Financial Mathematics

Piecewise constant processes

$\boxed{\Delta t\text{-PCP}}$

Definition: Fix $\Delta t > 0$.

A $\boxed{\textbf{$\Delta t$-piecewise constant process}}$ is a function $X : [0,\infty) \times [0,1] \to \mathbb{R}$

such that

for all $t \in [0,\infty)$,

$$X(t,\cdot) : [0,1] \to \mathbb{R} \text{ is a PCRV,}$$

Notation: $\boxed{X_t} := X(t,\cdot)$

and such that

for all $n \in \{0,1,2,3,\ldots\}$,

for all $t,u \in [\quad n(\Delta t) \quad , \quad (n+1)(\Delta t) \quad)$,

$$X(t,\cdot) = X(u,\cdot).$$

$\boxed{X_t = X_u}$

Idea: An evolving PCRV, but can only change at times $\Delta t, 2(\Delta t), 3(\Delta t), \ldots$.

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for all $t, u \in [\ n(\Delta t)\ ,\ (n + 1)(\Delta t)\ )$,

$X(t, \cdot) = X(u, \cdot)$. $\boxed{X_t = X_u}$

Remark: If X is a $\boxed{2}$-PCP,

then X is also $\boxed{1}$-PCP.

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Remark: If X is a Δt-PCP,

then X is also $\frac{1}{2}(\Delta t)$-PCP.

PCP

Definition:

A **piecewise constant process** consists of:

a positive number Δt,

together with

a Δt-PCP.

the **step** of the PCP

Notation:

If $(\Delta t, X)$ is a PCP,

then we often use X to denote $(\Delta t, X)$.

Definition:

If X is a PCP of step Δt,

then $\boxed{\Delta X}$ is the PCP of step Δt defined by

$$(\Delta X)_t = X_{t+\Delta t} - X_t$$

Remark: If X is a Δt-PCP,

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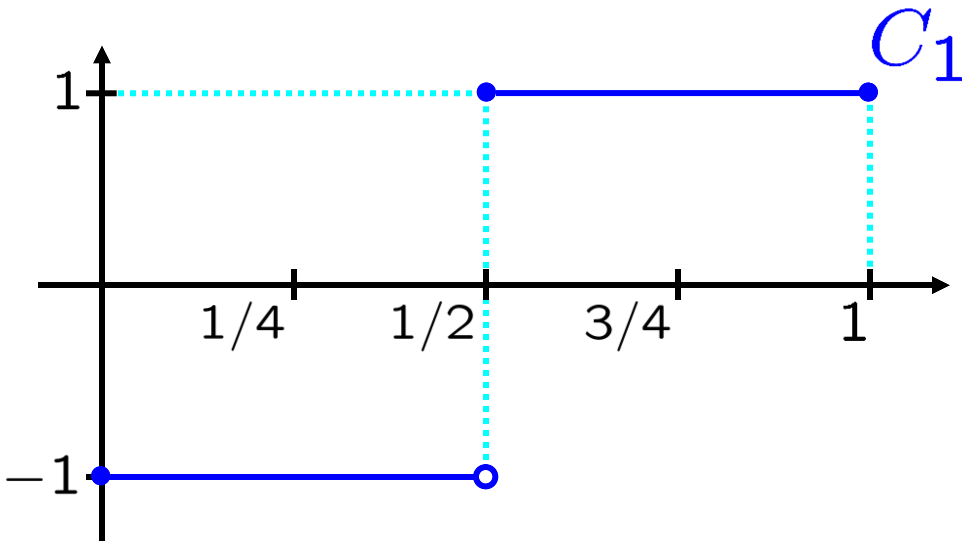
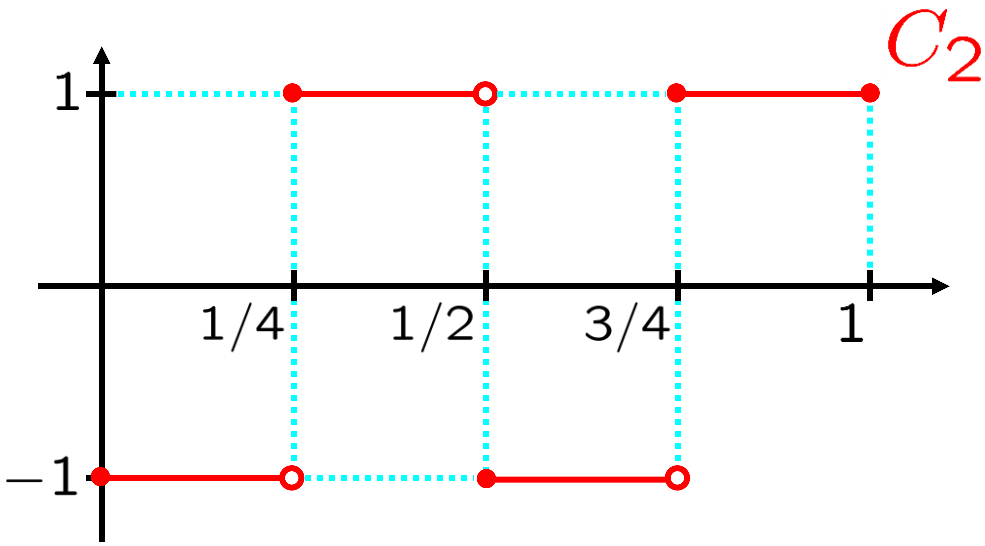
PCRV | Distribution



C_1

1	$1/2$
-1	$1/2$

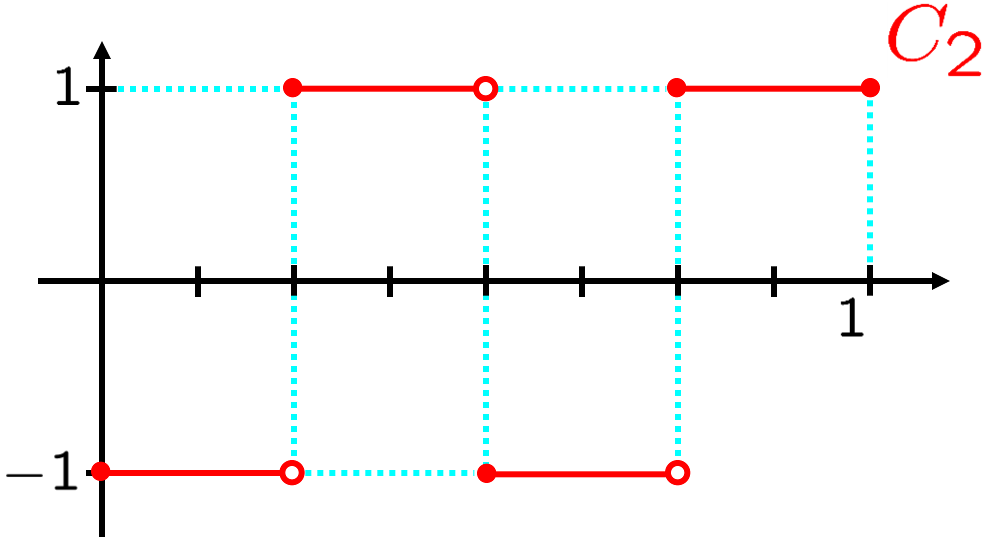


C_2

1	$1/2$
-1	$1/2$

Note: C_1 and C_2 are identically distributed, but are not equal.

PCRV | Distribution



C_2

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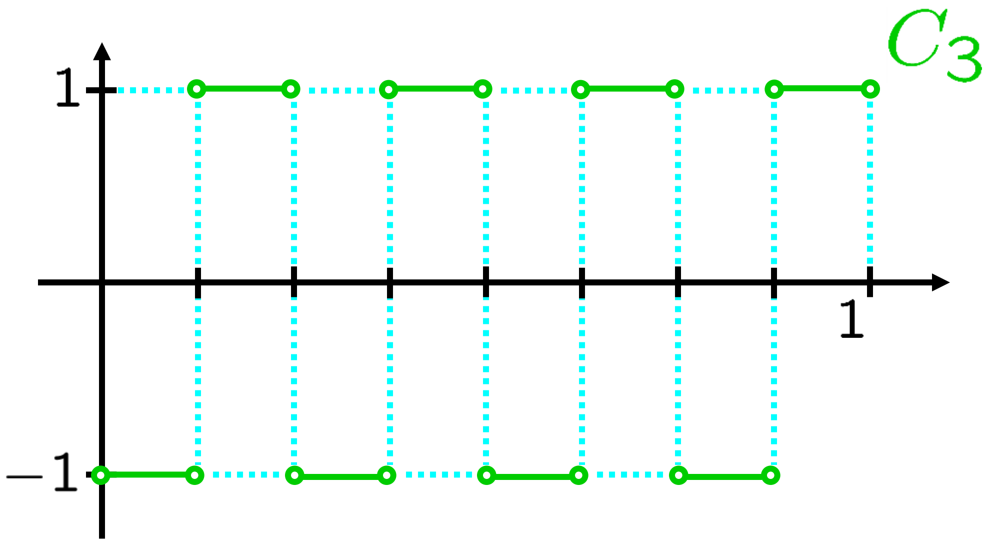

C_3

1	1/2
-1	1/2

Note: C_1, C_2, C_3 are $\boxed{\text{i.i.d.}}$ ← independent, identically distributed

Note: C_1, C_2, C_3 are $\boxed{\text{i.i.d.}}$ independent, identically distributed

Continuing, we can form an i.i.d. sequence

$$C_1, C_2, C_3, C_4, C_5, \ldots$$

of PCRVs, modeling a sequence of coin-flips:

Tyche picks $\omega \in \Omega := [0,1]$ at random.

Knowledge of any finite number of $C_1(\omega), C_2(\omega), \ldots$ tells us nothing about any other.

Knowledge of any finite number of coin flips tells us nothing about any other.

Note: C_1, C_2, C_3 are $\boxed{\text{i.i.d.}}$ independent, identically distributed

Continuing, we can form an i.i.d. sequence

$$C_1, C_2, C_3, C_4, C_5, \ldots$$

of PCRVs.

Let $M := 10^{10^{100}}$ = a googol plex.
Let $\Delta t := 1/M$.
Let X be the PCP of step Δt defined by

$$\begin{aligned}
X_0 &= 0, \\
X_{\Delta t} &= [\sqrt{\Delta t}][C_1], \\
X_{2(\Delta t)} &= [\sqrt{\Delta t}][C_1 + C_2], \\
X_{3(\Delta t)} &= [\sqrt{\Delta t}][C_1 + C_2 + C_3], \\
&\vdots
\end{aligned}$$

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Without these, X_1 would be very large.
With these, X_1 is "approx. normally distributed".
That is, $\forall$ "reasonable" g, we have:

$$\mathrm{E}[g(X_1)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)][e^{-x^2/2}]\,dx$$

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\end{aligned}
$$

CENTRAL LIMIT THEOREM

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PCRV approximations

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is **bounded** if there is some $C > 0$ such that, for all $x \in \mathbb{R}$,

$$|f(x)| \leq C.$$

Definition:

A **PCRV approximation** is a sequence $X^{(1)}, X^{(2)}, \ldots$ of PCRVs such that,

$\forall$continuous, $\boxed{\text{bdd}}$ $f : \mathbb{R} \to \mathbb{R}$,

bounded

the sequence $\mathrm{E}[f(X^{(1)})], \mathrm{E}[f(X^{(2)})], \ldots$ is convergent.

Multivariable boundedness

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is **bounded** if there is some $C > 0$ such that, for all $x \in \mathbb{R}$,

$$|f(x)| \leq C.$$

Definition: A function $f : \mathbb{R}^k \to \mathbb{R}$ is **bounded** if there is some $C > 0$ such that, for all $x \in \mathbb{R}^k$,

$$|f(x)| \leq C.$$

Definition: A function $f : X \to \mathbb{R}$ is **bounded** if there is some $C > 0$ such that, for all $x \in X$,

$$|f(x)| \leq C.$$

PCP approximations

Definition:

A **PCP approximation** is a sequence $X^{(1)}, X^{(2)}, \ldots$ of PCPs such that,

$\forall k$, $\forall t_1, \ldots, t_k \in [0, \infty)$,

bounded

$\forall$continuous $\boxed{\text{bdd}}$ $f : \mathbb{R}^k \to \mathbb{R}$,

the sequence

$$\mathsf{E}[f(X_{t_1}^{(1)}, \ldots, X_{t_k}^{(1)})],$$
$$\mathsf{E}[f(X_{t_1}^{(2)}, \ldots, X_{t_k}^{(2)})],$$
$$\mathsf{E}[f(X_{t_1}^{(3)}, \ldots, X_{t_k}^{(3)})],$$
$$\ldots$$

is convergent.

For every $N = 1, 2, 3, \ldots$,

let $X^{(N)}$ be the PCP of step $1/N$ def'd by

$$X_0^{(N)} = 0,$$

$$X_{1/N}^{(N)} = C_1/\sqrt{N},$$

$$X_{2/N}^{(N)} = [C_1 + C_2]/\sqrt{N},$$

$$X_{3/N}^{(N)} = [C_1 + C_2 + C_3]/\sqrt{N},$$

$$\vdots \qquad\qquad \vdots$$

$$X_t^{(N)} = \frac{C_1 + \cdots + C_?}{\sqrt{N}}$$

$$? = \lfloor tN \rfloor$$

Def'n: The above PCP approximation is the **standard $\boxed{\text{Brownian Motion}}$ approximation**.

$\boxed{\text{BM}}$

Problem: Compute $\lim_{N\to\infty} \mathsf{E}[(e^{X_t^{(N)}} - 5)_+]$.

$X_t^{(N)}$:= std BM approx

Subproblem: Compute $\lim_{N\to\infty} \mathsf{E}[(e^{X_3^{(N)}} - 5)_+]$.

Solution: $$X_3^{(N)} = \frac{C_1 + \cdots + C_{3N}}{\sqrt{N}}$$

$$\frac{X_3^{(N)}}{\sqrt{3}} = \frac{C_1 + \cdots + C_{3N}}{\sqrt{3N}}$$

$$\mathsf{E}\left[\phi\left(\boxed{\frac{X_3^{(N)}}{\sqrt{3}}}\right)\right] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\, dx$$

$$\phi(x) = (e^{\sqrt{3}x} - 5)_+$$

Problem: Compute $\lim_{N\to\infty} \mathrm{E}[(e^{X_t^{(N)}} - 5)_+]$.

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$$\boxed{e^{-a\sqrt{3}} - 5 = 0}$$

$$\mathrm{E}[(e^{X_3^{(N)}} - 5)_+]$$

$$\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[(e^{\sqrt{3}x} - 5)_+]} [e^{-x^2/2}] \, dx$$

$$\mathrm{E}\left[\phi\left(\boxed{\frac{X_3^{(N)}}{\sqrt{3}}}\right)\right] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[\phi(x)]} [e^{-x^2/2}] \, dx$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} [(e^{\sqrt{3}x} - 5)_+][e^{-x^2/2}]\,dx$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} [e^{\sqrt{3}x} - 5][e^{-x^2/2}]\, dx \qquad \boxed{e^{-a\sqrt{3}} - 5 = 0}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} [e^{\sqrt{3}x}][e^{-x^2/2}]\, dx - 5[\Phi(a)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} [e^{\sqrt{3}x} - 5][e^{-x^2/2}]\, dx$$

Problem: Compute $\lim_{N\to\infty} \mathrm{E}[(e^{X_t^{(N)}} - 5)_+]$.

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$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} [e^{\sqrt{3}x}][e^{-x^2/2}]\,dx - 5[\Phi(a)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a-\sqrt{3}}^{\infty} [e^{(\sqrt{3})^2/2}][e^{-x^2/2}]\,dx - 5[\Phi(a)]$$

$$= \frac{e^{3/2}}{\sqrt{2\pi}} \int_{-a-\sqrt{3}}^{\infty} [e^{-x^2/2}]\,dx - 5[\Phi(a)]$$

$$= e^{3/2}[\Phi(a+\sqrt{3})] - 5[\Phi(a)]$$

Problem: Compute $\lim_{N\to\infty} E[(e^{X_t^{(N)}} - 5)_+]$.

$X_t^{(N)}$:= std BM approx

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$$e^{-a\sqrt{3}} - 5 = 0$$

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$X_t^{(N)} :=$ std BM approx

Subproblem: Compute $\lim_{N\to\infty} \mathrm{E}[(e^{X_3^{(N)}} - 5)_+]$.

$$= e^{3/2}[\Phi(a+\sqrt{3})] - 5[\Phi(a)]$$

$$= e^{3/2}\left[\Phi\left(-\frac{\ln 5}{\sqrt{3}} + \sqrt{3}\right)\right] - 5\left[\Phi\left(-\frac{\ln 5}{\sqrt{3}}\right)\right]$$

$$e^{-a\sqrt{3}} = 5$$

$$-a\sqrt{3} = \ln 5$$

$$a = -\frac{\ln 5}{\sqrt{3}}$$

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$$e^{-a\sqrt{3}} = 5$$
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Problem: Compute $\lim_{N\to\infty} \mathsf{E}[(e^{X_{\boxed{t}}^{(N)}} - 5)_+]$.

π

$$\lim_{N\to\infty} \mathsf{E}[(e^{X_t^{(N)}} - 5)_+] \overset{??}{=} e^{t/2}\left[\Phi\left(-\frac{\ln 5}{\sqrt{t}} + \sqrt{t}\right)\right] - 5\left[\Phi\left(-\frac{\ln 5}{\sqrt{t}}\right)\right]$$

$$\lim_{N\to\infty} \mathsf{E}[(e^{X_3^{(N)}} - 5)_+] = e^{3/2}\left[\Phi\left(-\frac{\ln 5}{\sqrt{3}} + \sqrt{3}\right)\right] - 5\left[\Phi\left(-\frac{\ln 5}{\sqrt{3}}\right)\right]$$

$$e^{-a\sqrt{3}} = 5$$
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Problem: Compute $\lim_{N\to\infty} \mathsf{E}[(e^{X_{\boxed{t}}^{(N)}} - 5)_+]$.

π

$X_t^{(N)}$:= std BM approx

Solution: $$X_\pi^{(N)} = \frac{C_1 + \cdots + C_{\lfloor \pi N \rfloor}}{\sqrt{N}}$$

$\sqrt{\lfloor \pi N \rfloor}$

$$\frac{X_\pi^{(N)}}{\sqrt{\pi}} \sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}} = \frac{C_1 + \cdots + C_{\lfloor \pi N \rfloor}}{\sqrt{\lfloor \pi N \rfloor}}$$

$$\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}} \sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}\right)\right] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\, dx$$

$$\phi(x) = (e^{\sqrt{\pi}x} - 5)_+$$

Problem: Compute $\lim_{N\to\infty} \mathsf{E}[(e^{X_t^{(N)}} - 5)_+]$.

evaluate the integral just as before

$$\boxed{\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\right)\right]} \to \boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\,dx}$$

ADD THE LAST TWO LIMITS

$$\boxed{\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\right)\right]} \quad - \quad \cancel{\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}\right)\right]} \xrightarrow{\boxed{\text{IOU}}} 0$$

$$\cancel{\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}\right)\right]} \xrightarrow{\text{CLT}} \boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\,dx}$$

$$\phi(x) = (e^{\sqrt{\pi}x} - 5)_+$$

$$\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\right)\right] \quad - \quad \mathsf{E}\left[\phi\left(\underbrace{\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}}_{\substack{\| \\ : \\ S_N}}\right)\right] \overset{\text{IOU}}{\longrightarrow} 0$$

$$\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\right)\right] \quad - \quad \mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}\right)\right] \overset{\text{IOU}}{\longrightarrow} 0$$

$$\mathsf{E}\left[\phi\left(\underbrace{\frac{X_\pi^{(N)}}{\sqrt{\pi}}}_{\displaystyle \| \atop \displaystyle S_N \cdot t_N}\right)\right] - \mathsf{E}\left[\phi\left(\underbrace{\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}}_{\displaystyle \| \atop \displaystyle \ddot{S}_N}\right)\right] \overset{\mathsf{IOU}}{\longrightarrow} 0$$

$$\boxed{t_N := \sqrt{\frac{\lfloor \pi N \rfloor}{\pi N}}}$$

$$\boxed{\begin{array}{l} M := 10^{10^{100}} \\ \alpha(x) = Me^{M|x|} \end{array}}$$

$$-|Y| \le Y \le |Y|$$

$$-\mathsf{E}[|Y|] \le \mathsf{E}[Y] \le \mathsf{E}[|Y|]$$

$$\mathsf{E}[|Y|] \ge |\mathsf{E}[Y]|$$

$$\mathsf{E}[|(\phi(S_N \cdot t_N)) - (\phi(S_N))|] \overset{??}{\longrightarrow} 0$$

$$|\vee$$

Want: $|\mathsf{E}[(\phi(S_N \cdot t_N)) - (\phi(S_N))]| \to 0$

$$|\vee$$

$$0$$

Want: $\mathsf{E}[|(\phi(S_N \cdot t_N)) - (\phi(S_N))|] \to 0$

$$\underbrace{\mathsf{E}\left[\phi\left(\frac{X_\pi^{(N)}}{\sqrt{\pi}}\right)\right]}_{\textstyle \| \atop S_N \cdot t_N} - \mathsf{E}\left[\phi\left(\underbrace{\frac{X_\pi^{(N)}}{\sqrt{\pi}}\sqrt{\frac{\pi N}{\lfloor \pi N \rfloor}}}_{\textstyle \| \atop \because S_N}\right)\right] \overset{\text{IOU}}{\longrightarrow} 0$$

$$\boxed{t_N := \sqrt{\frac{\lfloor \pi N \rfloor}{\pi N}}} \searrow 1$$

$$\boxed{\begin{aligned} M &:= 10^{10^{100}} \\ \alpha(x) &= M e^{M|x|} \end{aligned}}$$

$$\mathsf{E}[\alpha(S_N)] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\alpha(x)][e^{-x^2/2}]\, dx < \infty$$

Want: $\mathsf{E}[(\alpha(S_N))\boxed{(1 - t_N)}] \longrightarrow 0$ ☺

$$|(\phi(S_N \cdot t_N)) - (\phi(S_N))| \leq (\alpha(S_N))(1 - t_N)$$

$$\boxed{\text{IOU}}$$

$\forall s \in \mathbb{R}, \forall t \in (0,1), |(\phi(st)) - (\phi(s))| \leq (\alpha(s))(1 - t)$

Want: $\mathsf{E}[|(\phi(S_N \cdot t_N)) - (\phi(S_N))|] \to 0$

$$\phi(x) = (e^{\sqrt{\pi}x} - 5)_+ \qquad |a_+ - b_+| \le |a - b|$$

$$|(\phi(x)) - (\phi(y))| \le |(e^{\sqrt{\pi}x} - \cancel{5}) - (e^{\sqrt{\pi}y} - \cancel{5})|$$

$$\le |e^{\sqrt{\pi}x} - e^{\sqrt{\pi}y}|$$

$\exists z \in (x, y)$ s.t. $$\frac{|(\phi(x)) - (\phi(y))|}{|x - y|} \le \sqrt{\pi} e^{\sqrt{\pi}z}$$

$$\boxed{\begin{array}{l} M := 10^{10^{100}} \\ \alpha(x) = M e^{M|x|} \end{array}}$$

$$\frac{|(\phi(x)) - (\phi(y))|}{|x - y|} \le \left| \frac{e^{\sqrt{\pi}x} - e^{\sqrt{\pi}y}}{x - y} \right| \overset{\text{MVT}}{=} \left| \sqrt{\pi} e^{\sqrt{\pi}z} \right|,$$

for some $z \in (x, y)$

$$\forall s \in \mathbb{R}, \forall t \in (0, 1), |(\phi(st)) - (\phi(s))| \overset{\text{IOU}}{\le} (\alpha(s))(1 - t)$$

$$\phi(x) = (e^{\sqrt{\pi}x} - 5)_+ \qquad |a_+ - b_+| \leq |a - b|$$

$$|(\phi(x)) - (\phi(y))| \leq |(e^{\sqrt{\pi}x} - \cancel{5}) - (e^{\sqrt{\pi}y} - \cancel{5})|$$

$$\leq |e^{\sqrt{\pi}x} - e^{\sqrt{\pi}y}|$$

$\exists z \in (x, y)$ s.t.

$$\frac{|(\phi(x)) - (\phi(y))|}{|x - y|} \leq \sqrt{\pi}e^{\sqrt{\pi}z}$$

$$\leq \sqrt{\pi}e^{\sqrt{\pi}a}$$

$$\boxed{\begin{aligned} M &:= 10^{10^{100}} \\ \alpha(x) &= Me^{M|x|} \end{aligned}}$$

$|s| \geq st$

$$\left.\begin{aligned} a &\geq x \\ &\text{and} \\ a &\geq y \end{aligned}\right\} \Rightarrow |(\phi(x)) - (\phi(y))| \leq \sqrt{\pi}e^{\sqrt{\pi}a}|x - y|$$

$|s| \geq s$

$$\boxed{x := st \qquad y := s \qquad a := |s|}$$

$$\forall s \in \mathbb{R}, \forall t \in (0, 1), |(\phi(st)) - (\phi(s))| \leq (\alpha(s))(1 - t)$$

$$
\begin{aligned}
|(\phi(st)) - (\phi(s))| &\le \sqrt{\pi}e^{\sqrt{\pi}|s|}|st - s| \\
&\le \sqrt{\pi}e^{\sqrt{\pi}|s|}\boxed{|s|} \cdot \boxed{|t-1|} \\
&\le \sqrt{\pi}\boxed{e^{\sqrt{\pi}|s|}}\boxed{e^{|s|}}(\boxed{1-t}) \\
&\le \sqrt{\pi}\boxed{e^{(1+\sqrt{\pi})|s|}}(1-t) \\
&\le \boxed{Me^{M|s|}}(1-t) \\
&= (\boxed{\alpha(s)})(1-t)
\end{aligned}
$$

$$\boxed{\forall x \ge 0,\quad x \le e^x}$$

$$\boxed{\begin{array}{l} M := 10^{10^{100}} \\ \alpha(\overset{s}{x}) = Me^{M|\overset{s}{x}|} \end{array}}$$

$$
\left.\begin{array}{c} |s| \ge st \\ a \ge x \\ \text{and} \\ a \ge y \\ |s| \ge s \end{array}\right\} \Rightarrow |(\phi(x)) - (\phi(y))| \le \sqrt{\pi}e^{\sqrt{\pi}a}|x-y|
$$

$$\boxed{x := st \qquad y := s \qquad a := |s|}$$

IOU

$$\forall s \in \mathbb{R}, \forall t \in (0,1), |(\phi(st)) - (\phi(s))| \le (\alpha(s))(1-t)$$

Definition: For any PCP Y of step Δt, ΔY is the PCP of step Δt defined by: $(\Delta Y)_t = Y_{t+\Delta t} - Y_t$.

Let $C_1, C_2, C_3, \ldots$ be our usual i.i.d. PCRVs with distr.: 50% prob. of 1, 50% prob. of -1.

For all integers $N \geq 1$, $\Delta t^{(N)} := 1/N$

$W^{(N)}$ is the PCP of step $1/N$ defined by:

$$\Delta W_{k/N}^{(N)} = C_k/\sqrt{N}, \quad \forall \text{integers } k \geq 0.$$

$$W_0^{(N)} = 0,$$

Problem: For all integers $N \geq 1$, find a PCP $X^{(N)}$ s.t.

$$\Delta X^{(N)} = [(0.01)(\Delta W^{(N)}) + (0.05)(\Delta t^{(N)})]X^{(N)},$$

$$[X^{(N)}]_{t=0} = 1.$$

The idea:

Every Nth of a year, we add on interest,
but the rate is random:

$$\text{It's } \pm\frac{0.01}{\sqrt{N}} + \frac{0.05}{N},$$

with 50% chance of $+$
and 50% chance of $-$.

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Fact: As $N \to \infty$, the PCPs $X^{(N)}$ converge to a "process" X, and we say that X satisfies the SDE:

$$dX = [(0.01)(dW) + (0.05)(dt)]X$$

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The "random bank": CRR model or discrete Black-Scholes model

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One sol'n involves the STOCHASTIC CHAIN RULE, a.k.a. ITO'S LEMMA.

Sometimes written:

$$dX = [\qquad\qquad\qquad (0.05)(dt)]X$$

Cf: $$\frac{dX}{dt} = (0.05)X$$

One sol'n involved the CHAIN RULE.

STOP