Financial Mathematics
Conditional probability, independence and the Central Limit Theorem
Conditional prob. of one event, given another

We win if \( Y(\omega) \) turns out to be 1. Tyche tells us \( X(\omega) \), then \( Y(\omega) \). In between?
We win if $Y(\omega)$ turns out to be 1. Tyche tells us $X(\omega)$, then $Y(\omega)$. In between? 90% of time, $X(\omega) = -1$, & we very likely lose.

$$\text{Pr}[Y = -1 | X = -1] = \frac{8}{9}$$

8/9 of the time
We win if $Y(\omega)$ turns out to be 1. Tyche tells us $X(\omega)$, then $Y(\omega)$. In between? 90% of time, $X(\omega) = -1$, & we very likely lose.

$\Pr[X = -1] = 0.9$

$\Pr[(Y = -1) \& (X = -1)] = 0.8$

$\Pr[Y = -1 | X = -1] = \frac{8}{9}$

8/9 of the time
Conditional prob. of one event, given another

We win if $Y(\omega)$ turns out to be 1. Tyche tells us $X(\omega)$, then $Y(\omega)$. 10% of time, $X(\omega) = 1$, & we very likely win.

$\Pr[Y = 1 | X = 1] = \frac{9}{10}$
Conditional prob. of one event, given another

\[ \Pr[X = 1] \]

\[ \Pr[(Y = 1) \& (X = 1)] \]

We win if \( Y(\omega) \) turns out to be 1. Tyche tells us \( X(\omega), \text{then } Y(\omega) \). 10% of time, \( X(\omega) = 1, \) & we very likely win.

\[ \Pr[Y = 1|X = 1] = \frac{9}{10} \]

\[ \frac{0.09}{0.10} = \frac{9}{10} \]

9/10 of the time
Definition: The conditional probability of $P$ given $Q$ is

$$\Pr[P \mid Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}$$

Warning: Only defined when $\Pr[Q] \neq 0$.

Is $P$ likely or unlikely?

Given that you’re told $Q$ happened, is $P$ likely or unlikely?
Definition: The **conditional probability** of $P$ given $Q$ is

$$\Pr[P \mid Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}$$

**Warning:** Only defined when $\Pr[Q] \neq 0$.

---

$C_1 := \begin{cases} 1 & \text{same distr.} \\ 0 & \text{coin-flipping standard} \end{cases}$

$C_2 := \begin{cases} 0 & \text{1/4} \\ 1 & \text{1/2} \end{cases}$

**Key point:** Finding out $C_1 = 1$ has **no** influence on the prob. that $C_2 = 1$.

$$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5 = \Pr[C_2 = 1]$$
Definition: The **conditional probability** of \( P \) given \( Q \) is

\[
\Pr[P | Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}
\]

**Warning:** Only defined when \( \Pr[Q] \neq 0 \).

Definition: Assume \( \Pr[Q] \neq 0 \).

\( P \) & \( Q \) are **independent** (events) if

\[ \Pr[P | Q] = \Pr[P] \]

\text{i.e.: if} \quad \frac{\Pr[P \& Q]}{\Pr[Q]} = \Pr[P] \text{,}

\text{i.e.: if} \quad \Pr[P \& Q] = (\Pr[P])(\Pr[Q]) \text{.}

Key point: Finding out \( C_1 = 1 \) has **no influence on** the prob. that \( C_2 = 1 \).

\[
\Pr[(C_2 = 1) | (C_1 = 1)] = \frac{0.25}{0.5} = 0.5
\]

these are independent
Definition: The **conditional probability** of $P$ given $Q$ is

$$\Pr[P \mid Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}$$

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If $\Pr[P \& Q] = (\Pr[P])(\Pr[Q])$.

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\[\text{Warning: Only defined when } \Pr[Q] \neq 0.\]

Definition: \( P \& Q \) are **independent** (events) if

\[
\Pr[P \& Q] = (\Pr[P])(\Pr[Q]).
\]

"The probability of both is the product of the probabilities"

Key point: Finding out \( C_1 = 1 \) has **no influence** on the prob. that \( C_2 = 1 \).

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\[\text{these are independent}\]
Definition: The **conditional probability** of $P$ given $Q$ is

$$\Pr[P \mid Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}$$

**Warning:** Only defined when $\Pr[Q] \neq 0$.

Definition: $P \& Q$ are **independent** (events) if $\Pr[P \& Q] = (\Pr[P])(\Pr[Q])$.

Definition: $S \& T$ are **independent** (PCRVs) if, $\forall A, B \subseteq \mathbb{R}$, $S \in A$ is independent of $T \in B$.

Key point: Finding out $C_1 = 1$ has no influence on the prob. that $C_2 = 1$.

$$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5$$

these are independent

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Definition: The **conditional probability** of \( P \) given \( Q \) is
\[
\Pr[P|Q] = \frac{\Pr[P \& Q]}{\Pr[Q]}
\]

**Warning:** Only defined when \( \Pr[Q] \neq 0 \).

Definition: \( P \& Q \) are **independent** (events)
if \( \Pr[P \& Q] = (\Pr[P])(\Pr[Q]) \).

Definition: \( S \& T \) are **independent** (PCRVs)
if, \( \forall A, B \subseteq \mathbb{R}, S \in A \) is independent of \( T \in B \).

\[
\Pr[(C_2 = 1) \,(|\,(C_1 = 1))] = \frac{0.25}{0.5} = 0.5
\]

\( \text{these are independent} \)
Def’ns: $P$, $Q$, $R$ are independent (events) if $P$, $Q$, $R$ are pairwise-independent and $\Pr[P \& Q \& R] = (\Pr[P])(\Pr[Q])(\Pr[R])$.

$S$, $T$, $U$ are independent (PCRVs) if, $\forall A, B, C \subseteq \mathbb{R}, S \in A$, $T \in B$ and $U \in C$ are indep. etc., etc., etc.

**Definition:**

$P \& Q$ are independent (events) if $\Pr[P \& Q] = (\Pr[P])(\Pr[Q])$.

**Definition:**

$S \& T$ are independent (PCRVs) if, $\forall A, B \subseteq \mathbb{R}$, $S \in A$ is independent of $T \in B$.

$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5$

these are independent

$C_1 \in \{1\}$ is independent of $C_2 \in \{1\}$.

$C_1 \in \{-1\}$ is independent of $C_2 \in \{1\}$.

$C_1$ and $C_2$ independent
Exercise: Graph $C_4$.

Fact: $C_1, C_2, C_3, \ldots$ are pairwise independent.

Stronger: Any finite set of $C_1, C_2, \ldots$ is an independent set.
Def’n: Let $S$ and $T$ be PCRVs. Let $F \equiv \{(a, b) \in \mathbb{R}^2 \mid \Pr[(S = a) \& (T = b)] > 0\}$. The **joint distribution** of $(S, T)$ associates, to each element $(a, b) \in F$, the value $\Pr[(S = a) \& (T = b)]$.

**Remark:** To compute the distribution of $S + T$, you need to know the **joint** distr. of $(S, T)$. Knowing both the distribution of $S$ and the distribution of $T$ is insufficient. Same for $ST$. However, if $S$ and $T$ are independent, then their joint distribution is determined by their individual distributions, because

\[
\Pr[(S = a) \& (T = b)] = (\Pr[S = a])(\Pr[T = b]).
\]

All this generalizes to $\geq 2$ PCRVs.
Fact: independent $\Rightarrow$ uncorrelated

Pf: Let $S$, $T$ be independent PCRVs.

Want: $E[ST] = (E[S])(E[T])$

Define:

$A := \{a \in \mathbb{R} | Pr[S = a] > 0\}$

$B := \{b \in \mathbb{R} | Pr[T = b] > 0\}$

$E[ST] = \sum_{a \in A} \sum_{b \in B} (Pr[(S = a) \& (T = b)])ab$

$= \sum_{a \in A} \sum_{b \in B} (Pr[S = a])(Pr[(T = b)])ab$

$= \left(\sum_{a \in A} (Pr[S = a])a\right)\left(\sum_{b \in B} (Pr[(T = b)])b\right)$

$= (E[S])(E[T])$ QED
Fact:
Let $X$ and $Y$ be independent PCRVs. Then, for any functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(X)$ and $g(Y)$ are independent.

The idea:

Flip a $\pm 1$ fair coin twice.

If I tell you the first flip, you get no useful info about the second.

If I tell you $3 \times (\text{the first flip}) + 7$, you get no useful info about $5 \times (\text{the second flip}) - 1$. 

coin has $+1$ and $-1$ instead of $H$ and $T$. 

Fact:
Let $X$ and $Y$ be independent PCRVs. Then, for any functions $f, g : \mathbb{R} \to \mathbb{R}$, $f(X)$ and $g(Y)$ are independent.

Proof:
Given $S, T \subseteq \mathbb{R}$.

Want: $\Pr[(f(X) \in S) \& (g(Y) \in T)]$

$\equiv (\Pr[f(X) \in S])(\Pr[g(Y) \in T])$

$\Pr[(f(X) \in S) \& (g(Y) \in T)]$

$\equiv \Pr[(X \in f^{-1}(S)) \& (Y \in g^{-1}(T))]$

$\equiv (\Pr[X \in f^{-1}(S)])(\Pr[Y \in g^{-1}(T)])$

$\equiv (\Pr[f(X) \in S])(\Pr[g(Y) \in T])$

QED
Fact:
Let $X$ and $Y$ be independent PCRVs. Then, for any functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(X)$ and $g(Y)$ are independent.

Fact: independent $\Rightarrow$ uncorrelated

Restatement:
Let $A$ and $B$ be independent PCRVs. Then $E[AB] = (E[A])(E[B])$.

Corollary:
Let $X$ and $Y$ be independent PCRVs. Then, for any functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $E[(f(X))(g(Y))] = (E[f(X)])(E[g(Y)])$.

Rmk: Converse is true, too. pf omitted
Definition: \( \forall n > 0, D_n := C_1 + \cdots + C_n \) models (\#heads) – (\#tails) after \( n \) flips of a fair coin.

- \( C_1 := \)
  -\( -1 \quad 1/2 \quad 1 \)
  \(-1 \quad 50\% \)
  \( 1/2 \quad 50\% \)
  \( 1 \quad 50\% \)

- \( C_2 := \)
  -\( -1 \quad 1/4 \quad 1/2 \quad 3/4 \quad 1 \)
  \(-1 \quad 50\% \)
  \( 1/4 \quad 50\% \)
  \( 1/2 \quad 50\% \)
  \( 3/4 \quad 50\% \)
  \( 1 \quad 50\% \)

- \( D_2 := \)
  -\( -2 \quad -1 \quad 1 \quad 1/4 \quad 3/4 \quad 1 \)
  \(-2 \quad 25\% \)
  \(-1 \quad 25\% \)
  \( 1 \quad 50\% \)
  \( 1/4 \quad 50\% \)
  \( 3/4 \quad 50\% \)
  \( 1 \quad 50\% \)
  \(-2 \quad 25\% \)
Definition: \( \forall \text{integers } n > 0, \)
\[
D_n := C_1 + \cdots + C_n
\]
models (\(#\text{heads}\) – (\(#\text{tails}\)) after \(n\) flips of a fair coin

Fact: independent \(\Rightarrow\) uncorrelated,
i.e., \(S, T\) independent \(\Rightarrow\)
\[
\text{Var}[S + T] = \text{Var}[S] + \text{Var}[T].
\]

\(C_1, \ldots, C_n\) are all standard (i.e., mean 0, variance 1)
\[
\mathbb{E}[D_n] = (\mathbb{E}[C_1]) + \cdots + (\mathbb{E}[C_n])
\]
\[
= 0 + \cdots + 0 = 0
\]
\[
\text{Var}[D_n] = (\text{Var}[C_1]) + \cdots + (\text{Var}[C_n])
\]
\[
= 1 + \cdots + 1 = n
\]
\[
\mathbb{E}\left[\frac{D_n}{\sqrt{n}}\right] = 0 \quad \text{and} \quad \text{Var}\left[\frac{D_n}{\sqrt{n}}\right] = 1,
\]
i.e., \(\frac{D_n}{\sqrt{n}}\) is standard.
Definition: ∀ integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]
models \((\#\text{heads}) - (\#\text{tails})\) after \( n \) flips of a fair coin.

Preview of the **Central Limit Theorem**:

\[
\frac{D_n}{\sqrt{n}} \rightarrow Z \quad \text{in distribution}, \quad \text{as } n \rightarrow \infty.
\]

**Definition?**

Standard normal random variable

\[
E \left[ \frac{D_n}{\sqrt{n}} \right] = 0 \quad \text{and} \quad \text{Var} \left[ \frac{D_n}{\sqrt{n}} \right] = 1,
\]

i.e., \( \frac{D_n}{\sqrt{n}} \) is standard.
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]
models \((\#\text{heads}) - (\#\text{tails})\) after \( n \) flips of a fair coin.

Preview of the Central Limit Theorem:
\[
\frac{D_n}{\sqrt{n}} \rightarrow Z
\]
in distribution, as \( n \rightarrow \infty \).

\( \forall \) test functions \( \psi \),
\[
E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow E[\psi(Z)]
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]
Definition: $\forall$ integers $n > 0$,  
$$D_n := C_1 + \cdots + C_n$$

Preview of the Central Limit Theorem:

$\forall$ test functions $\psi$,

$$E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx$$

Relatively easy: “test function” $=$

“continuous, compactly supported function”

$\forall$ test functions $\psi$,

$$E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx$$
Definition: \( \forall \text{integers } n > 0, \ D_n := C_1 + \cdots + C_n \)

models \((\#\text{heads}) - (\#\text{tails})\) after \(n\) flips of a fair coin

Preview of the **Central Limit Theorem**:
\[ \forall \text{test functions } \psi, \]
\[ \mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx \]

Relatively easy: “test function” = “continuous, compactly supported function”

Harder to prove: “test function” = “continuous, exponentially-bounded function”

\( f \text{ exponentially bounded means:} \)
\[ \exists A, B > 0 \text{ s.t. } \forall x \in \mathbb{R}, \ |f(x)| \leq Ae^{B|x|} \]
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]

Preview of the Central Limit Theorem:
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
\mathbb{E} \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Exercise: Compute \( \lim_{n \to \infty} \mathbb{E} \left[ \left( e^{D_n/\sqrt{n}} - 7 \right)_+ \right] \).

\( f \) exponentially bounded means:
\[
\exists A, B > 0 \text{ s.t. } \forall x \in \mathbb{R}, \ |f(x)| \leq Ae^{B|x|}
\]
Definition: \( \forall \) integers \( n > 0 \), \( D_n := C_1 + \cdots + C_n \)

Preview of the Central Limit Theorem:
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]

Solution:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(e^x - 7)_+] [e^{-x^2/2}] \, dx = \ldots
\]
\[
\text{exp-bdd} \quad \rightarrow \quad \psi(x) = (e^x - 7)_+
\]

Exercise: Compute \( \lim_{n \to \infty} E \left[ (e^{D_n/\sqrt{n}} - 7)_+ \right] \).

\( f \) exponentially bounded means:
\[
\exists A, B > 0 \text{ s.t. } \forall x \in \mathbb{R}, \quad |f(x)| \leq Ae^{B|x|}
\]
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n : = C_1 + \cdots + C_n
\]
models (#heads) – (#tails) after \( n \) flips of a fair coin

Preview of the Central Limit Theorem:
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
E \left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \xrightarrow{n \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]
Hint: \( \psi(x) : = e^{ax+b} \)

Def’n: \( \forall X \), the **augmented expectation of** \( X \) is defined by
\[
[\text{AE}[X]] : = (E[X]) + \frac{1}{2}(\text{Var}[X]).
\]

“asymptotically normal”

Fact: Fix \( a, b \in \mathbb{R} \). Let \( R_n : = a \left( \frac{D_n}{\sqrt{n}} \right) + b \).

“\( E \) almost asymptotically commutes with \( e^{\bullet} \)”

Then \( \lim_{n \to \infty} E[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]} \).

Pf: \( \lim_{n \to \infty} E[e^{R_n}] \xrightarrow{\text{CLT}} e^{b}e^{a^2/2} \xrightarrow{\text{exercise}} \lim_{n \to \infty} e^{\text{AE}[R_n]} \).
Definition: \( \forall \) integers \( n > 0 \),
\[
D_n := C_1 + \cdots + C_n
\]

Preview of the Central Limit Theorem:
\( \forall \) continuous, exponentially-bounded \( \psi \),
\[
E\left[ \psi \left( \frac{D_n}{\sqrt{n}} \right) \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}] \, dx
\]
Hint: \( \psi(x) := (ax + b)^{1/2} \)

Def’n: \( \forall X \), the augmented expectation of \( X \)
is defined by \( \text{AE}[X] := (E[X]) + \frac{1}{2}(\text{Var}[X]) \).

“asymptotically normal”

Fact: Fix \( a, b \in \mathbb{R} \). Let \( R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b \).

“E almost asymptotically commutes with \( e^x \)”

Then
\[
\lim_{n \to \infty} E[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]}
\]

Pf: 
\[
\lim_{n \to \infty} E[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n \to \infty} e^{\text{AE}[R_n]}
\]

\( \blacksquare \)
Fact: Fix \( a, b \in \mathbb{R} \). Let \( R := aZ + b \).

"E almost commutes with \( e^\cdot \)…

Then \( \mathbb{E}[e^R] = e^{\mathbb{A}\mathbb{E}[R]} \).

…but we need to go from the expectation to the augmented expectation

Next subtopic: mean/var of summand from mean/var of iid sum

Def’n: \( \forall X \), the **augmented expectation** of \( X \) is defined by \( \mathbb{A}\mathbb{E}[X] := (\mathbb{E}[X]) + \frac{1}{2}(\mathbb{V}[X]) \).

Fact: Fix \( a, b \in \mathbb{R} \). Let \( R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b \).

"E almost asymptotically commutes with \( e^\cdot \)"

Then \( \lim_{n \to \infty} \mathbb{E}[e^{R_n}] = \lim_{n \to \infty} e^{\mathbb{A}\mathbb{E}[R_n]} \).

Pf: \( \lim_{n \to \infty} \mathbb{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n \to \infty} e^{\mathbb{A}\mathbb{E}[R_n]} \). \( \square \)
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]$$
$$\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Def’n: $\forall X$, the augmented expectation of $X$ is defined by $\text{AE}[X] := (\mathbb{E}[X]) + \frac{1}{2}(\text{Var}[X])$.

“asymptotically normal”

Fact: Fix $a, b \in \mathbb{R}$. Let $R_n := a \left( \frac{D_n}{\sqrt{n}} \right) + b$.

“$E$ almost asymptotically commutes with $e$”

Then $\lim_{n \to \infty} \mathbb{E}[e^{R_n}] = \lim_{n \to \infty} e^{\text{AE}[R_n]}$.

Pf: $\lim_{n \to \infty} \mathbb{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n \to \infty} e^{\text{AE}[R_n]}$. QED
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]$$

$$\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Solution:

$$0.225181512$$

$$\therefore$$

$$\mathbb{E}[S] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$$

$$= n\mu = (12)\mu,$$

so $\mu = 0.225181512/12$
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

\[
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n]
\]
\[
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
\]

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Solution: $\mu = \frac{0.225181512}{12}$

\[
\text{Var}[S] = \text{Var}[X_1] + \cdots + \text{Var}[X_n]
\]

\[
\mu = \frac{0.225181512}{12}
\]
Exercise: Let \( n := 12 \). Assume \( X_1, \ldots, X_n \) i.i.d.

\[
\mu := E[X_1] = \cdots = E[X_n] \\
\sigma := SD[X_1] = \cdots = SD[X_n]
\]

Let \( S := X_1 + \cdots + X_n \).

Assume \( E[S] = 0.225181512 \), \( SD[S] = 0.158877565 \). Find \( \mu \) and \( \sigma \).

Solution: \( \mu = 0.225181512/12 \)

\[
(0.158877565)^2
\]

\[
\Var[S] = \Var[X_1] + \cdots + \Var[X_n]
\]

\[
= n\sigma^2 = (12)\sigma^2,
\]

so \( \sigma^2 = (0.158877565)^2/12 \)

so \( \sigma = 0.158877565/\sqrt{12} \)
Exercise: Let \( n := 12 \). Assume \( X_1, \ldots, X_n \) iid.

\[
\begin{align*}
\mu &:= \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n] \\
\sigma &:= \text{SD}[X_1] = \cdots = \text{SD}[X_n]
\end{align*}
\]

Let \( S := X_1 + \cdots + X_n \).

Assume \( \mathbb{E}[S] = 0.225181512 \),
\[
\text{SD}[S] = 0.158877565.
\]
Find \( \mu \) and \( \sigma \).

Solution: \[
\begin{align*}
\mu &= 0.225181512/12 \\
\sigma &= 0.158877565/\sqrt{12}
\end{align*}
\]
Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n \text{ iid}$. 

$$
\mu := \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n] \\
\sigma := \text{SD}[X_1] = \cdots = \text{SD}[X_n]
$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathbb{E}[S] = 0.225181512$, $\text{SD}[S] = 0.158877565$. Find $\mu$ and $\sigma$.

Solution: $\mu = 0.225181512/12$

$$
\sigma = 0.158877565/\sqrt{12}
$$

Mean and variance are cut by a factor of 12. Standard deviation is cut by a factor of $\sqrt{12}$.

Conversely, on adding $n$ uncorrelated PCRVs, SD increases by a factor of $\sqrt{n}$, NOT $n$. A portfolio of uncorrelated assets is better . . .

Let’s explore this . . .
\[ \text{Var}[A + B] = (\text{Var}[A]) + (\text{Var}[B]) + 2(\text{Cov}[A, B]) \]

\[ \text{E}[A + B] = (\text{E}[A]) + (\text{E}[B]) \]

Say \( A \) and \( B \) are prices, one month from now, of two financial assets.

If \( \text{E}[A] \) is large, then \( A \) becomes attractive.
If \( \text{E}[B] \) is large, then \( B \) becomes attractive.
If \( \text{Var}[A] \) is small, then \( A \) becomes attractive.
If \( \text{Var}[B] \) is small, then \( B \) becomes attractive.

If \( \text{Cov}[A, B] \) is small or, even better, negative, then the portfolio of \( A \) and \( B \) becomes attractive.
Cauchy-Schwarz:
\[-\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]} \leq \text{Cov}[A, B] \leq \sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}\]

Definition:

A and B are **perfectly correlated** if
\[
\text{Cov}[A, B] = \sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}
\]

Definition:

A and B are **perfectly anti-correlated** if
\[-\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]} = \text{Cov}[A, B]\]
Cauchy-Schwarz: 
\[-\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]} \leq \text{Cov}[A, B] \leq \sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}\]

\(\forall\) non-deterministic PCRVs \(A, B\),

\[
\text{Corr}[A, B] := \frac{\text{Cov}[A, B]}{\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}}
\]

\(-1 \leq \text{Corr}[A, B] \leq 1\)

Suppose \(A\) and \(B\) are non-deterministic PCRVs.

\(\text{Corr}[A, B] = 1\) if and only if \(A\) and \(B\) are perfectly correlated.

\(\text{Corr}[A, B] = 0\) if and only if \(A\) and \(B\) are uncorrelated.

\(\text{Corr}[A, B] = -1\) if and only if \(A\) and \(B\) are perfectly anti-correlated.
Definition:

A and B are **positively correlated** if
\[ \text{Cov}[A, B] > 0 \]
(equiv., for non-det. A and B: \[ \text{Corr}[A, B] > 0 \]).

Definition:

A and B are **negatively correlated** if
\[ \text{Cov}[A, B] < 0 \]
(equiv., for non-det. A and B: \[ \text{Corr}[A, B] < 0 \]).

Definition:

A and B are **uncorrelated** if
\[ \text{Cov}[A, B] = 0 \]
(equiv., for non-det. A and B: \[ \text{Corr}[A, B] = 0 \]).

If A and B are uncorrelated, or, even better, negatively correlated then the portfolio of A and B becomes attractive.
Definition: **Standard deviation** \( := \sqrt{\text{Variance}} \)

\( \forall \text{PCRVs } X, \ [\text{SD}[X]] := \sqrt{\text{Var}[X]} \)

\[
\begin{align*}
\text{Var}[2X] &= 4(\text{Var}[X]) \\
\text{SD}[2X] &= 2(\text{SD}[X]) \\
\text{Var}[cX] &= c^2(\text{Var}[X]) \\
\text{SD}[cX] &= |c|(\text{SD}[X])
\end{align*}
\]

**Intuition**: Variance measures risk, but standard deviation measures risk better, because doubling the position really ought only to double the risk.
Definition: **Standard deviation** \( := \sqrt{\text{Variance}} \)

\[ \forall \text{PCRVs } X, \ [\text{SD}[X]] := \sqrt{\text{Var}[X]} \]

\[ \text{Var}[A + B] = (\text{Var}[A]) + (\text{Var}[B]) + 2(\text{Cov}[A, B]) \]

\[ \text{SD}[A + B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A, B])} \]
\[ \text{Corr}[A, B] := \frac{\text{Cov}[A, B]}{\sqrt{\text{Var}[A]} \sqrt{\text{Var}[B]}} \]

\[ \text{SD}[A + B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A, B])} \]
\[
\text{Corr}[A, B] := \frac{\text{Cov}[A, B]}{\sqrt{\text{Var}[A]} \sqrt{\text{Var}[B]}}
\]

\[
\text{SD}[A + B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A, B])}
\]

Assume \( \text{Corr}[A, B] = 1 \).

Then \( \text{Cov}[A, B] = 1 \sqrt{\text{Var}[A]} \sqrt{\text{Var}[B]} \)
\[= (\text{SD}[A])(\text{SD}[B]).\]

Then \( \text{SD}[A + B] = \sqrt{[(\text{SD}[A]) + (\text{SD}[B])]^2} \)
\[= (\text{SD}[A]) + (\text{SD}[B]).\]

For perfectly correlated PCRVs, standard deviations add.
$$\text{Corr}[A, B] := \frac{\text{Cov}[A, B]}{\sqrt{\text{Var}[A]} \sqrt{\text{Var}[B]}}$$

$$\text{Var}[A + B] = (\text{Var}[A]) + (\text{Var}[B]) + 2(\text{Cov}[A, B])$$

Assume $\text{Corr}[A, B] = 0$.

Then $\text{Cov}[A, B] = 0 \cdot \sqrt{\text{Var}[A]} \sqrt{\text{Var}[B]} = 0$.

Then $\text{Var}[A + B] = (\text{Var}[A]) + (\text{Var}[B])$

For uncorrelated PCRVs, variances add.

For perfectly correlated PCRVs, standard deviations add.