Financial Mathematics

Basics of piecewise constant random variables

PCRV

Definition:

A **piecewise constant random variable** is a piecewise constant function $\underbrace{[0,1]}_{\Omega} \to \mathbb{R}$.

PCRV: Finitely many pieces.

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

Five "pieces"

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

Intuition:

On each ... picks a point

$\omega \in$ ... $[0,1]$ ... m, and

...s.

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

$0.22 - 0.00 = 0.22$

$0.54 - 0.22 = 0.32$

$0.65 - 0.54 = 0.11$

$0.99 - 0.65 = 0.34$

$1.00 - 0.99 = 0.01$

Intuition:

On each "trial", Tyche picks a point $\omega \in \Omega := [0,1]$ at random, and reports back $X(\omega)$ to us.

Question: What is the probability that

$$1 < X < 2 \quad ?$$

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

$0.22 - 0.00 = 0.22$

$0.54 - 0.22 = 0.32$

$0.65 - 0.54 = 0.11$

$0.99 - 0.65 = 0.34$

$1.00 - 0.99 = 0.01$

$\Pr[1 < X < 2] = 0$

$\Pr[1 \le X < 2] = 0.22 = 22\%$

$\Pr[1 < X \le 2] = 0.32 = 32\%$

$\Pr[1 \le X \le 2] = 0.22 + 0.32 = 54\%$

$\Pr[X = 0] = 0.01 = 1\%$

Question: What is the probability that $1 < X < 2$? Ans: 0

Definition:

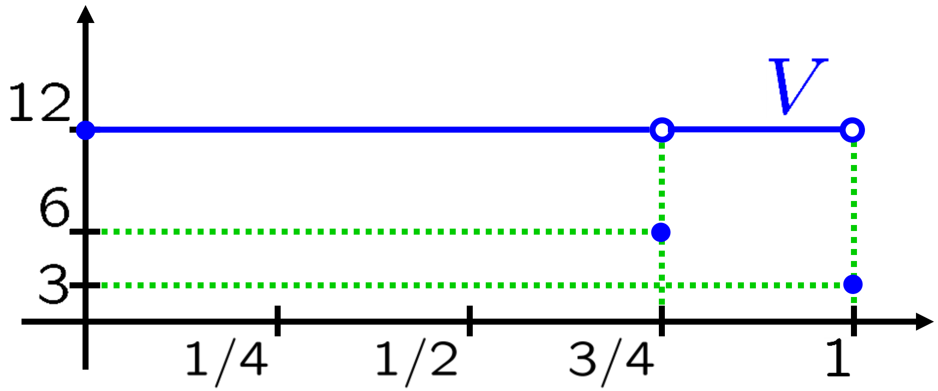
A PCRV is **deterministic** if it's constant, except (possibly) at finitely many points.

e.g.:

Let $U : \Omega \to \mathbb{R}$ be defined by $U(\omega) = 27$.

e.g.:

Let $V : \Omega \to \mathbb{R}$ be defined by $V(\omega) = 12$, except $V(0.75) = 6$ and $V(1) = 3$.



Note: "Pieces" can have zero length.

Let $Y : [0,1] \to \mathbb{R}$ be defined by

$$Y(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega < 0.54 \\ -7, & \text{if } \omega = 0.54 \\ 8, & \text{if } 0.54 < \omega < 1 \\ 0, & \text{if } \omega = 1.00 \end{cases}$$

$1, \text{ if } 0.00 \le \omega < 0.22$	0.22
$2, \text{ if } 0.22 \le \omega < 0.54$	0.32
$-7, \text{ if } \omega = 0.54$	0
$8, \text{ if } 0.54 < \omega < 1$	0.46
$0, \text{ if } \omega = 1.00$	0

$\Pr[Y > 0] = 1$, so we say:

$Y > 0$ almost surely

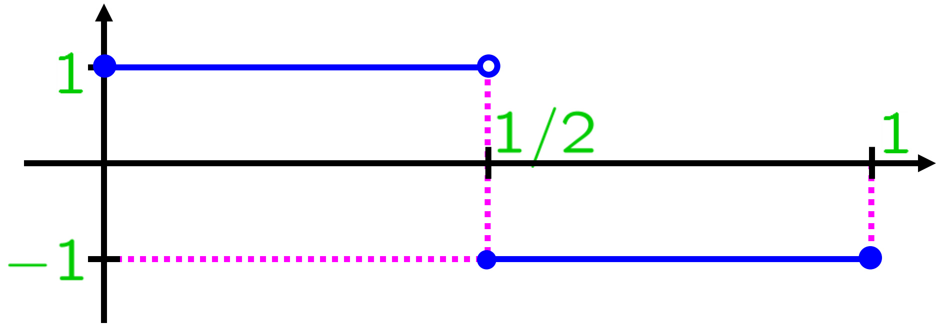
a.s.

It's not true that $Y > 0$ surely.

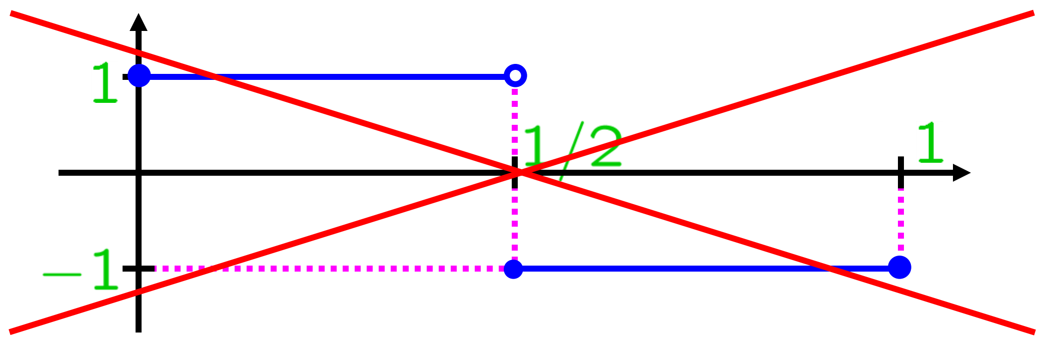
It's true that $Y \ge -7$ surely.

Modeling two coin flips

$$C_1 := \begin{cases} 1, & \text{if first flip is heads} \\ -1, & \text{if first flip is tails} \end{cases}$$

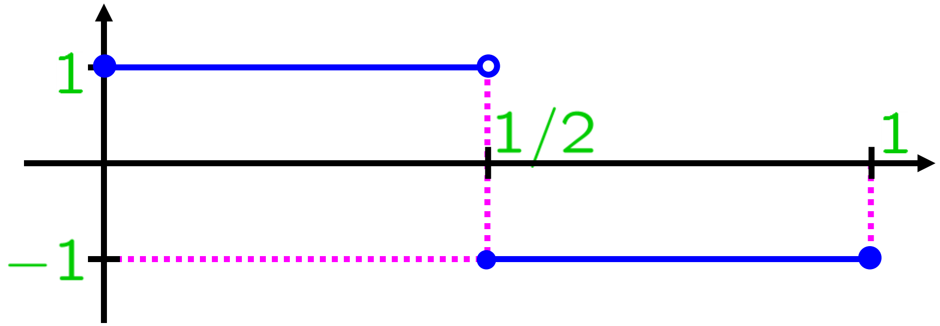
$C_1 :=$ 

$$C_2 := \begin{cases} 1, & \text{if second flip is heads} \\ -1, & \text{if second flip is tails} \end{cases}$$

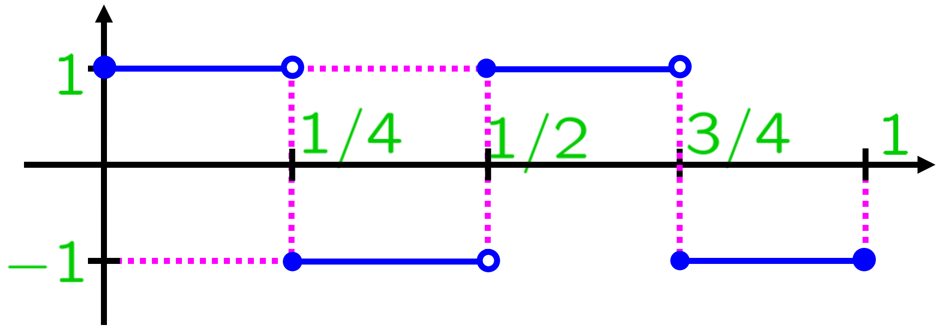


Modeling two coin flips

$$C_1 := \begin{cases} 1, & \text{if first flip is heads} \\ -1, & \text{if first flip is tails} \end{cases}$$

$C_1 :=$ 

$$C_2 := \begin{cases} 1, & \text{if second flip is heads} \\ -1, & \text{if second flip is tails} \end{cases}$$

$C_2 :=$ 

The distribution of a PCRV

Definition:

Let $S : [0,1] \to \mathbb{R}$ be a PCRV.

Let $F := \{a \in \mathbb{R} \mid \Pr[S = a] > 0\}$

The **distribution of** S associates

to any $a \in F$,

the value $\Pr[S = a]$.

Note:

Can be thought of

as a function $F \to (0,1]$,

or as a "measure" on $\mathbb{R}$,

which is "supported" on F.

e.g.: Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 \le \omega \le 1.00 \end{cases}$$

1	0.22
2	0.32
-7	0.11
8	0.34
0	0.01

The distribution of X is:

1	0.22
2	0.32
-7	0.11
8	0.34
0	0.01

e.g.: Let $Y : [0,1] \to \mathbb{R}$ be defined by

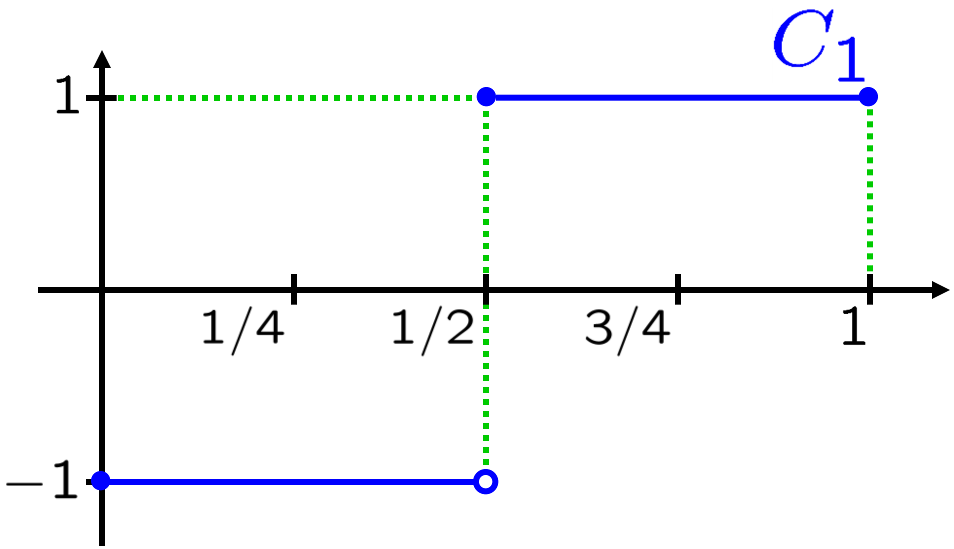
$$Y(\omega) = \begin{cases} 1, & \text{if } 0.00 \leq \omega < 0.22 \\ 2, & \text{if } 0.22 \leq \omega < 0.54 \\ -7, & \text{if } \omega = 0.54 \\ 8, & \text{if } 0.54 < \omega < 1 \\ 0, & \text{if } \omega = 1.00 \end{cases}$$

$1, \text{ if } 0.00 \leq \omega < 0.22$	0.22
$2, \text{ if } 0.22 \leq \omega < 0.54$	0.32
$-7, \text{ if } \omega = 0.54$	0
$8, \text{ if } 0.54 < \omega < 1$	0.46
$0, \text{ if } \omega = 1.00$	0

The distribution of Y is:

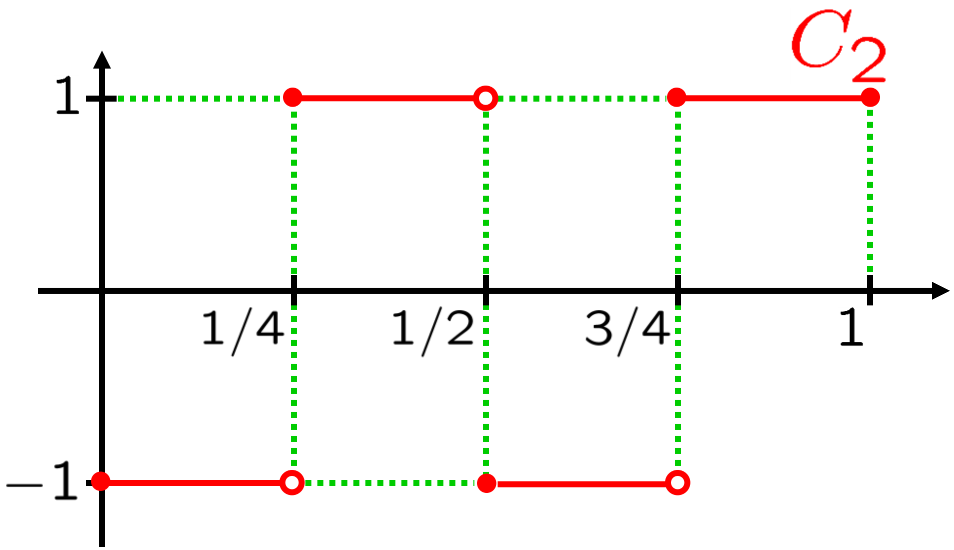
1	0.22
2	0.32
8	0.46

PCRV	Distribution



C_1

1	1/2
−1	1/2



C_2

1	1/2
−1	1/2

Note: C_1 and C_2 are identically distributed, but are not equal.

Def'n: Let S and T be PCRVs.

Let $F := \{(a,b) \in \mathbb{R}^2 \mid \Pr[(S=a)\&(T=b)] > 0\}$.

The joint distribution of (S,T) associates, to each element $(a,b) \in F$, the value $\Pr[(S=a)\&(T=b)]$.

Note: Can be thought of as a function $F \longrightarrow (0,\infty)$, or as a "measure" on $\mathbb{R}^2$, which is "supported" on F.

Remark: To compute the distribution of $S+T$, you need to know the JOINT distr. of (S,T). Knowing both the distribution of S and the distribution of T is insufficient. Same for ST.

Def'n: Let S and T be PCRVs.
Let $F := \{(a,b) \in \mathbb{R}^2 \mid \Pr[(S = a)\&(T = b)] > 0\}$.
The joint distribution of (S,T)
associates, to each element $(a,b) \in F$,
the value $\Pr[(S = a)\&(T = b)]$.

---

e.g.: Let $A := C_1$, $B := C_2$, $A' := C_1$, $B' := C_1$.
Then A and A' have the same distribution,
and B and B' have the same distribution,
but $A + B$ and $A' + B'$ do not have
the same distribution.
Note: (A,B) and (A',B') do not have
the same joint distribution.

Remark: To compute the distribution of $S + T$,
you need to know the JOINT distr. of (S,T).
Knowing both the distribution of S
and the distribution of T
is insufficient. Same for ST.

Let A be a PCRV,

so $A : [0,1] \to \mathbb{R}$ is piecewise constant.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

Definition: $\boxed{f(A)} := f \circ A : [0,1] \to \mathbb{R}$

Note: $f(A)$ is a PCRV as well.

---

e.g.:

$$A(\omega) = \begin{cases} 4, & \text{if } 0.00 \le \omega < 0.45 \\ 2, & \text{if } 0.45 \le \omega \le 0.75 \\ -8, & \text{if } 0.75 < \omega \le 1 \end{cases}$$

$$f(x) = x^2 \qquad\qquad B := f(A)$$

Then
$$B(\omega) = \begin{cases} 16, & \text{if } 0.00 \le \omega < 0.45 \\ 4, & \text{if } 0.45 \le \omega \le 0.75 \\ 64, & \text{if } 0.75 < \omega \le 1 \end{cases}$$

$$[0,1] = \ddot{\Omega}$$

Definition: For any PCRV $T : \ddot{\Omega} \to \mathbb{R}$,

the **mean** **of** T is $\boxed{\mathsf{E}[T]} := \int_0^1 T(\omega)\, d\omega.$

Note: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$
and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

For any PCRV T, let $\boxed{T^\circ} := T - (\mathsf{E}[T])$.
For any PRCV T, the **variance** **of** T is
$\boxed{\mathrm{Var}[T]} := \mathsf{E}[(T^\circ)^2]$.

Remark: For any PCRV T, $\mathrm{Var}[T] \geq 0$.
T is deterministic iff $\mathrm{Var}[T] = 0$.

Fact: $\mathrm{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Pf of fact: $\mu := \mathrm{E}[T]$ $\quad T^\circ = T - \mu$

$$\mathrm{Var}[T] = \mathrm{E}[(T^\circ)^2] = \mathrm{E}[(T-\mu)^2]$$
$$= \mathrm{E}[T^2 - 2\mu T + \mu^2]$$

Note: $\mathrm{E}[\bullet]$ is linear,

i.e., $\mathrm{E}[S+T] = (\mathrm{E}[S]) + (\mathrm{E}[T])$

and $\mathrm{E}[cS] = c(\mathrm{E}[S])$.

For any PCRV T, let $T^\circ := T - (\mathrm{E}[T])$.

For any PRCV T, the **variance of** T is

$\mathrm{Var}[T] := \mathrm{E}[(T^\circ)^2]$.

Remark: For any PCRV T, $\mathrm{Var}[T] \geq 0$.

T is deterministic iff $\mathrm{Var}[T] = 0$.

Fact: $\mathrm{Var}[T] = (\mathrm{E}[T^2]) - (\mathrm{E}[T])^2$

Pf of fact: $\mu := \mathsf{E}[T]$ $\qquad T^\circ = T - \mu$

$$\mathsf{Var}[T] = \mathsf{Var}[\hat{T}] - 2\mu T + \mu^2]$$
$$= (\mathsf{E}[T^2]) - 2\mu(\mathsf{E}[T]) + \mu^{2,2} - 2\mu T + \mu^2]$$

Note: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$
and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

For any PCRV T, let $T^\circ := T - (\mathsf{E}[T])$.
For any PRCV T, the **variance of** T is
$\mathsf{Var}[T] := \mathsf{E}[(T^\circ)^2]$.

Remark: For any PCRV T, $\mathsf{Var}[T] \geq 0$.
T is deterministic iff $\mathsf{Var}[T] = 0$.

Fact: $\mathsf{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Pf of fact: $\mu := \mathsf{E}[T]$ $\qquad T^\circ = T - \mu$

$$\begin{aligned}\mathrm{Var}[T] &= \mathsf{E}[T^2 - 2\mu T + \mu^2] \\ &= (\mathsf{E}[T^2]) - 2\mu(\mathsf{E}[T]) + \mu^2 \\ &= (\mathsf{E}[T^2]) - 2\mu^2 + \mu^2 = (\mathsf{E}[T^2]) - \mu^2\end{aligned}$$

Note: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

For any PCRV T, let $T^\circ := T - (\mathsf{E}[T])$.

For any PRCV T, the **variance of** T is

$$\mathrm{Var}[T] := \mathsf{E}[(T^\circ)^2].$$

Remark: For any PCRV T, $\mathrm{Var}[T] \geq 0$.

T is deterministic iff $\mathrm{Var}[T] = 0$.

Fact: $\mathrm{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Pf of fact: $\mu := \mathsf{E}[T]$ $\quad T^\circ = T - \mu$

$\mathsf{Var}[T] = (\mathsf{E}[T^2]) - \mu^2$

$= (\mathsf{E}[T^2]) - \mu^2$

Note: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

For any PCRV T, let $T^\circ := T - (\mathsf{E}[T])$.

For any PRCV T, the **variance of** T is

$\mathsf{Var}[T] := \mathsf{E}[(T^\circ)^2]$.

Remark: For any PCRV T, $\mathsf{Var}[T] \geq 0$.

T is deterministic iff $\mathsf{Var}[T] = 0$.

Fact: $\mathsf{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Pf of fact: $\mu := \mathsf{E}[T]$ $\quad T^\circ = T - \mu$

$$\mathrm{Var}[T] = (\mathsf{E}[T^2]) - \mu^2$$
$$= (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2 \qquad \text{QED}$$

Note: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

For any PCRV T, let $T^\circ := T - (\mathsf{E}[T])$.

For any PRCV T, the **variance of** T is

$$\mathrm{Var}[T] := \mathsf{E}[(T^\circ)^2].$$

Remark: For any PCRV T, $\mathrm{Var}[T] \geq 0$.

T is deterministic iff $\mathrm{Var}[T] = 0$.

Fact: $\mathrm{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 < \omega \le 1.00 \end{cases}$$

0.22 0.32 0.11 0.34 0.01

The expectation or mean of X is

$$\mathrm{E}[X] := \int_0^1 X(\omega)\, d\omega$$

$$= 1(0.22) + 2(0.32) - 7(0.11) + 8(0.34) + 0(0.01)$$

$$= 2.81$$

Intuition: Measure of (average) size.

Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \leq \omega < 0.22 \\ 2, & \text{if } 0.22 \leq \omega \leq 0.54 \\ -7, & \text{if } 0.54 < \omega \leq 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 < \omega \leq 1.00 \end{cases}$$

0.22
0.32
0.11
0.34
0.01

$\mathrm{E}[X] = 2.81$

$\mathrm{E}[X]$

$\mathrm{Var}[U] := \mathrm{E}[(U^\circ)^2]$

$= 2.81$

Intuition: Measure of (average) size.

Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \le \omega < 0.22 \\ 2, & \text{if } 0.22 \le \omega \le 0.54 \\ -7, & \text{if } 0.54 < \omega \le 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 < \omega \le 1.00 \end{cases}$$

0.22
0.32
0.11
0.34
0.01

$\mathrm{E}[X] = 2.81$

The variance of X is

$U^\circ := U - (\mathrm{E}[U])$

$$\mathrm{Var}[X] := \mathrm{E}[(X^\circ)^2] = \mathrm{E}[(X - (\mathrm{E}[X]))^2]$$

$$= (1 - 2.81)^2(0.22) + (2 - 2.81)^2(0.32) + (-7 - 2.81)^2(0.11) + (8 - 2.81)^2(0.34) + (0 - 2.81)^2(0.01) = \text{Exercise}$$

Intuition: Measure of risk.

Let $X : [0,1] \to \mathbb{R}$ be defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0.00 \leq \omega < 0.22 \\ 2, & \text{if } 0.22 \leq \omega \leq 0.54 \\ -7, & \text{if } 0.54 < \omega \leq 0.65 \\ 8, & \text{if } 0.65 < \omega < 0.99 \\ 0, & \text{if } 0.99 < \omega \leq 1.00 \end{cases}$$

0.22	
0.32	
0.11	
0.34	
0.01	

$E[X] = 2.81$

Key idea: "Most investors are return-loving, but risk-averse."

If X is the price, one month from now, of some financial asset, then investors typically hope for X to have large mean and small variance.

Fact: $\mathrm{E}[\bullet]$ is linear,

i.e., $\mathrm{E}[S+T] = (\mathrm{E}[S]) + (\mathrm{E}[T])$

and $\mathrm{E}[cS] = c(\mathrm{E}[S])$.

WARNING: $\mathrm{Var}[\bullet]$ is NOT linear,

but rather quadratic.

$$(2S)^\circ = 2S - (\mathrm{E}[2S]) = 2S - 2(\mathrm{E}[S])$$

$$= 2(S - (\mathrm{E}[S])) = 2(S^\circ)$$

$U^\circ := U - (\mathrm{E}[U])$

$$\mathrm{Var}[2S] = \mathrm{E}[((2S)^\circ)^2] = \mathrm{E}[(2(S^\circ))^2]$$

$$= \mathrm{E}[4(S^\circ)^2] = 4(\mathrm{E}[(S^\circ)^2]) = 4(\mathrm{Var}[S])$$

$\mathrm{Var}[U] := \mathrm{E}[(U^\circ)^2]$

Fact: $\mathsf{E}[\bullet]$ is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

WARNING: $\mathsf{Var}[\bullet]$ is NOT linear,

but rather quadratic.

$\mathsf{Cov}[S,T]$ is defined by:

$$\mathsf{Var}[S+T] = (\mathsf{Var}[S]) + (\mathsf{Var}[T]) + 2(\mathsf{Cov}[S,T])$$

Cauchy-Schwarz: $-1 \leq \frac{\mathsf{Cov}[S,T]}{\underbrace{(\sqrt{\mathsf{Var}[S]}\sqrt{\mathsf{Var}[T]})}} \leq 1$

$\mathrel{\stackrel{\cdot\cdot}{=}}$

Def'n: $\mathsf{Corr}[S,T]$

WARNING:
$\mathsf{Corr}[S,T]$ is not defined if S or T is deterministic

---

Definition: S and T are **uncorrelated** if $\mathsf{Cov}[S,T] = 0$.

Fact: E[•] is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

WARNING: Var[•] is NOT linear,

but rather quadratic.

$\mathrm{Cov}[S,T]$ is defined by:

$$\mathrm{Var}[S+T] = (\mathrm{Var}[S]) + (\mathrm{Var}[T]) + 2(\mathrm{Cov}[S,T])$$

Definition:

S and T are **uncorrelated** if $\mathrm{Cov}[S,T] = 0$,

i.e., if $\mathrm{Var}[S+T] = (\mathrm{Var}[S]) + (\mathrm{Var}[T])$,

Definition: S and T are **uncorrelated** if $\mathrm{Cov}[S,T] = 0$.

Fact: E[•] is linear,

i.e., $\mathsf{E}[S+T] = (\mathsf{E}[S]) + (\mathsf{E}[T])$

and $\mathsf{E}[cS] = c(\mathsf{E}[S])$.

WARNING: Var[•] is NOT linear,

but rather quadratic.

$\mathsf{Cov}[S,T]$ is defined by:

$$\mathsf{Var}[S+T] = (\mathsf{Var}[S]) + (\mathsf{Var}[T]) + 2(\mathsf{Cov}[S,T])$$

Definition:

S and T are **uncorrelated** if $\mathsf{Cov}[S,T] = 0$,

i.e., if $\mathsf{Var}[S+T] = (\mathsf{Var}[S]) + (\mathsf{Var}[T])$,

i.e., if $\mathsf{E}[ST] = (\mathsf{E}[S])(\mathsf{E}[T])$.

Fact: $\mathsf{Var}[T] = (\mathsf{E}[T^2]) - (\mathsf{E}[T])^2$

Fact: $\mathsf{Cov}[S,T] = (\mathsf{E}[ST]) - (\mathsf{E}[S])(\mathsf{E}[T])$

Fact: If $T = S + 3$, then $\text{Var}[T] = \text{Var}[S]$.

Proof: $\text{E}[T] = (\text{E}[S]) + 3$

$U^\circ := U - (\text{E}[U])$

$$\begin{aligned} T^\circ &= T - (\text{E}[T]) \\ &= (S + \not{3}) - ((\text{E}[S]) + \not{3}) \\ &= S - (\text{E}[S]) \\ &= S^\circ \end{aligned}$$

$\text{Var}[U] := \text{E}[(U^\circ)^2]$

Definition:

S and T are **uncorrelated** if $\text{Cov}[S, T] = 0$,

i.e., if $\text{Var}[S + T] = (\text{Var}[S]) + (\text{Var}[T])$,

i.e., if $\text{E}[ST] = (\text{E}[S])(\text{E}[T])$.

Fact: $\text{Var}[T] = (\text{E}[T^2]) - (\text{E}[T])^2$

Fact: $\text{Cov}[S, T] = (\text{E}[ST]) - (\text{E}[S])(\text{E}[T])$

Fact: If $T = S + 3$, then $\text{Var}[T] = \text{Var}[S]$.

Proof: $\text{E}[T] = (\text{E}[S]) + 3$

$U^\circ := U - (\text{E}[U])$

$$
\begin{aligned}
T^\circ &= T - (\text{E}[T]) \\
&= (S + \not{3}) - ((\text{E}[S]) + \not{3}) \\
&= S - (\text{E}[S]) \\
&= S^\circ
\end{aligned}
$$

$\text{Var}[U] := \text{E}[(U^\circ)^2]$

$\text{Var}[T] = \text{E}[(T^\circ)^2] = \text{E}[(S^\circ)^2] = \text{Var}[S]$ QED

Fact: If $T - S$ is constant,
then $\text{Var}[T] = \text{Var}[S]$.

Fact: If $T - S$ is deterministic,
then $\text{Var}[T] = \text{Var}[S]$.

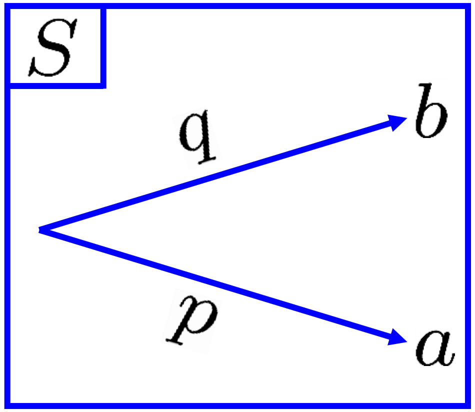
Problem: Suppose $0 \le p \le 1$ and $a, b \in \mathbb{R}$.

Suppose $\Pr[S = b] = q := 1 - p$

and $\Pr[S = a] = p$.

Find $\text{Var}[S]$.

binary PCRV



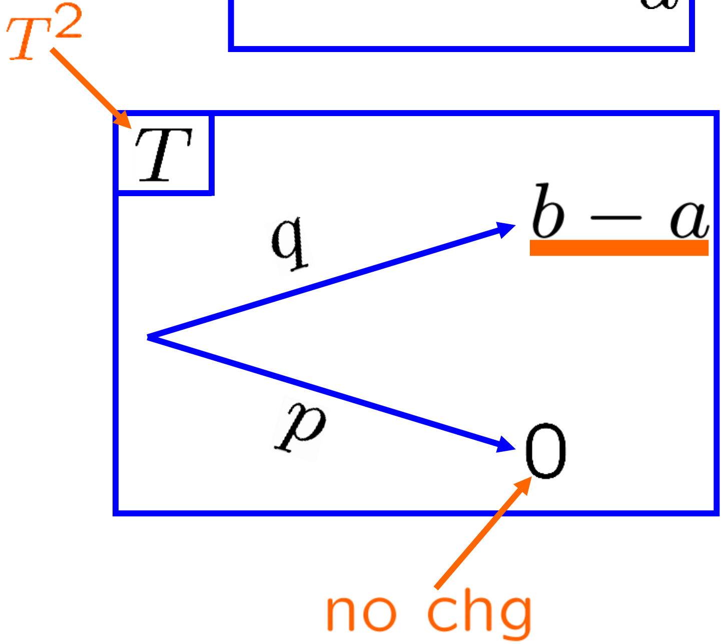
Solution: Let $T := S - a$.

Then $\text{Var}[S] = \text{Var}[T]$,

$\Pr[T = b - a] = q$

and $\Pr[T = 0] = p$.

Then $E[T] = q(b - a)$



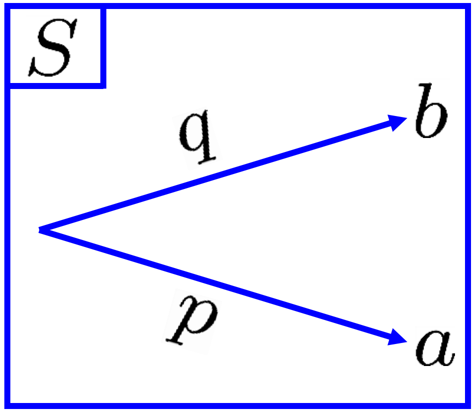
Problem: Suppose $0 \le p \le 1$ and $a, b \in \mathbb{R}$.

Suppose $\Pr[S = b] = q := 1 - p$

and $\Pr[S = a] = p$.

Find $\mathrm{Var}[S]$.

binary PCRV



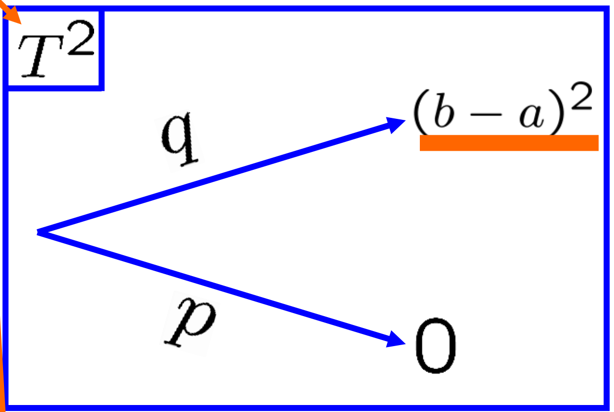
Solution: Let $T := S - a$.

Then $\mathrm{Var}[S] = \mathrm{Var}[T]$,

$\Pr[T = b - a] = q$

and $\Pr[T = 0] = p$.

T^2



Then $\mathrm{E}[T] = q(b - a)$

and $\mathrm{E}[T^2] = q(b - a)^2$.

Then $\mathrm{Var}[S] = \mathrm{Var}[T]$

$= (\mathrm{E}[T^2]) - (\mathrm{E}[T])^2$

$= q(b - a)^2 - q^2(b - a)^2$

$= (q - q^2)(b - a)^2 = (1 - q)q(b - a)^2 = pq(b - a)^2$ ∎

Problem: Suppose $0 \le p \le 1$ and $a, b \in \mathbb{R}$.

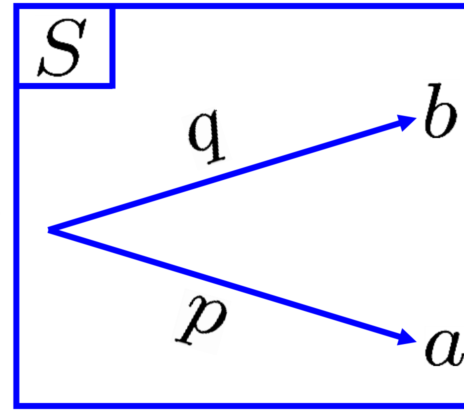
Suppose $\Pr[S = b] = q := 1 - p$

and $\Pr[S = a] = p$.

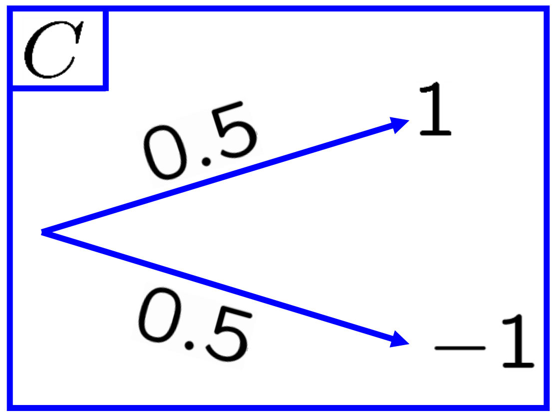
Find $\mathrm{Var}[S]$.

Solution: $\mathrm{Var}[S] = pq(b-a)^2$

binary PCRV



e.g., coin-flipping:



$\mathrm{Var}[S] =$

$pq(b-a)^2$

Problem: Suppose $0 \le p \le 1$ and $a, b \in \mathbb{R}$.

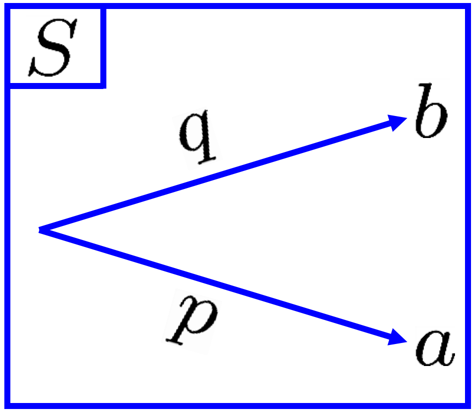
Suppose $\Pr[S = b] = q := 1 - p$

and $\Pr[S = a] = p$.

Find $\mathrm{Var}[S]$.

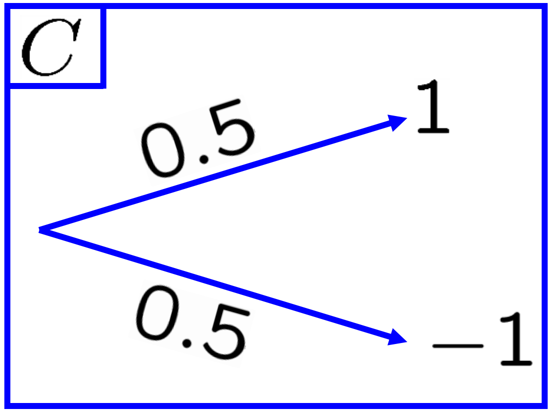
Solution: $\mathrm{Var}[S] = pq(b-a)^2$

binary PCRV



---

e.g., coin-flipping:



$\mathrm{E}[C] = 0$ and $\mathrm{Var}[C] = 1$,

i.e., C is standard.

Definition: S is **standard** means:

$\mathrm{E}[S] = 0$ and $\mathrm{Var}[S] = 1$.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\mathrm{SD}[S] := \sqrt{\mathrm{Var}[S]}$$

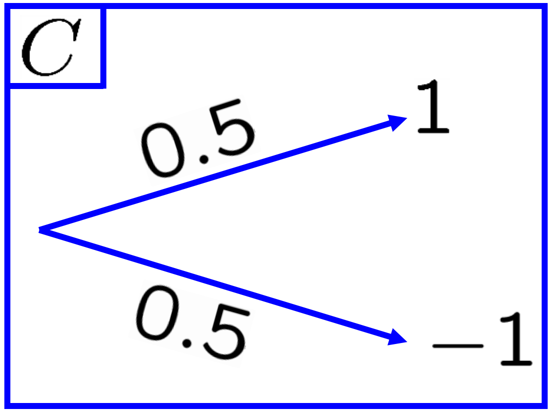
Pf: exercise

Fact: $\forall U$, $U_\circ := \dfrac{U - (\mathrm{E}[U])}{\mathrm{SD}[U]}$ is standard.

non-deterministic

A renormalization of U

e.g., coin-flipping:



$\mathrm{E}[C] = 0$ and $\mathrm{Var}[C] = 1$,

i.e., C is standard.

Definition: S is **standard** means:

$\mathrm{E}[S] = 0$ and $\mathrm{Var}[S] = 1$.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\boxed{\text{SD}[S]} := \sqrt{\text{Var}[S]}$$

Pf: exercise

Fact: $\forall U$, $\boxed{U_\circ} := \boxed{\dfrac{U - (\text{E}[U])}{\text{SD}[U]}}$ is standard.

non-deterministic

A renormalization of U

$$\mu = \text{E}[U]$$

$$\sigma = \text{SD}[U]$$

$$U_\circ = \frac{U - \mu}{\sigma} \quad \Rightarrow \quad U = \sigma U_\circ + \mu$$

Any (non-deterministic) U is "almost" standard.

Definition: S is **standard** means:

$$\text{E}[S] = 0 \quad \text{and} \quad \text{Var}[S] = 1.$$

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\boxed{\text{SD}[S]} := \sqrt{\text{Var}[S]}$$

$$(2S)^\circ = 2S - (\text{E}[2S]) = 2S - 2(\text{E}[S])$$
$$= 2(S - (\text{E}[S])) = 2(S^\circ)$$

$\boxed{U^\circ} := U - (\text{E}[U])$

TAKE SQUARE ROOT

$$\text{Var}[2S] = \text{E}[((2S)^\circ)^2] = \text{E}[(2(S^\circ))^2]$$
$$= \text{E}[4(S^\circ)^2] = 4(\text{E}[(S^\circ)^2]) = 4(\text{Var}[S])$$

$$\text{SD}[2S] = 2(\text{SD}[S])$$

$\boxed{\text{Var}[U]} := \text{E}[(U^\circ)^2]$

Intuition: Variance measures risk, but standard deviation measures risk better, because doubling the position really ought only to double the risk, not quadruple it.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\boxed{\text{SD}[S]} := \sqrt{\text{Var}[S]}$$

Remark: identically distributed $\Rightarrow$
same mean,
same variance,
same standard deviation.

If I hold a portfolio with variance 4,
and if I give you half of each asset,
then we both hold portfolios of variance 1.

Has our risk gone from 4 to $1 + 1$? NO!

Intuition: Variance measures risk, but
standard deviation measures risk better,
because doubling the position really ought
only to double the risk, not quadruple it.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\boxed{\mathrm{SD}[S]} := \sqrt{\mathrm{Var}[S]}$$

Remark: identically distributed $\Rightarrow$
same mean,
same variance,
same standard deviation.

WARNING: identical JOINT distribution
is needed for same covariance,
same correlation.

Intuition: Variance measures risk, but
standard deviation measures risk better,
because doubling the position really ought
only to double the risk, not quadruple it.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\boxed{\text{SD}[S]} := \sqrt{\text{Var}[S]}$$

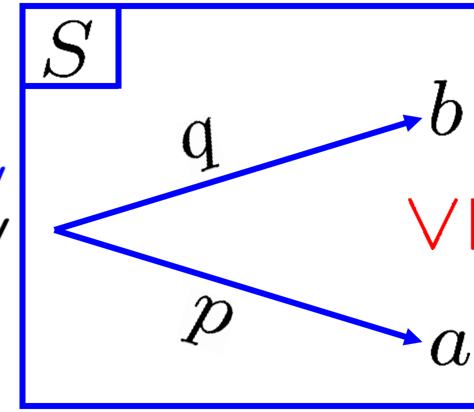
Problem: Suppose $0 \le p \le 1$ and $a \le b$.

Suppose $\Pr[S = b] = q := 1 - p$

and $\Pr[S = a] = p$.

Find SD$[S]$.

binary PCRV



Solution: $\text{Var}[S] = pq(b-a)^2$,

so $\text{SD}[S] = \underbrace{\sqrt{pq}}_{\text{the geometric mean of } p \text{ and } q}\underbrace{(b-a)}_{\text{naïve risk}}$.

arithmetic mean of p and q is $(p+q)/2$.

geometric mean of p and q is $\sqrt{pq}$.

Next topic: probabilities are expectations

Probabilities are expectations!

Definition: Let $A \subseteq B$.

The **indicator function of A (in B)** is the function $\mathbf{1}_A^B : B \to \{0,1\}$ defined by

$$\mathbf{1}_A^B(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B \backslash A. \end{cases}$$

Fact:

If f is the indicator function of S in $\mathbb{R}$, then $\mathrm{E}[f(X)] = \Pr[X \in S]$.

Pf:

$$f(X) = f \circ X : \Omega \to [0,1]$$

Probabilities are expectations!

Definition: Let $A \subseteq B$.

The **indicator function of A (in B)** is the function $\mathbf{1}_A^B : B \to \{0,1\}$ defined by

$$\mathbf{1}_A^B(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B \setminus A. \end{cases}$$

Fact:

If f is the indicator function of S in $\mathbb{R}$, then $\mathrm{E}[f(X)] = \Pr[X \in S]$.

Pf:

$$\mathrm{E}[f(X)] = \Pr[f(X) = 1] = \Pr[X \in S]$$

$f(X) = 1$ iff $X \in S$

$f(X) \in \{0,1\}$ a.s.

QED

$$f(X) = f \circ X : \Omega \to \{0,1\}$$

Probabilities are expectations!

Definition: Let $A \subseteq B$.

The **indicator function of A (in B)** is the function $\mathbf{1}_A^B : B \to \{0,1\}$ defined by

$$\mathbf{1}_A^B(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B \backslash A. \end{cases}$$

Fact:

If f is the indicator function of S in $\mathbb{R}$, then $\mathrm{E}[f(X)] = \Pr[X \in S]$.

$$\Pr[(f(X))(g(Y)) = 1] = \Pr[(f(X) = 1) \,\&\, (g(Y) = 1)]$$
$$= \Pr[(X \in S) \,\&\, (Y \in T)] \quad \text{QED}$$

Fact:

If f is the indicator function of S in $\mathbb{R}$, and if g is the indicator function of T in $\mathbb{R}$, then

$\mathrm{E}[(f(X))(g(Y))] = \Pr[(X \in S) \,\&\, (Y \in T)]$.

Applications to finance: Reduction of risk

Let $X_1, \ldots, X_{100}$ be the dollar prices,
one month from now,
of 100 assets, all costing 1 dollar now.

Suppose, for all integers $j \in [1, 100]$, that
$\mathrm{E}[X_j] = 1.1$ and $\mathrm{SD}[X_j] = 1$.

Let $S = X_1 + \cdots + X_{100}$. Cost: 100 dollars.

Then $\mathrm{E}[S] = 110$.

If, $\forall$integers $j, k \in [1, 100]$, $\mathrm{Corr}[X_j, X_k] = 0$,
then $\mathrm{SD}[S] = \sqrt{100} = 10$.

If, $\forall$integers $j, k \in [1, 100]$, $\mathrm{Corr}[X_j, X_k] = 1$,
then $\mathrm{SD}[S] = 100$.

Let $X_1, \ldots, X_N$ be the dollar prices,
one month from now,
of N assets, all costing 1 dollar now.

Suppose we have 1 dollar to spend.

Suppose, $\forall$integers $j \in [1, N]$,
$\mathrm{E}[X_j]$ is known.

Suppose we want expected return to be ρ.

Suppose, $\forall$integers $j, k \in [1, N]$,
$\mathrm{Cov}[X_j, X_k]$ is known.

Goal of "Modern Portfolio Theory":
Choose $c_1, \ldots, c_N$ such that
$\mathrm{Var}[c_1 X_1 + \cdots + c_N X_N]$ is minimized,
subject to $\mathrm{E}[c_1 X_1 + \cdots + c_N X_N] = 1 + \rho$
and $c_1 + \cdots + c_N = 1$.

Problem: We trade in two assets.

$A :=$ return, one month from now, on the first asset

$B :=$ return, one month from now, on the second asset

Assume $\text{SD}[A] = 0.5$,

$\text{SD}[B] = 0.3$

and $\text{Corr}[A, B] = 0.8$.

We need to hold \$10 of the first asset.
We want to short the second asset,
so as to reduce our risk as much as possible.

Find x such that $\text{SD}[10A - xB]$ is minimized.

Sol'n: Same as minimizing $(\text{SD}[10A - xB])^2$,

i.e., $\text{Var}[10A - xB]$

$= \text{Var}[10A] + \text{Var}[-xB]$

$+ 2(\text{Cov}[10A, -xB])$

min: $\text{Var}[10A] + \text{Var}[-xB] + 2(\text{Cov}[10A, -xB])$

$$\begin{aligned} \text{Assume } \text{SD}[A] &= 0.5, \\ \text{SD}[B] &= 0.3 \\ \text{and } \text{Corr}[A, B] &= 0.8. \end{aligned}$$

We need to hold \$10 of the first asset.
We want to short the second asset,
so as to reduce our risk as much as possible.

Find x such that $\text{SD}[10A - xB]$ is minimized.

Sol'n: Same as minimizing $(\text{SD}[10A - xB])^2$,

$$\begin{aligned} & \textit{i.e.}, \ \text{Var}[10A - xB] \\ = \quad & \text{Var}[10A] + \text{Var}[-xB] \\ & + 2(\text{Cov}[10A, -xB]) \end{aligned}$$

$$\text{min}: \underbrace{\text{Var}[\underline{10}A]}_{\underline{100}(\underbrace{\text{Var}[A]}_{(0.5)^2})} + \underbrace{\text{Var}[\underline{-x}B]}_{\underline{x^2}(\underbrace{\text{Var}[B]}_{(0.3)^2})} + \underbrace{\underline{2}(\text{Cov}[\underline{10}A, \underline{-x}B])}_{\underline{-20x}(\underbrace{\text{Cov}[A,B]}_{(0.8)(0.5)(0.3)})}$$

Assume $\text{SD}[A] = \boxed{0.5}$,

$\text{SD}[B] = \boxed{0.3}$

and $\text{Corr}[A, B] = \boxed{0.8}$.

covariance
is
correlation
×
prod. of std devs

We need to hold \$10 of the first asset.
We want to short the second asset,
so as to reduce our risk as much as possible.

Find x such that $\text{SD}[10A - xB]$ is minimized.

Sol'n: Same as minimizing $(\text{SD}[10A - xB])^2$,

i.e., $\text{Var}[10A - xB]$

$$= \text{Var}[10A] + \text{Var}[-xB] + 2(\text{Cov}[10A, -xB])$$

$$\text{min: } \underbrace{\text{Var}[10A]}_{100(\text{Var}[A])} + \underbrace{\text{Var}[-xB]}_{x^2(\text{Var}[B])} + \underbrace{2(\text{Cov}[10A, -xB])}_{-20x(\text{Cov}[A,B])}$$

$$100\underbrace{(\text{Var}[A])}_{(0.5)^2} \quad x^2\underbrace{(\text{Var}[B])}_{(0.3)^2} \quad -20x\underbrace{(\text{Cov}[A,B])}_{(0.8)(0.5)(0.3)}$$

$$\text{min: } 25 + (0.09)x^2 - (2.4)x$$

$$\text{deriv.: } (0.18)x - (2.4) = 0$$

STOP

$$x = \frac{2.4}{0.18} = \frac{40}{3} = 13.33$$

Short \$13.33 of the second asset.

Find x such that $\text{SD}[10A - xB]$ is minimized.

Sol'n: Same as minimizing $(\text{SD}[10A - xB])^2$,

i.e., $\text{Var}[10A - xB]$

$= \text{Var}[10A] + \text{Var}[-xB]$