Financial Mathematics

The heat equation

Definition:

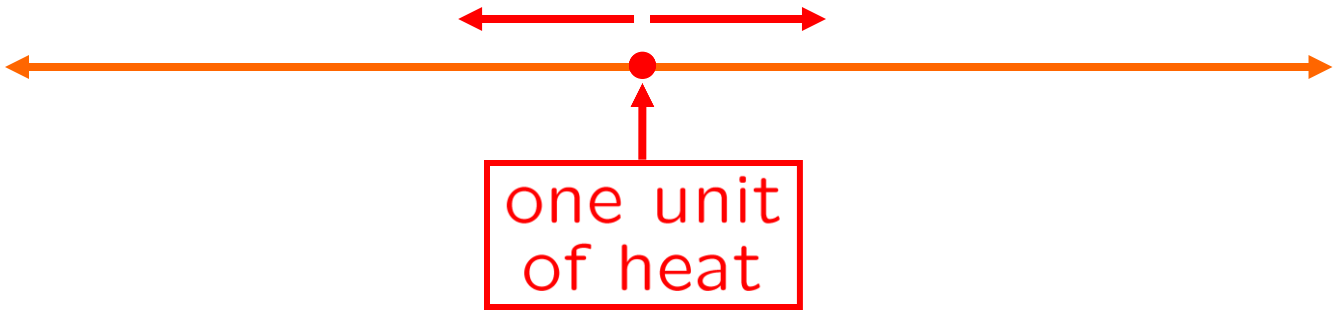
For any "reasonable" $p, q : \mathbb{R} \to \mathbb{R}$,
the **convolution of** p **and** q is the function
$p * q : \mathbb{R} \to \mathbb{R}$ defined by

$$(p * q)(x) = \int_{-\infty}^{\infty} [p(x - y)][q(y)] \, dy$$

Intuition: Translate $p(x)$ to right y units,
giving $p(x - y)$.
Weight the result by $q(y) \, dy$,
giving $[p(x - y)][q(y)] \, dy$.
Add up the results, via $\int_{-\infty}^{\infty}$.

SKILL: Given "reasonable" p, q, find $p * q$.

Heat on an infinite bar



$f_t(x)$ =

$f(x,t) :=$ the temp. at x at time t.

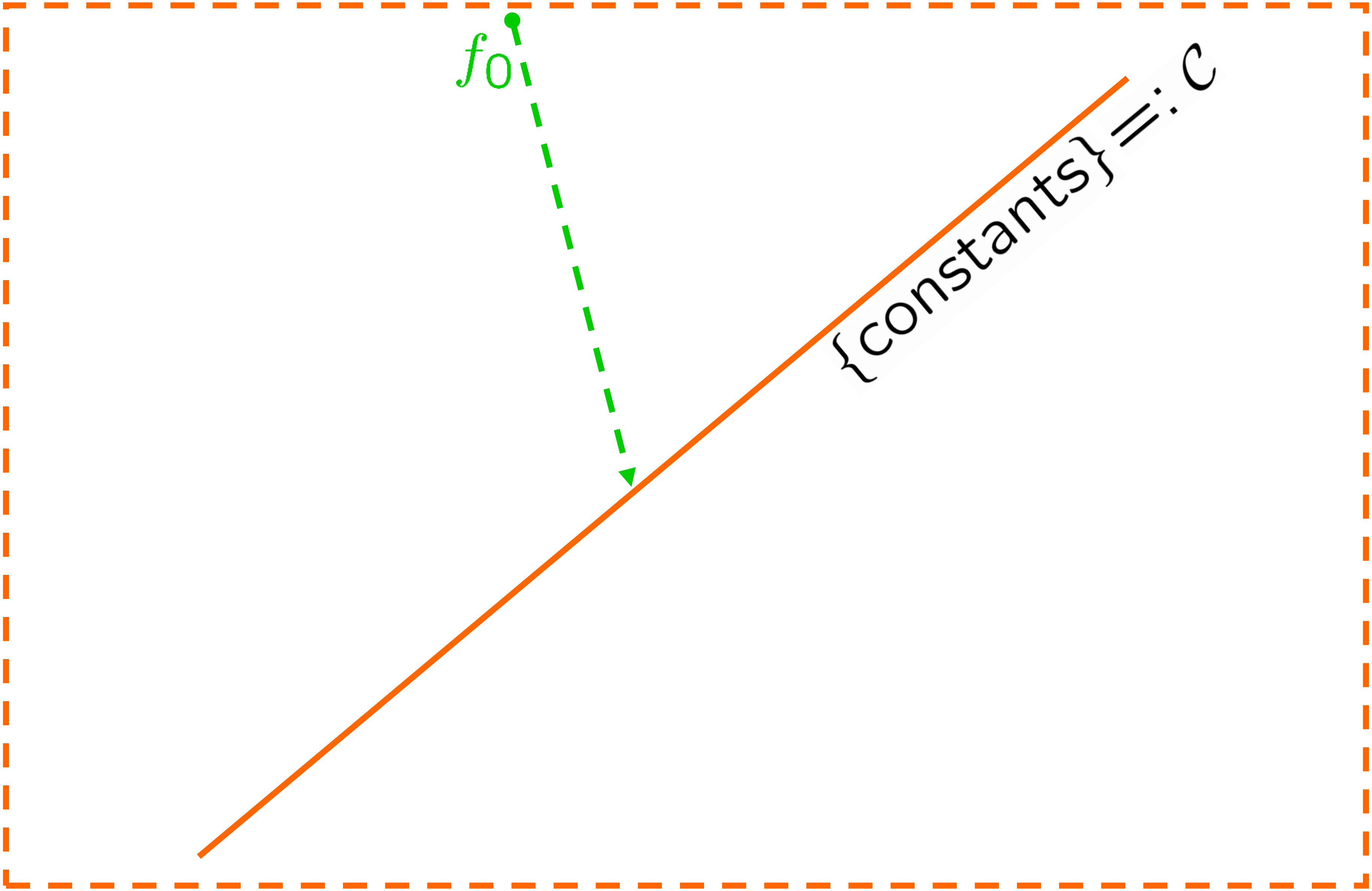
The heat distribution flows
inside of a function space
toward being evenly distributed.

$$t \mapsto f_t : (0,\infty) \to \overbrace{\{\text{inf. diff. functions } \mathbb{R} \to \mathbb{R}\}}^{\text{smooth}}$$

$$\cup\!|$$

$$\{\text{constants}\}$$

$\mathcal{F} := \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\}$



Question: How to travel toward $\mathcal{C}$?

$$\mathcal{F} := \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\}$$

$$\cup\!|$$

$$\mathcal{C} := \{\text{constants } \mathbb{R} \to \mathbb{R}\}$$

Let $\mathcal{D} := \mathcal{F} \to \mathcal{F}$ be defined by $\mathcal{D}f = f'$.

Then $\mathcal{C} = \ker(\mathcal{D})$, *i.e.*: $f \in \mathcal{C}$ iff $\mathcal{D}f = 0$.

If $L : V \to W$ is linear, how can we naturally "travel" in V toward $\ker(L)$?

One method:

Find a smooth function $H : V \to \mathbb{R}$, which is 0 on $\ker(L)$ and positive off $\ker(L)$.

Then flow along $-\nabla H$.

How can we create this $H : V \to \mathbb{R}$?

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Define $H : V \to \mathbb{R}$ by $H(v) = |L(v)|^2$.

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Define $H : V \to \mathbb{R}$ by $H(v) = |L(v)|^2$.

Note: $v \mapsto |L(v)|$ isn't smooth.

Note: $v \mapsto |L(v)|^2/2$ is often preferred, because it has a slightly simpler grad.

If $L : V \to W$ is linear, how can we naturally "travel" in V toward ker(L)?

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How can we calculate ∇H?

Then flow along $-\nabla H$.

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How can we calculate ∇H?

Recall:

$\nabla H : V \to V$ is defined by

$$[(\nabla H)(v)] \cdot v' = [d/dt]_{t=0}[\underbrace{H(v + tv')}_{|L(v+tv')|^2}]$$

$$|L(v + tv')|^2$$

$$\|$$

$$\begin{matrix} |L(v)|^2 + t^2|L(v')|^2 \\ +2t[L(v)] \cdot [L(v')] \end{matrix} = \begin{matrix} |L(v)|^2 + |L(tv')|^2 \\ +2[L(v)] \cdot [L(tv')] \end{matrix}$$

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$$= 2[L(v)] \cdot [L(v')]$$

$$= 2[L^t L(v)] \cdot v'$$

If $L : V \to W$ is linear, how can we naturally "travel" in V toward $\ker(L)$?

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$$2[L^t L(v)] \cdot v'$$

If $L : V \to W$ is linear, how can we naturally "travel" in V toward $\ker(L)$?

One method: divide by 2

Define $H : V \to \mathbb{R}$ by $H(v) = |L(v)|^2$. ←

Then flow along $-\nabla H$.

How can we calculate ∇H?

Recall:

$\nabla H : V \to V$ is defined by

$$[(\nabla H)(v)] \cdot v' = 2[L^t L(v)] \cdot v' \quad \longleftarrow \quad \forall v, v'$$

$$(\nabla H)(v) = 2[L^t L(v)] \quad \longleftarrow \quad \forall v$$

$$\nabla H = 2[L^t L]$$

If $L : V \to W$ is linear, how can we naturally "travel" in V toward $\ker(L)$?

One method:

divide by 2

Define $H : V \to \mathbb{R}$ by $H(v) = |L(v)|^2/2$.

Then flow along $-\nabla H$.

How can we calculate ∇H?

Recall:

$\nabla H : V \to V$ is defined by

$$[(\nabla H)(v)] \cdot v' = 2[L^t L(v)] \cdot v'/2 \qquad \forall v, v'$$

$$(\nabla H)(v) = 2[L^t L(v)]/2 \qquad \forall v$$

$$\nabla H = L^t L$$

$$\mathcal{F} := \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\}$$

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Let $\mathcal{D} := \mathcal{F} \to \mathcal{F}$ be defined by $\mathcal{D}f = f'$.

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Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.

Then flow along $-\nabla\mathcal{H}$.

$$\mathcal{D}(f) \in \mathcal{F}$$

Question: What is meant by $|\mathcal{D}(f)|^2$?

Question: $\exists$a "dot product" in $\mathcal{F}$?

$\mathcal{F} := \{$smooth functions $\mathbb{R} \to \mathbb{R}\}$

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N very large

Question: Is there a "dot product" in $\mathbb{R}^N$?

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N very large

Question: Is there a "dot product" in $\mathbb{R}^N$?

$$v \cdot w = v_1 w_1 + \cdots + v_N w_N$$

$$f \cdot g = \int_{-\infty}^{\infty} [f(x)][g(x)]\, dx$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} [f(x)][g(x)]\, dx$$

The L^2 "scalar product"

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Question: $\exists$a "dot product" in $\mathcal{F}$?

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The L^2 "scalar product"

Warning: Technical problem: What if $[f(x)][g(x)]$ is not integrable?

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The L^2 "scalar product"

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Definition: A function $f : \mathbb{R} \to \mathbb{R}$ has **compact support** if there is some $C > 0$ such that, for all $x \in \mathbb{R} \backslash [-C, C]$, $f(x) = 0$.

"f vanishes off some compact interval"

$\mathcal{F} := \{$smooth functions $\mathbb{R} \to \mathbb{R}\}$

Question: $\exists$a "dot product" in $\mathcal{F}$?

$$\langle f, g \rangle = \int_{-\infty}^{\infty} [f(x)][g(x)]\, dx$$

The L^2 "scalar product"

Warning: Technical problem: What if $[f(x)][g(x)]$ is not integrable?

$\mathcal{F}_C := \{$smooth functions $\mathbb{R} \to \mathbb{R}$ with compact support$\}$

$\langle f, g \rangle$ is defined only for $f \in \mathcal{F}$, $g \in \mathcal{F}_C$ and $f \in \mathcal{F}_C$, $g \in \mathcal{F}$.

$$\mathcal{F} := \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\} \supseteq \mathcal{F}_C$$

$$\cup|$$

$$\mathcal{C} := \{\text{constants } \mathbb{R} \to \mathbb{R}\}$$

Let $\mathcal{D} := \mathcal{F} \to \mathcal{F}$ be defined by $\mathcal{D}f = f'$.

Then $\mathcal{C} = \ker(\mathcal{D})$, *i.e.*: $f \in \mathcal{C}$ iff $\mathcal{D}f = 0$.

How can we naturally

"travel" in $\mathcal{F}$ toward $\ker(\mathcal{D})$?

One method:

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.

Then flow along $-\nabla\mathcal{H}$.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} [f(x)][g(x)]\, dx,$$

$\forall f \in \mathcal{F}, \forall g \in \mathcal{F}_C$

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$$\langle f, g \rangle = \int_{-\infty}^{\infty} [f(x)][g(x)]\, dx, \qquad \forall f \in \mathcal{F}, \forall g \in \mathcal{F}_C$$

or vice versa

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.
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$$|f| := \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\infty}^{\infty} [f(x)]^2\, dx}, \qquad \forall f \in \mathcal{F}_C$$

$$\langle (\nabla\mathcal{H})(f), g \rangle = [d/dt]_{t=0}[\mathcal{H}(f + tg)]$$

$$= [d/dt]_{t=0}[|\mathcal{D}(f + tg)|^2/2]$$

$$= [d/dt]_{t=0}[|\mathcal{D}(f)|^2/2 + |\mathcal{D}(tg)|^2/2 + 2\langle \mathcal{D}(f), \mathcal{D}(tg) \rangle/2]$$

$\forall f, g \in \mathcal{F}_C$

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or vice versa

$$|f| := \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\infty}^{\infty} [f(x)]^2\, dx}, \qquad \forall f \in \mathcal{F}_C$$

$$\langle (\nabla\mathcal{H})(f), g \rangle = \langle \mathcal{D}(f), \mathcal{D}(g) \rangle, \qquad \forall f, g \in \mathcal{F}_C$$

$$= \langle \mathcal{D}^t \mathcal{D}(f), g \rangle$$

meaning?

$$= \langle \mathcal{D}(f), \mathcal{D}(g) \rangle,$$

Question: Meaning of $\mathcal{D}^t$????? $\forall f, g \in \mathcal{F}_C$

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.
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$\forall f, g \in \mathcal{F}_C,$ $\qquad \langle f, \mathcal{D}(g)\rangle$

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$\forall f, g \in \mathcal{F}_C,$

$$\langle f, \mathcal{D}(g) \rangle$$

$$= \int_{-\infty}^{\infty} [f(x)][g'(x)]\, dx$$

~~up up~~ minus the integral of new new

$$= -\int_{-\infty}^{\infty} [f'(x)][g(x)]\, dx$$

$$= \langle -\mathcal{D}f, g \rangle$$

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$\forall f, g \in \mathcal{F}_C$, $\qquad \langle f, \mathcal{D}(g)\rangle = \langle -\mathcal{D}f, g\rangle$

Definition: $\quad \mathcal{D}^t := -\mathcal{D}$

$$= \langle -\mathcal{D}f, g\rangle$$

DONE

Question: Meaning of $\mathcal{D}^t$?????

Define $\mathcal{H}:\mathcal{F}\to\mathbb{R}$ by $\mathcal{H}(f)=|\mathcal{D}(f)|^2/2$.
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Definition: $\mathcal{D}^t:=-\mathcal{D}$

Int. by parts says d/dx is antisymmetric

$\forall f,g\in\mathcal{F}_C,\qquad \langle f,\mathcal{D}(g)\rangle=\langle\mathcal{D}^t f,g\rangle$

$$\langle(\nabla\mathcal{H})(f),g\rangle=\langle\mathcal{D}(f),\mathcal{D}(g)\rangle=\langle\mathcal{D}^t\mathcal{D}(f),g\rangle$$

$$\nabla\mathcal{H}=\mathcal{D}^t\mathcal{D}$$

$$=-\mathcal{D}^2$$

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.
Then flow along $-\nabla\mathcal{H}$.

$$\nabla\mathcal{H} = -\mathcal{D}^2$$

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$$\nabla\mathcal{H} = -\mathcal{D}^2$$

$$-\nabla\mathcal{H} = \mathcal{D}^2$$

Then flow along $\mathcal{D}^2$.

Definition:

v_t flows along M if

$$\dot{v}_t = M v_t.$$

$$\dot{f}_t = \mathcal{D}^2 f_t$$

$$\dot{f}_t = f_t''$$

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.
Then flow along $-\nabla\mathcal{H}$.

$$\dot{f}_t = f_t''$$

$$\dot{f}_t(x) = f_t''(x)$$

$$\dot{f}_t = f_t''$$

Define $\mathcal{H} : \mathcal{F} \to \mathbb{R}$ by $\mathcal{H}(f) = |\mathcal{D}(f)|^2/2$.
Then flow along $-\nabla\mathcal{H}$.

$$\dot{f}_t = f_t''$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

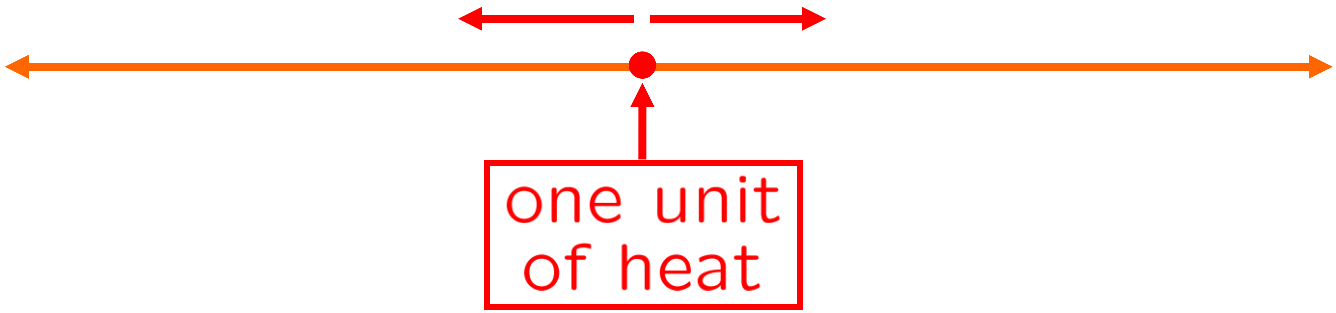
$$\dot{f}_t(x) = f_t''(x)$$

$$f_t(x) = f(x,t)$$

$$\dot{f}_t(x) = \frac{\partial}{\partial t}[f(x,t)]$$

$$f_t''(x) = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Heat on an infinite bar



$f_t(x) =: f(x,t) :=$ the temp. at x at time t.

Another model: The unit of heat breaks up into N parts, with N very large. Each part moves randomly to left or right by $1/\sqrt{N}$ each $1/N$ units of time. Let $N \to \infty$.

Probability thy: THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

SDEs

Feynman-Kač

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

THE HEAT EQUATION

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$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)]e^{-ixX}\,dX$$

$$\frac{\partial}{\partial t}[f(x,t)] = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t}[F(X,t)]\right) e^{-ixX}\,dX$$

$$\frac{\partial}{\partial x}[f(x,t)] = \int_{-\infty}^{\infty} -iX[F(X,t)]e^{-ixX}\,dX$$

$$\frac{\partial^2}{\partial x^2}[f(x,t)] = \int_{-\infty}^{\infty} -X^2[F(X,t)]e^{-ixX}\,dX$$

THE HEAT EQUATION

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Solution
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$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)] e^{-ixX} \, dX$$

$$\frac{\partial}{\partial t}[f(x,t)] = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t}[F(X,t)] \right) e^{-ixX} \, dX$$

$$||$$

inverse Fourier transforms

$$\frac{\partial^2}{\partial x^2}[f(x,t)] = \int_{-\infty}^{\infty} -X^2 [F(X,t)] e^{-ixX} \, dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2 [F(X,t)]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)] e^{-ixX}\, dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2 [F(X,t)]$$

$$H(t) := F(6,t)$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2 [F(X,t)]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)]e^{-ixX}\,dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2[F(X,t)] \longleftarrow X := 6$$

$$H(t) := F(6,t)$$

$$H'(t) = -36[H(t)]$$

$$R(t) := \ln(H(t))$$

$$\frac{d}{dt}\left[H(t) = e^{R(t)} \right]$$

$$[e^{R(t)}][R'(t)] = -36[e^{R(t)}]$$

$$R'(t) = -36$$

$$R(t) = -36t + C$$

$$H(t) = Ke^{-36t}$$

$$K = e^C$$

$$F(6,t) = Ke^{-36t}$$

$$F(X,t) = K_X e^{-X^2 t}$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)] e^{-ixX}\, dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2[F(X,t)]$$

$$F(X,t) = K_X e^{-X^2 t}$$

Pick any expression $Q(X)$ and set

$$F(X,t) = [Q(X)] e^{-X^2 t}.$$

$$F(X,t) = K_X e^{-X^2 t}$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)] e^{-ixX}\, dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2[F(X,t)]$$

DONE

$$F(X,t) = K_X e^{-X^2 t}$$

Pick any expression $Q(X)$ and set

$$F(X,t) = [Q(X)]e^{-X^2 t}.$$

Then $$\frac{\partial}{\partial t}[F(X,t)] = [Q(X)][e^{-X^2 t}][-X^2]$$
$$= -X^2[F(X,t)]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
??????

$$f(x,t) = \int_{-\infty}^{\infty} [F(X,t)] e^{-ixX}\, dX$$

$$\frac{\partial}{\partial t}[F(X,t)] = -X^2[F(X,t)]$$

DONE

$$F(X,t) = K_X e^{-X^2 t}$$

Pick any expression $Q(X)$ and set

$$F(X,t) = [Q(X)]e^{-X^2 t}.$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$(Q(X) = 1)$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX} \, dX$$

$$f(x,t) = \int_{-\infty}^{\infty} e^{-X^2 t} e^{-ixX} \, dX \qquad (Q(X) = 1)$$

$$f(5,7) = \int_{-\infty}^{\infty} e^{-7X^2} e^{-5iX} \, dX$$

$$f(x,t) = \int_{-\infty}^{\infty} e^{-X^2 t} e^{-ixX}\, dX$$

$$f(5,7) = \int_{-\infty}^{\infty} e^{-7X^2} e^{-5iX}\, dX \qquad X \to X - \frac{5i}{14}$$

$$= \int_{-\infty}^{\infty} e^{-7(X-5i/14)^2} e^{-5i(X-5i/14)}\, dX$$

$$e^{-7(2X(-5i/14))} = e^{5iX}$$

$$= \int_{-\infty}^{\infty} e^{-7(X^2+(5i/14)^2)} e^{5i(5i/14)}\, dX$$

$$e^{-5iX}$$

$$= e^{-7(5i/14)^2} e^{(5i)^2/14} \int_{-\infty}^{\infty} e^{-7X^2}\, dX$$

$$f(x,t) = \int_{-\infty}^{\infty} e^{-X^2 t} e^{-ixX}\, dX$$

$$f(5,7) = \int_{-\infty}^{\infty} e^{-7X^2} e^{-5iX}\, dX$$

$$= e^{-7(5i/14)^2} e^{(5i)^2/14} \int_{-\infty}^{\infty} e^{-7X^2}\, dX \qquad X \to \frac{X}{\sqrt{7}}$$

$$= e^{-(14/2)(5i/14)^2} e^{(5i)^2/14} \int_{-\infty}^{\infty} e^{-X^2} \frac{dX}{\sqrt{7}}$$

$$= \frac{e^{-(1/2)(5i)^2/14} e^{(5i)^2/14}}{\sqrt{7}} \int_{-\infty}^{\infty} e^{-X^2}\, dX$$

$$= \frac{e^{(1/2)(5i)^2/14}}{\sqrt{7}} \sqrt{\pi} \quad = e^{(1/2)(5i)^2/14} \sqrt{\pi/7}$$

$$f(x,t) = \int_{-\infty}^{\infty} e^{-X^2 t} e^{-ixX}\, dX$$

$$f(5,7) = \int_{-\infty}^{\infty} e^{-7X^2} e^{-5iX}\, dX$$

$$= e^{(1/2)(5i)^2/14}\sqrt{\pi/7}$$

$$= e^{(1/4)(-5^2)/7}\sqrt{\pi/7}$$

$$f(x,t) = e^{(1/4)(-x^2)/t}\sqrt{\pi/t} \;= \frac{e^{-x^2/(4t)}}{\sqrt{t}}\sqrt{\pi}$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}} \sqrt{\pi} \qquad (Q(X) = 1)$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}} \qquad (Q(X) = 1/\sqrt{\pi})$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}}$$

$$\frac{\partial}{\partial t}f(x,t) = \frac{\sqrt{t}e^{-x^2/(4t)}x^2/(4t^2) - e^{-x^2/(4t)}/(2\sqrt{t})}{t}$$

$$\frac{\partial}{\partial x}f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}}\left[\frac{-2x}{4t}\right]$$

$$\frac{\partial^2}{\partial x^2}f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}}\left[\frac{-2x}{4t}\right]^2 + \frac{e^{-x^2/(4t)}}{\sqrt{t}}\left[\frac{-2}{4t}\right]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

$$\frac{\partial}{\partial t} f(x,t) = \frac{\sqrt{t} e^{-x^2/(4t)} x^2/(4t^2) - e^{-x^2/(4t)}/(2\sqrt{t})}{t}$$

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}} \left[\frac{-2x}{4t}\right]^2 + \frac{e^{-x^2/(4t)}}{\sqrt{t}} \left[\frac{-2}{4t}\right]$$

$$\frac{\sqrt{t} x^2/(4t^2) - 1/(2\sqrt{t})}{t} \overset{?}{=} \frac{1}{\sqrt{t}} \left[\frac{-2x}{4t}\right]^2 + \frac{1}{\sqrt{t}} \left[\frac{-2}{4t}\right]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

$$\frac{\sqrt{t}}{\sqrt{t}}\,\frac{\sqrt{t}x^2/(4t^2) - 1/(2\sqrt{t})}{t} \stackrel{?}{=} \frac{1}{\sqrt{t}}\left[\frac{-2x}{4t}\right]^2 + \frac{1}{\sqrt{t}}\left[\frac{-2}{4t}\right]$$

$$\frac{tx^2/(4t^2) - 1/2}{t\sqrt{t}} \stackrel{?}{=} \frac{1}{\sqrt{t}}\left[\frac{x^2}{4t^2}\right] + \frac{1}{\sqrt{t}}\left[\frac{-1}{2t}\right]$$

$$\|$$

$$\frac{x^2/(4t)}{t\sqrt{t}} - \frac{1/2}{t\sqrt{t}} = \frac{x^2}{4t^2\sqrt{t}} - \frac{1}{2t\sqrt{t}}$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution

?????

$$\frac{\sqrt{t}}{\sqrt{t}}\,\frac{\sqrt{t}x^2/(4t^2) - 1/(2\sqrt{t})}{t} \stackrel{?}{=} \frac{1}{\sqrt{t}}\left[\frac{-2x}{4t}\right]^2 + \frac{1}{\sqrt{t}}\left[\frac{-2}{4t}\right]$$

$$\frac{x^2}{4t^2\sqrt{t}} - \frac{1}{2t\sqrt{t}} \stackrel{?}{=} \frac{1}{\sqrt{t}}\left[\frac{x^2}{4t^2}\right] + \frac{1}{\sqrt{t}}\left[\frac{-1}{2t}\right]$$

☺

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Solution
?????

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}} \sqrt{\pi} \qquad (Q(X) = 1)$$

☺ $$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}} \qquad (Q(X) = 1/\sqrt{\pi})$$

$$f(x,t) = \frac{e^{-(x-a)^2/(4t)}}{\sqrt{t}} \qquad (Q(X) = e^{iaX}/\sqrt{\pi})$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}}\sqrt{\pi}$$ ($Q(X) = 1$)

$$f(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{t}}$$ ($Q(X) = 1/\sqrt{\pi}$)
(Total heat $2\sqrt{\pi}$)

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$ ($Q(X) = 1/(2\pi)$)
(Total heat 1)

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

THE FUNDAMENTAL SOLUTION

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$(Q(X) = 1/(2\pi))$
(Total heat 1)

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$(Q(X) = 1/(2\pi))$
(Total heat 1)

$$R(X) = e^{3iX}[Q(X)]$$

$$g(x,t) = \int_{-\infty}^{\infty} [R(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$f(x,t)$

$\|$

$$g(x+3,t) = \int_{-\infty}^{\infty} [e^{3iX}][Q(X)] e^{-X^2 t} e^{-i(x+3)X}\, dX$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$(Q(X) = 1/(2\pi))$
(Total heat 1)

$R(X) = e^{3iX}[Q(X)]$

$$g(x,t) = \int_{-\infty}^{\infty} [R(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$g(x+3,t) = f(x,t)$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$ $(Q(X) = 1/(2\pi))$
(Total heat 1)

$$g(x,t) = f(x-3,t)$$ (Total heat 1)

$$g(x+3,t) = f(x,t)$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

Pick any "reasonable" expression $Q(X)$ and set

$$f(x,t) = \int_{-\infty}^{\infty} [Q(X)] e^{-X^2 t} e^{-ixX}\, dX$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$(Q(X) = 1/(2\pi))$
(Total heat 1)

$$g(x,t) = f(x-3,t)$$

(Total heat 1)

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)]$$

(Total heat 9)

$$\frac{\partial}{\partial t}[h(x,t)] = \frac{\partial^2}{\partial x^2}[h(x,t)]$$

$$\frac{\partial}{\partial t}[g(x,t)] = \frac{\partial^2}{\partial x^2}[g(x,t)]$$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

The heat equation is **linear**, *i.e.*,
a linear combination of solutions is a solution,
e.g., $\partial_2(4f+5g) = \partial_{11}(4f+5g)$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

EXAMPLES OF NONLINEAR DIFFEQS

$$\frac{\partial}{\partial t}[f(x,t)] = \left[\frac{\partial}{\partial x}[f(x,t)]\right]^2$$

$$f' = e^f$$

$$f' = f^2$$

EXAMPLES OF LINEAR DIFFEQS

$$\partial_2^2 f = \partial_1^2 f$$

$$f'(x) = [x^2][f(x)]$$

The heat equation is **linear**, *i.e.*, a linear combination of solutions is a solution, *e.g.*, $\partial_2(4f + 5g) = \partial_{11}(4f + 5g)$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

SKILL:
Recognize if a given differential equation is linear or nonlinear.

The heat equation is **linear**, *i.e.*,
a linear combination of solutions is a solution,
e.g., $\partial_2(4f + 5g) = \partial_{11}(4f + 5g)$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

The heat equation is **linear**, *i.e.*,
a linear combination of solutions is a solution,
e.g., $\partial_2(4f+5g) = \partial_{11}(4f+5g)$

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

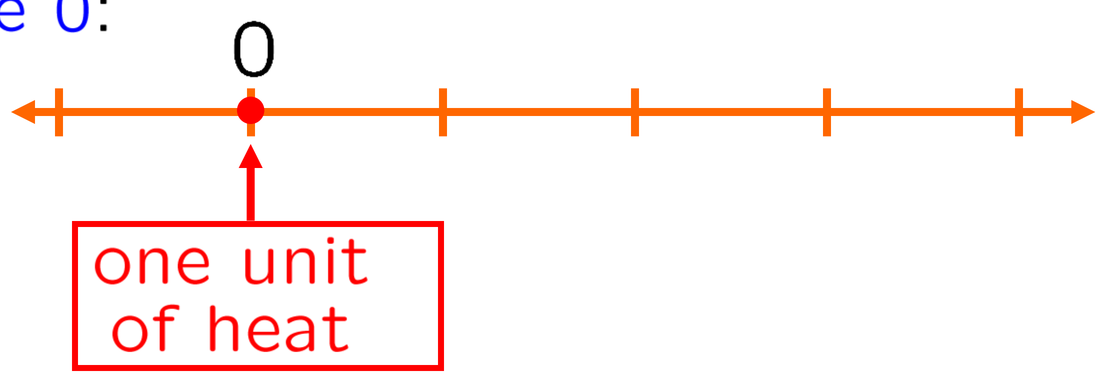
$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

time 0:



$f(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

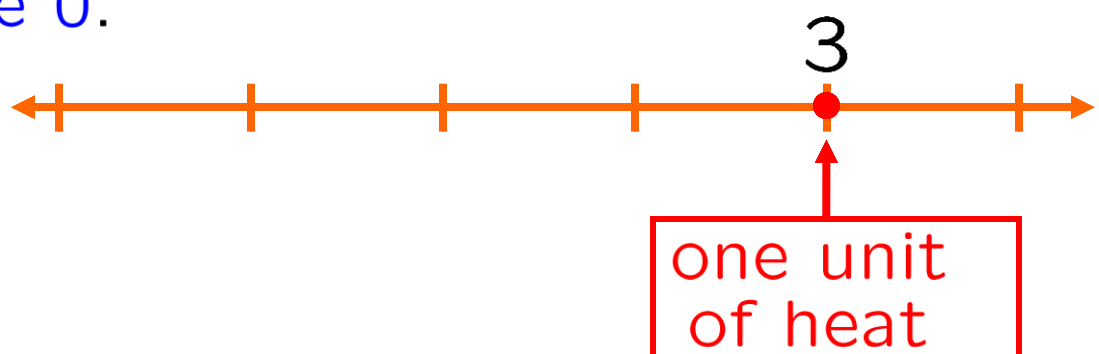
$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

time 0:



$g(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

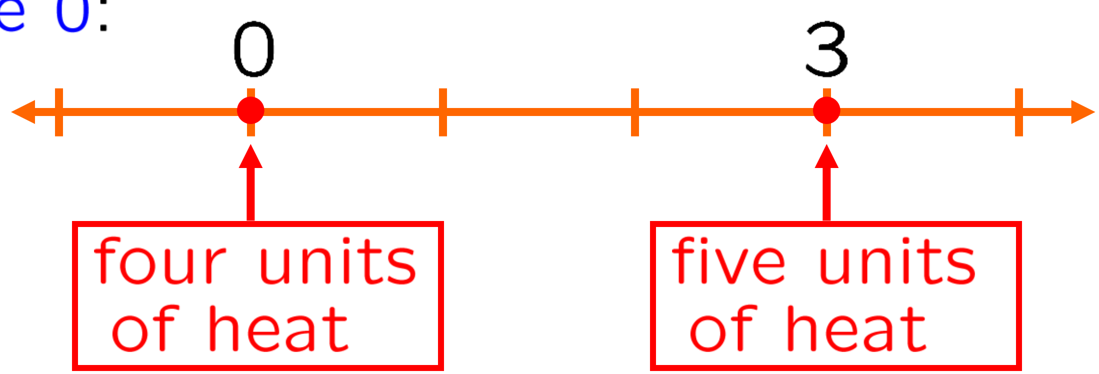
$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

time 0:



$h(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

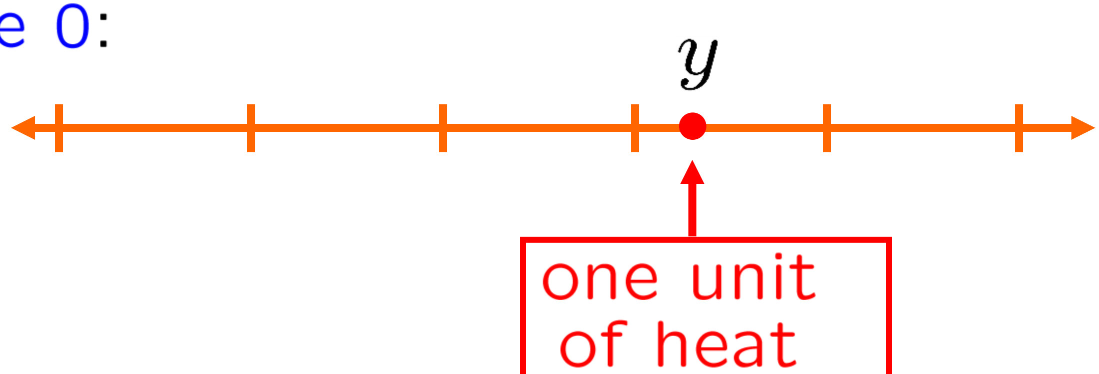
$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}} \qquad \partial_2 f = \partial_{11} f$$

$$g(x,t) = f(x-3,t) \qquad \partial_2 g = \partial_{11} g$$

$$h(x,t) = 4[f(x,t)] + 5[g(x,t)] \qquad \partial_2 h = \partial_{11} h$$

time 0:



$f(x-y,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

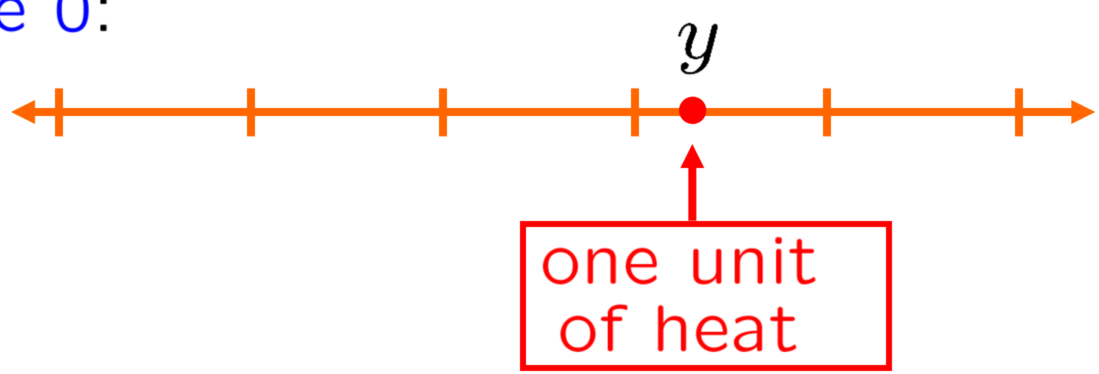
$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

1 unit of heat at y yields: $f(x-y,t)$

time 0:



$f(x-y,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

1 unit of heat at y yields: $f(x-y,t)$

time 0: Let $\phi(y)$ be the temperature at the point y at time 0, $\forall y$.

Amt of heat between a and b:

$$\int_a^b \phi(y)dy$$

$c(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

$\phi(y)\,dy$ units of heat at y yields: $[f(x-y,t)][\phi(y)]\,dy$

1 unit of heat at y yields: $f(x-y,t)$

time 0: Let $\phi(y)$ be the temperature at the point y at time 0, $\forall y$.

Amt of heat at y:

$$\phi(y)dy$$

$c(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

$\phi(y)\,dy$ units of heat at y yields:

$$[f(x-y,t)][\phi(y)]\,dy$$

Add up via $\int_{-\infty}^{\infty}$ yields:

$$c(x,t)$$

time 0: Let $\phi(y)$ be the temperature at the point y at time 0, $\forall y$.

Amt of heat at y:

$$\phi(y)dy$$

$c(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

Add up via $\int_{-\infty}^{\infty}$ yields: $\int_{-\infty}^{\infty}[f(x-y,t)][\phi(y)]\,dy$

Add up via $\int_{-\infty}^{\infty}$ yields: $c(x,t)$

time 0: Let $\phi(y)$ be the temperature at the point y at time 0, $\forall y$.

Amt of heat at y:

$$\phi(y)dy$$

$c(x,t)$ is the temperature at the point x at time t

THE HEAT EQUATION

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\partial_2 f = \partial_{11} f$$

$$f_t(x) = f(x,t) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

$$\partial_2 f = \partial_{11} f$$

Add up via $\int_{-\infty}^{\infty}$ yields: $\int_{-\infty}^{\infty}[f(x-y,t)][\phi(y)]\,dy$

$\|$

Add up via $\int_{-\infty}^{\infty}$ yields: $c(x,t) = c_t(x)$

time 0: Let $\phi(y)$ be the temperature at the point y at time 0, $\forall y$.

Amt of heat at y:

$$\phi(y)dy$$

$c(x,t)$ is the temperature at the point x at time t

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$$\int_{-\infty}^{\infty} [f_t(x-y)][\phi(y)]\,dy = \int_{-\infty}^{\infty} [f(x-y,t)][\phi(y)]\,dy$$

$$= c(x,t) = c_t(x)$$

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Definition:

For any "reasonable" $p, q : \mathbb{R} \to \mathbb{R}$,
the **convolution of** p **and** q is the function
$p * q : \mathbb{R} \to \mathbb{R}$ defined by

$$(p * q)(x) = \int_{-\infty}^{\infty} [p(x-y)][q(y)]\, dy$$

$$\int_{-\infty}^{\infty} [f_t(x-y)][\phi(y)]\, dy = \int_{-\infty}^{\infty} [f(x-y,t)][\phi(y)]\, dy$$

$$\| \qquad\qquad\qquad \|$$

$$(f_t * \phi)(x) \qquad\qquad c(x,t) = c_t(x)$$

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$$\int_{-\infty}^{\infty} [f_t(x-y)][\phi(y)]\,dy = (f_t * \phi)(x)$$

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SOLUTION

$$c_t(x) = (f_t * \phi)(x)$$

$$(f_t * \phi)(x) = c_t(x)$$

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FUNDAMENTAL SOLUTION

SOLUTION

$$c_t(x) = (f_t * \phi)(x)$$

Key idea: Every "reasonable" solution comes from the fundamental solution via translation and adding.

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FUNDAMENTAL SOLUTION

$$f_{1/4}(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$$

is e^{-x^2} normalized to have integral 1

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FUNDAMENTAL SOLUTION

$$f_{1/4}(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$$

is e^{-x^2} normalized to have integral 1

$$f_{t/4}(x) = \frac{e^{-x^2/t}}{\sqrt{\pi t}}$$

Eq'n has $\partial/\partial t$, $\partial^2/\partial x^2$.

Sol'n has t, x^2.

is $e^{-x^2/t}$ normalized to have integral 1

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FUNDAMENTAL SOLUTION

How to remember the fundamental solution:

1. Start with $e^{-x^2/t}$.
2. Normalize by dividing by $\sqrt{\pi t}$.
3. Quadruple speed by replacing t by $4t$.

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Note: If you skip Step 3, then you get the fundamental solution to

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{1}{4}\frac{\partial^2}{\partial x^2}[f(x,t)]$$

Rod's conductance quartered

Conservation of total heat

$$\int_{-\infty}^{\infty} f(x,t)\,dx = \int_{-\infty}^{\infty} \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}\,dx$$

$$\int_{-\infty}^{\infty} f(x,6)\,dx = \int_{-\infty}^{\infty} \frac{e^{-x^2/(4\cdot 6)}}{2\sqrt{6\pi}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-(x\sqrt{4\cdot 6})^2/(4\cdot 6)}}{2\sqrt{6\pi}}\,dx\sqrt{4\cdot 6}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$$

Change 6 to t.

Conservation of total heat

$$\int_{-\infty}^{\infty} f(x,t)\,dx = \int_{-\infty}^{\infty} \frac{e^{-x^2/(4t)}}{\sqrt{t}}\,dx$$

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$$= \int_{-\infty}^{\infty} \frac{e^{-(x\sqrt{4\cdot t})^2/(4\cdot t)}}{2\sqrt{t\pi}}\,dx\sqrt{4\cdot t}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$$

Change 6 to t.

Conservation of total heat

$$\frac{\partial}{\partial t}[f(x,t)] = \frac{\partial^2}{\partial x^2}[f(x,t)]$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} f(x,t)\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[f(x,t)]\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}[f(x,t)]\,dx = \int_{-\infty}^{\infty} \overbrace{\frac{\partial^2}{\partial x^2}[f(x,t)]}\cdot \underbrace{1}\,dx$$

$$= 0 - \int_{-\infty}^{\infty} \frac{\partial}{\partial x}[f(x,t)] \cdot \underbrace{\frac{\partial}{\partial x}[1]}_{0}\,dx = 0$$

STOP