \wedge dy)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2 + y^2}$$
$$c = r/20$$
$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] + \frac{1}{20} + \frac{1}{20} - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right]\right)(dx \wedge dy)$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] + \frac{1}{20} + \frac{1}{20} - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right]\right)(dx \wedge dy)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] + \frac{1}{20} + \frac{1}{20} - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right]\right)(dx \wedge dy)$$

$$= \left(\left[\frac{\partial}{\partial y}\left(\frac{sx}{r}\right)\right] - \left[\frac{\partial}{\partial x}\left(\frac{sy}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[x\frac{\partial}{\partial y}\left(\frac{s}{r}\right)\right] - \left[y\frac{\partial}{\partial x}\left(\frac{s}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$\frac{s}{r} = \frac{\sqrt{1 - c^2}}{r} = \frac{\sqrt{1 - (r^2/(400))}}{\sqrt{r^2}} = \sqrt{\frac{1}{r^2} - \frac{1}{400}} = F(r^2)$$

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}} \qquad \frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2+y^2}$$
$$c = r/20$$
$$s = \sqrt{1-c^2}$$

$$\omega = \left(\frac{-sx-cy}{r}\right)dx + \left(\frac{cx-sy}{r}\right)dy$$

$$d\omega = \left(\left[x\frac{\partial}{\partial y}\left(\frac{s}{r}\right)\right] - \left[y\frac{\partial}{\partial x}\left(\frac{s}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[x\frac{\partial}{\partial y}\left(\frac{s}{r}\right)\right] - \left[y\frac{\partial}{\partial x}\left(\frac{s}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}}$$

$$\frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL → $A/10$

$$r := \sqrt{x^2 + y^2}$$
$$c = r/20$$
$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left(\left[x \frac{\partial}{\partial y}\left(\frac{s}{r}\right)\right] - \left[y \frac{\partial}{\partial x}\left(\frac{s}{r}\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[x \frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y \frac{\partial}{\partial x}\left(F(r^2)\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

Chain Rule

Chain Rule

$$\left[x[F'(r^2)]\left[\frac{\partial}{\partial y}\left(r^2\right)\right]\right] - \left[y[F'(r^2)]\left[\frac{\partial}{\partial x}\left(r^2\right)\right]\right]$$

$$= \left[x[F'(r^2)]\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right] - \left[y[F'(r^2)]\frac{\partial}{\partial x}\left(x^2 + y^2\right)\right]$$

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}}$$

$$\frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL ⟶ $A/10$

$$r := \sqrt{x^2+y^2}$$

$$c = r/20$$

$$s = \sqrt{1-c^2}$$

$$\omega = \left(\frac{-sx-cy}{r}\right) dx + \left(\frac{cx-sy}{r}\right) dy$$

$$d\omega = \left(\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left(\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$= \left[x[F'(r^2)]\,(2y)\right] - \left[y[F'(r^2)]\,(2x)\right]$$

$$\left[x[F'(r^2)]\frac{\partial}{\partial y}\left(x^2+y^2\right)\right] - \left[y[F'(r^2)]\frac{\partial}{\partial x}\left(x^2+y^2\right)\right]$$

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}}$$

$$\frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left(\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right] + \frac{1}{10}\right)(dx \wedge dy)$$

$$\left[x[F'(r^2)]\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right] - \left[y[F'(r^2)]\frac{\partial}{\partial x}\left(x^2 + y^2\right)\right]$$

$$= \left[x[F'(r^2)]\,(2y)\right] - \left[y[F'(r^2)]\,(2x)\right]$$

$$= \left[[F'(r^2)]\,(2xy)\right] - \left[[F'(r^2)]\,(2xy)\right]$$

$$= 0$$

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}}$$

$$\frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega$$

$$\|$$

GOAL

$$A/10$$

$$r := \sqrt{x^2 + y^2}$$

$$c = r/20$$

$$s = \sqrt{1 - c^2}$$

$$\omega = \left(\frac{-sx - cy}{r}\right) dx + \left(\frac{cx - sy}{r}\right) dy$$

$$d\omega = \left(\underbrace{\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right]}_{0} + \frac{1}{10}\right)(dx \wedge dy)$$

$$d\omega = \frac{1}{10}(dx \wedge dy)$$

0

$$F(q) := \sqrt{\frac{1}{q} - \frac{1}{400}}$$

$$\frac{s}{r} = F(r^2)$$

$$A = \int_R dx \wedge dy$$

Total amount the wheel turns:

$$\int_R d\omega \;=\; A/10$$

GOAL

STOP

$$r := \sqrt{x^2+y^2}$$
$$c = r/20$$
$$s = \sqrt{1-c^2}$$

$$\omega = \left(\frac{-sx-cy}{r}\right)dx + \left(\frac{cx-sy}{r}\right)dy$$

$$d\omega = \left(\underbrace{\left[x\frac{\partial}{\partial y}\left(F(r^2)\right)\right] - \left[y\frac{\partial}{\partial x}\left(F(r^2)\right)\right]}_{0} + \frac{1}{10}\right)(dx\wedge dy)$$

$$d\omega = \frac{1}{10}(dx \wedge dy)$$

$$\int_R d\omega = \frac{1}{10}\int_R dx \wedge dy = \frac{1}{10}A$$

$$A = \int_R dx \wedge dy$$