

Financial Mathematics

Problems in integration

Definition:

$x \rightarrow -2$

SNCDF

$\Phi(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

NOT when x is upper limit

$\Phi(-2)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2} e^{-t^2/2} dt$$

$t \rightarrow -t$

$dt \rightarrow (-1)dt$

MULTIPLY BY $\sqrt{2\pi}$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^2 e^{-t^2/2} (-1) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_2^{\infty} e^{-t^2/2} dt$$

$$\int_2^{\infty} e^{-t^2/2} dt = \sqrt{2\pi} [\Phi(-2)]$$

$2 \rightarrow x$

do not forget

$$\int_x^{\infty} e^{-t^2/2} dt = \sqrt{2\pi} [\Phi(-x)]$$

minus needed

when x is lower limit

2

Problem:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_3^\infty e^{4x} e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_{3-4}^\infty e^{4^2/2} e^{-x^2/2} dx \\ &= \frac{e^8}{\sqrt{2\pi}} \int_{-1}^\infty e^{-x^2/2} dx \\ &= e^8 [\Phi(1)] \blacksquare \end{aligned}$$

$$\begin{aligned} [4x - (x^2/2)]_{x \rightarrow x+4} &= 4(x+4) - ((x+4)^2/2) \\ &= -(x^2/2) + (4^2/2) \end{aligned}$$

Problem:

$$\frac{1}{\sqrt{2\pi}} \int_3^{\infty} (e^{4x} - 7)e^{-x^2/2} dx$$
$$= \left[\frac{1}{\sqrt{2\pi}} \int_{3-4}^{\infty} e^{\frac{4^2/2}{4x}} e^{-x^2/2} dx \right] - \left[\frac{7}{\sqrt{2\pi}} \int_3^{\infty} e^{-x^2/2} dx \right]$$
$$e^8[\Phi(1)] \qquad 7[\Phi(-3)]$$



Problem: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{4x} - 7)_+ e^{-x^2/2} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^{\infty} (e^{4x} - 7)_+ e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^{\infty} (e^{4x} - 7) e^{-x^2/2} dx$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{a-4}^{\infty} e^{4x} e^{-x^2/2} dx \right] - \left[\frac{7}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2} dx \right]$$

$e^8 [\Phi(4-a)] \qquad 7[\Phi(-a)]$

$$e^{4a} - 7 = 0$$

$$e^{4a} = 7$$

$$4a = \ln 7$$

$$a = (\ln 7)/4 \approx 0.486477537$$

$$\Phi(-a) \approx 0.31331$$

Problem: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{4x} - 7)_+ e^{-x^2/2} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^{\infty} (e^{4x} - 7)_+ e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^{\infty} (e^{4x} - 7) e^{-x^2/2} dx$$

$$= \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{a-4}^{\infty} e^{4x} e^{-x^2/2} dx \right]}_{e^8 [\Phi(4-a)]} - \underbrace{\left[\frac{7}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2} dx \right]}_{7[\Phi(-a)]}$$

≈ 2978 ■

$\Phi(4-a) \approx 0.99978$

$\Phi(-a) \approx 0.31331$

$a = (\ln 7)/4 \approx 0.486477537$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 0$: $\int e^{-x^2/2} dx = \sqrt{2\pi}(\Phi(x)) + C$

Solution: $\frac{d}{dx} [\sqrt{2\pi}(\Phi(x))] = e^{-x^2/2}$

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \left[e^{-t^2/2} \right]_{t \rightarrow x} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

REMEMBER:

chg x to $-$

$k = 1$.

$$\int x e^{-x^2/2} dx$$

Simplest antiderivative:

$$-e^{-x^2/2}$$

only works for $k = 1$

Solution:

$$\int e^{-u} du = -e^{-u} + C$$

$$= -e^{-x^2/2} + C$$

$$u = x^2/2$$

$$du = x dx$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 1:$ $\int x e^{-x^2/2} dx$

Simplest
antiderivative:
 $-e^{-x^2/2}$

$$\int_{-\infty}^{\infty} x e^{-x^2/2} dx = 0$$

$$-e^{-x^2/2} \rightarrow 0, \text{ as } x \rightarrow \pm\infty$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 2$: $\int x^2 e^{-x^2/2} dx$

easier: $\downarrow -e^{-x^2/2}$

$k = 1$

$$\int x^2 e^{-x^2/2} dx = \int x \cdot x e^{-x^2/2} dx$$

easier: $\downarrow 1$

$$= -x e^{-x^2/2} - \int -e^{-x^2/2} dx$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$$k = 2: \int x^2 e^{-x^2/2} dx$$

$$= -x e^{-x^2/2} - \int -e^{-x^2/2} dx$$

$$\int x^2 e^{-x^2/2} dx$$

$$= -x e^{-x^2/2} - \int -e^{-x^2/2} dx$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 2$: $\int x^2 e^{-x^2/2} dx$

$= -x e^{-x^2/2} - \int -e^{-x^2/2} dx$

$= -x e^{-x^2/2} + \int e^{-x^2/2} dx$

$= -x e^{-x^2/2} + \sqrt{2\pi} [\Phi(x)] + C$

$k = 0$

do not forget $\sqrt{2\pi}$

DONE

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 3$: $\int x^3 e^{-x^2/2} dx$

$$\begin{aligned} \int x^3 e^{-x^2/2} dx &= \int x^2 \boxed{x e^{-x^2/2}} dx \\ &\quad \begin{array}{c} \boxed{2x} \\ \uparrow \end{array} \\ &\quad \begin{array}{c} \boxed{-e^{-x^2/2}} \\ \downarrow \end{array} \\ &= -x^2 e^{-x^2/2} - \int -2x e^{-x^2/2} dx \end{aligned}$$

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$k = 3$: $\int x^3 e^{-x^2/2} dx$

$$\begin{aligned} &= -x^2 e^{-x^2/2} - \int -2x e^{-x^2/2} dx \\ &= -x^2 e^{-x^2/2} + 2 \int x e^{-x^2/2} dx \\ &= -x^2 e^{-x^2/2} - 2 e^{-x^2/2} + 2C \end{aligned}$$

Annotations: Orange arrows point from the $-$ sign and the $2x$ term in the second line to the 2 in the third line. A blue box labeled $k=1$ has an arrow pointing to the integral in the third line. The final result is enclosed in an orange box.

Compute: $\int x^k e^{-x^2/2} dx$, for $k = 0, 1, 2, \dots$

$$\begin{aligned} \int x^k e^{-x^2/2} dx &= \int x^{k-1} x e^{-x^2/2} dx \\ &= -x^{k-1} e^{-x^2/2} - \int (k-1)x^{k-2} e^{-x^2/2} dx \\ &= -x^{k-1} e^{-x^2/2} + (k-1) \int x^{k-2} e^{-x^2/2} dx \end{aligned}$$

The diagram includes several annotations: a pink box with the number '4' points to the x^k term in the first line; orange boxes highlight x^k , x^{k-1} , $x e^{-x^2/2}$, $(k-1)x^{k-2}$, $(k-1)$, and x^{k-2} ; orange arrows show the flow of terms and the integration process; a pink box with the number '2' points to the x^{k-2} term in the final line; a separate orange box at the top right contains the expression $-e^{-x^2/2}$.

Exercise: $\int x^4 e^{-x^2/2} dx$

$$\int x^k e^{-x^2/2} dx = \int x^{k-1} x e^{-x^2/2} dx$$

$$= -x^{k-1} e^{-x^2/2} - \int -(k-1)x^{k-2} e^{-x^2/2} dx$$

$$= -x^{k-1} e^{-x^2/2} + (k-1) \int x^{k-2} e^{-x^2/2} dx$$

\forall integers $k \geq 1$, $-x^{k-1}e^{-x^2/2} \rightarrow 0$, as $x \rightarrow \pm\infty$

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2} dx = (k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-x^2/2} dx$$

$$\int x^k e^{-x^2/2} dx = \int x^{k-1} x e^{-x^2/2} dx$$

$$= -x^{k-1} e^{-x^2/2} - \int -(k-1)x^{k-2} e^{-x^2/2} dx$$

$$= -x^{k-1} e^{-x^2/2} + (k-1) \int x^{k-2} e^{-x^2/2} dx$$

∀ integers $k \geq 1$,

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2} dx = (k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-x^2/2} dx$$

∀ integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2} dx = (2n-1)(2n-3) \cdots (3)(1)$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2} dx &= (2n-1) \int_{-\infty}^{\infty} x^{2n-2} e^{-x^2/2} dx \\ &= (2n-1)(2n-3) \int_{-\infty}^{\infty} x^{2n-4} e^{-x^2/2} dx \\ &= \cdots = (2n-1)(2n-3) \cdots 3 \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \\ &= (2n-1)(2n-3) \cdots 3 \cdot 1 \int_{-\infty}^{\infty} e^{-x^2/2} dx \\ &= [(2n-1)(2n-3) \cdots 3 \cdot 1] \sqrt{2\pi} \end{aligned}$$

∀ integers $k \geq 1$,

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2} dx = (k-1) \int_{-\infty}^{\infty} x^{k-2} e^{-x^2/2} dx$$

∀ integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2} dx = (2n) \int_{-\infty}^{\infty} x^{2n-1} e^{-x^2/2} dx$$

$$= (2n)(2n-2) \int_{-\infty}^{\infty} x^{2n-3} e^{-x^2/2} dx$$

$$= \dots = (2n)(2n-2) \dots 4 \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx$$

$$= (2n)(2n-2) \dots 4 \cdot 2 \int_{-\infty}^{\infty} x e^{-x^2/2} dx$$

$$\int_{-\infty}^{\infty} x e^{-x^2/2} dx = 0$$

= 0

∀ integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2} dx = 0$$

∀ integers $n \geq 0$,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2} dx$$

∀ integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2} dx = (2n-1)(2n-3) \cdots (3)(1)$$

= 0

\forall integers $n \geq 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2} dx = 0$$

\forall integers $n \geq 1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2} dx = (2n-1)(2n-3) \cdots (3)(1)$$

