Financial Mathematics

Problems in integration

Definition:

$x :\longrightarrow -2$

$$\Phi(x) = \frac{1}{\boxed{\sqrt{2\pi}}} \int_{-\infty}^{x} e^{-t^2/2} \, dt$$

NOT when x is upper limit

SNCDF

$t :\longrightarrow -t$

$$\boxed{\Phi(-2)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2} e^{-t^2/2} \, dt$$

$dt :\longrightarrow (-1)dt$

MULTIPLY BY $\sqrt{2\pi}$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{2} e^{-t^2/2} \boxed{(-1)} \, dt$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-t^2/2} \, dt}$$

$$\int_{2}^{\infty} e^{-t^2/2} \, dt = \sqrt{2\pi}[\Phi(-2)]$$

$2 :\longrightarrow x$

do not forget $\sqrt{2\pi}$

$$\int_{x}^{\infty} e^{-t^2/2} \, dt = \boxed{\sqrt{2\pi}}[\Phi(-x)]$$

minus needed when x is lower limit

2

Problem:

$$\frac{1}{\sqrt{2\pi}}\int_3^\infty e^{4x}e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{3-4}^\infty e^{4^2/2}e^{-x^2/2}\,dx$$

$$= \frac{e^8}{\sqrt{2\pi}}\int_{-1}^\infty e^{-x^2/2}\,dx$$

$$= e^8[\Phi(1)] \blacksquare$$

$$[4x - (x^2/2)]_{x:\to x+4} = 4(x+4) - ((x+4)^2/2)$$

$$= -(x^2/2) + (4^2/2)$$

Problem:

$$\frac{1}{\sqrt{2\pi}}\int_3^\infty (e^{4x} - 7)e^{-x^2/2}\,dx$$

$$= \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{3-4}^\infty e^{\overset{4^2/2}{\cancel{4x}}} e^{-x^2/2}\,dx\right]}_{e^8[\Phi(1)]} - \underbrace{\left[\frac{7}{\sqrt{2\pi}}\int_3^\infty e^{-x^2/2}\,dx\right]}_{7[\Phi(-3)]}$$

Problem: $\displaystyle \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{4x}-7)_+ e^{-x^2/2}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(e^{4x}-7)_+ e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(e^{4x}-7) e^{-x^2/2}\,dx$$

$$= \left[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{a-4}^{\infty} e^{\overset{4^2/2}{\cancel{\boxed{4x}}}} e^{-x^2/2}\,dx}_{e^8[\Phi(4-a)]}\right] - \left[\underbrace{\frac{7}{\sqrt{2\pi}}\int_{a}^{\infty} e^{-x^2/2}\,dx}_{7[\Phi(-a)]}\right]$$

$$\begin{aligned} e^{4a}-7 &= 0 \\ e^{4a} &= 7 \\ 4a &= \ln 7 \\ a &= (\ln 7)/4 \approx 0.486477537 \end{aligned}$$

$\Phi(-a) \approx 0.31331$

Problem: $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{4x}-7)_+e^{-x^2/2}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(e^{4x}-7)_+e^{-x^2/2}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(e^{4x}-7)e^{-x^2/2}\,dx$$

$$=\underbrace{\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{a-4}^{\infty}e^{\boxed{\cancel{4x}}\,4^2/2}e^{-x^2/2}\,dx\right]}_{e^8[\Phi(4-a)]}-\underbrace{\left[\frac{7}{\sqrt{2\pi}}\int_{a}^{\infty}e^{-x^2/2}\,dx\right]}_{7[\Phi(-a)]}}_{\approx\ 2978}$$

$\Phi(4-a)\approx 0.99978$ $\Phi(-a)\approx 0.31331$

$a=(\ln 7)/4\approx 0.486477537$

Compute: $\int \boxed{x^k} e^{-x^2/2} \; dx, \quad \text{for } k = 0, 1, 2, \ldots$

$k = 0$: $\int e^{-x^2/2} \; dx = \sqrt{2\pi}(\Phi(x)) + C$

Solution: $$\frac{d}{dx}\left[\sqrt{2\pi}(\Phi(x))\right] = e^{-x^2/2}$$

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}\left[e^{-t^2/2}\right]_{t:\to x} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2} \; dt$$

Compute: $\int x^k e^{-x^2/2} \, dx$, for $k = 0, 1, 2, \ldots$

REMEMBER: chg x to $-$

$k = 1$: $\int \boxed{x} \; e^{-\boxed{x^2/2}} \; \boxed{dx}$

Simplest antiderivative: $\boxed{-e^{-x^2/2}}$ only works for $k = 1$

Solution: $\int e^{-\boxed{u}} \; \boxed{du} = -e^{-u} + C$

$= -e^{-x^2/2} + C$

C_0

$u = x^2/2$

$du = x \, dx$

Compute: $\int x^k e^{-x^2/2} \, dx,$ for $k = 0, 1, 2, \ldots.$

$k = 1$:	$\int x \; e^{-x^2/2} \; dx$	Simplest antiderivative: $-e^{-x^2/2}$

$$\int_{-\infty}^{\infty} x e^{-x^2/2} \, dx = 0$$

$$-e^{-x^2/2} \to 0, \text{ as } x \to \pm\infty$$

Compute: $\displaystyle\int x^k e^{-x^2/2}\, dx, \quad$ for $k = 0, 1, 2, \ldots$

$k = 2$: $\displaystyle\int x^2\, e^{-x^2/2}\, dx$

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easier: $\boxed{-e^{-x^2/2}}$

$\boxed{k = 1}$

$$\int \boxed{x^2}\, e^{-x^2/2}\, dx = \int \boxed{x}\, \boxed{x\, e^{-x^2/2}}\, dx$$

easier: $\boxed{1}$

$$= -x\, e^{-x^2/2} - \int -e^{-x^2/2}\, dx$$

Compute: $\int x^k e^{-x^2/2} \, dx$, for $k = 0, 1, 2, \ldots$.

$k = 2$: $\int x^2 \, e^{-x^2/2} \, dx$

$$= -x \, e^{-x^2/2} \quad - \int - e^{-x^2/2} \, dx$$

$$\int x^2 \, e^{-x^2/2} \, dx$$

$$= -x \, e^{-x^2/2} \quad - \int -e^{-x^2/2} \, dx$$

Compute: $\int x^k e^{-x^2/2} \; dx, \quad$ for $k = 0, 1, 2, \ldots$

$k = 2$: $\quad \int x^2 \, e^{-x^2/2} \; dx$

$$= -x \, e^{-x^2/2} \quad - \int - e^{-x^2/2} \; dx$$

$k = 0$

$$= -x \, e^{-x^2/2} \quad + \boxed{\int e^{-x^2/2} \; dx}$$

do not forget $\sqrt{2\pi}$

$$= -x \, e^{-x^2/2} \quad + \boxed{\sqrt{2\pi}} [\Phi(x)] \quad + \; C$$

DONE

Compute: $\int x^k e^{-x^2/2} \, dx, \quad$ for $k = 0, 1, 2, \ldots$.

$k = 3$: $\int x^3 e^{-x^2/2} \, dx$

$$-e^{-x^2/2}$$

$$\int \boxed{x^3} e^{-x^2/2} \, dx = \int \boxed{x^2} \, \boxed{x \, e^{-x^2/2}} \, dx$$

$$\boxed{2x}$$

$$= -x^2 \, e^{-x^2/2} \; - \int -2x e^{-x^2/2} \, dx$$

Compute: $\int x^k e^{-x^2/2}\, dx$, for $k = 0, 1, 2, \ldots$.

$k = 3$: $\int x^3\, e^{-x^2/2}\, dx$

$$= -x^2 e^{-x^2/2} - \int -2x e^{-x^2/2}\, dx$$

$$= -x^2 e^{-x^2/2} + \boxed{2}\boxed{\int x e^{-x^2/2}\, dx} \qquad \boxed{k = 1}$$

$$= \boxed{-x^2 e^{-x^2/2} - 2\, e^{-x^2/2} + \boxed{2C}}$$

Compute: $\displaystyle\int x^k e^{-x^2/2}\, dx$, for $k = \underline{0, 1, 2, \ldots}$.

$$\int x^{\overset{4}{k}} e^{-x^2/2}\, dx = \int \underbrace{x^{k-1}}_{(k-1)x^{k-2}} \overbrace{x\, e^{-x^2/2}}^{-e^{-x^2/2}}\, dx$$

$$= -x^{k-1}\, e^{-x^2/2} - \int - (k-1)x^{k-2} e^{-x^2/2}\, dx$$

$$= -x^{k-1}\, e^{-x^2/2} + (k-1) \int x^{\overset{2}{k-2}} e^{-x^2/2}\, dx$$

Exercise: $\int x^{\boxed{4}} e^{-x^2/2} \, dx$

$$\int x^k e^{-x^2/2} \, dx = \int x^{k-1} \, x \, e^{-x^2/2} \, dx$$

$$= -x^{k-1} \, e^{-x^2/2} - \int -(k-1) x^{k-2} e^{-x^2/2} \, dx$$

$$= -x^{k-1} \, e^{-x^2/2} + (k-1) \int x^{k-2} e^{-x^2/2} \, dx$$

$\forall$integers $k \geq 1$, $\quad -x^{k-1}e^{-x^2/2} \to 0$, as $x \to \pm\infty$

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2}\, dx = (k-1)\int_{-\infty}^{\infty} x^{k-2}e^{-x^2/2}\, dx$$

$$\int \; x^k e^{-x^2/2} \;\; dx \; = \int x^{k-1} \; x \, e^{-x^2/2} \; dx$$

$$= -x^{k-1} \; e^{-x^2/2} \; - \int -(k-1)x^{k-2}e^{-x^2/2}\, dx$$

$$= -x^{k-1} \; e^{-x^2/2} \; + (k-1)\int x^{k-2}e^{-x^2/2}\, dx$$

$\forall$integers $k \geq 1$,

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2}\, dx = (k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-x^2/2}\, dx$$

$\forall$integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = (2n-1)(2n-3)\cdots(3)(1)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx &= (2n-1) \int_{-\infty}^{\infty} x^{2n-2} e^{-x^2/2}\, dx \\
&= (2n-1)(2n-3) \int_{-\infty}^{\infty} x^{2n-4} e^{-x^2/2}\, dx \\
= \cdots &= (2n-1)(2n-3)\cdots 3 \int_{-\infty}^{\infty} x^{2} e^{-x^2/2}\, dx \\
&= (2n-1)(2n-3)\cdots 3 \cdot 1 \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \\
&= [(2n-1)(2n-3)\cdots 3 \cdot 1]\sqrt{2\pi}
\end{aligned}$$

$\forall$integers $k \geq 1$,

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2}\,dx = (k-1)\int_{-\infty}^{\infty} x^{k-2} e^{-x^2/2}\,dx$$

$\forall$integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2}\,dx = (2n)\int_{-\infty}^{\infty} x^{2n-1} e^{-x^2/2}\,dx$$

$$= (2n)(2n-2)\int_{-\infty}^{\infty} x^{2n-3} e^{-x^2/2}\,dx$$

$$= \cdots = (2n)(2n-2)\cdots 4\int_{-\infty}^{\infty} x^{3} e^{-x^2/2}\,dx$$

$$= (2n)(2n-2)\cdots 4\cdot 2\int_{-\infty}^{\infty} x e^{-x^2/2}\,dx$$

$$= 0$$

$$\boxed{\int_{-\infty}^{\infty} x e^{-x^2/2}\,dx = 0}$$

$\forall$integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2}\, dx = 0$$

$\forall$integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2}\, dx$$

$\forall$integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = (2n-1)(2n-3)\cdots(3)(1)$$

$$= 0$$

$\forall$integers $n \geq 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2}\, dx = 0$$

$\forall$integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = (2n-1)(2n-3)\cdots(3)(1)$$