Financial Mathematics

One period pricing and hedging

Dan wants 100 Euros one month from now.

He'll receive \$100 one month from now
from some source, but only has \$3 right now.

Current price is \$1/Euro. Worry: Rises to $>$ \$1.

Take out a loan? Loan rate: 1% per month!

Dan has poor credit . . . No loans for Dan!

Dollar price of a Euro a month from now is unknown.

Call it S. Dan wants a contract that will pay him

$$100(S - 1), \text{ if } S > 1.$$

Alice agrees to sell Dan a contract of this form.

What if $S \leq 1$?

(Money burns a hole in Dan's pocket, and he knows
he'll spend the \$3 by the end of the month.
if he doesn't spend it now.

(So he can't count on having
more than \$100 at the end of the month.)

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Call it S. Dan wants a contract that will pay him $100(S-1)$, if $S>1$.

Alice agrees to sell Dan a contract of this form.

What if $S \leq 1$?

Futures or forward: $100(S-1)$, if $S \leq 1$.

i.e., Dan pays Alice $100(1-S)$, if $S \leq 1$.

Knowing Dan is irresponsible, Alice refuses to agree to this.

Option: 0, if $S \leq 1$.

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pay him $\left\{ \begin{array}{r} 100(S-1), \text{ if } S > 1 \\ 0, \text{ if } S \leq 1 \end{array} \right\}$ one month from now.

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This is the payoff or claim.

The claim is contingent!

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$$= 100\left\{\begin{array}{r} S-1, \text{ if } S-1 > 0 \\ 0, \text{ if } S-1 \leq 0 \end{array}\right\} = 100(S-1)_+$$

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What price does she charge?

$$100(S-1)_+$$

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What price does she charge? More or less than \$3?

Step 1: Model "the underlying", *i.e.*, the Euro, *i.e.*, S.

Alice selects: A 1-subperiod $70-30$ CRR model,

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in which one ASSUMES that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has

a 70% chance of changing from 1 to $1 \times e^u$

and a 30% chance of changing from 1 to $1 \times e^d$.

Dollar price of a Euro a month from now is S.

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NOTE: S is a binary random variable,
whose distribution is described by:

$$\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.$$

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Step 2: Calibrate the model.
Alice asks her market analyst for the (one-month)
drift := $E[\ln S]$ and volatility := $SD[\ln S]$.
She gets this answer: vol is a std dev, NOT a var
drift $= 0.018765126$ and vol $= 0.045864002$
unrealistically high // CRR assumes independence //
$0.225181512/12$ low $0.158877565/\sqrt{12}$

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whos... describ...
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NOTE: $\ln S$ is a binary random variable, whose distribution is described by:

$\Pr[\ln S = u] = 0.7$ and $\Pr[\ln S = d] = 0.3$.

$$E[\ln S] = (0.7)u + (0.3)d$$

$$SD[\ln S] = \sqrt{(0.7)(0.3)}(u - d)$$

Step 2: Calibrate the model.

Alice asks her market analyst for the one-year

drift $:= \mathrm{E}[\ln S]$ and volatility $:= \mathrm{SD}[\ln S]$.

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$\Pr[\ln S = u] = 0.7$ and $\Pr[\ln S = d] = 0.3$.

$$\left.\begin{array}{l} 0.018765126 = (0.7)u + (0.3)d \\ 0.045864002 = \sqrt{(0.7)(0.3)}(u - d) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u = 0.0487902 \\ d = -0.0512933 \end{array}\right.$$

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drift $:= \mathrm{E}[\ln S]$ and volatility $:= \mathrm{SD}[\ln S]$.

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$u = 0.0487902$

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According to this model, $S \in \{1.05, 0.95\}$ a.s.

Recall: Dollar price of a Euro a month from now is S.

Step 3: Find a perfect hedging strategy.

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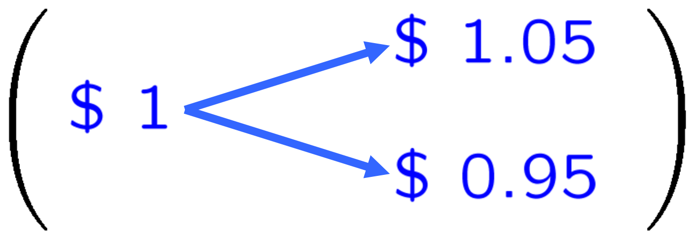
Alice sets up a hedging portfolio:

x Euros and a y dollar bank loan.

$x \quad \times$



NOTE: Alice does not have access to a bank that holds Euros. Her Euros all go "under the matress".

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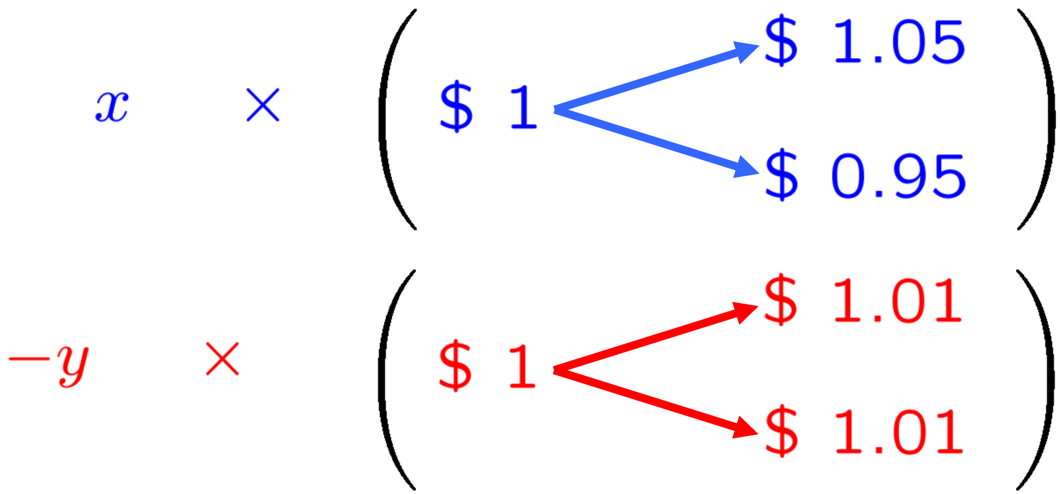

Loan rate: 1% per month!

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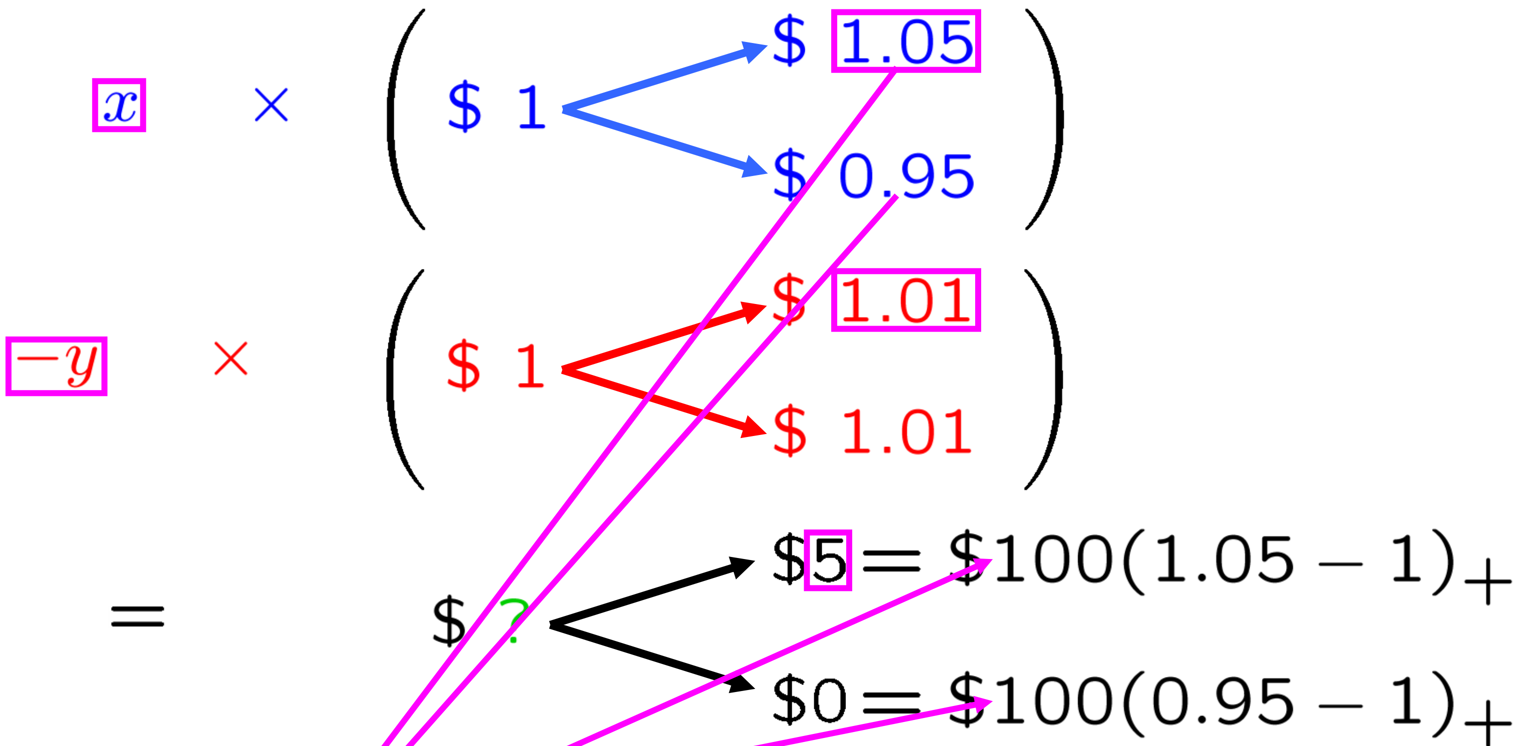

Dan and Alice agree on an option that will pay him $100(S-1)_+$ one month from now.

$$1.05\ x - 1.01\ y = 5$$

~~Alice sets up a hedging portfolio:~~
x Euros and a y dollar bank loan.

$$x \times \left(\$\,1 \to \begin{cases} \$\,1.05 \\ \$\,0.95 \end{cases} \right)$$

$$-y \times \left(\$\,1 \to \begin{cases} \$\,1.01 \\ \$\,1.01 \end{cases} \right)$$

$$= \quad \$\ ? \to \begin{cases} \$5 \\ \$0 \end{cases}$$

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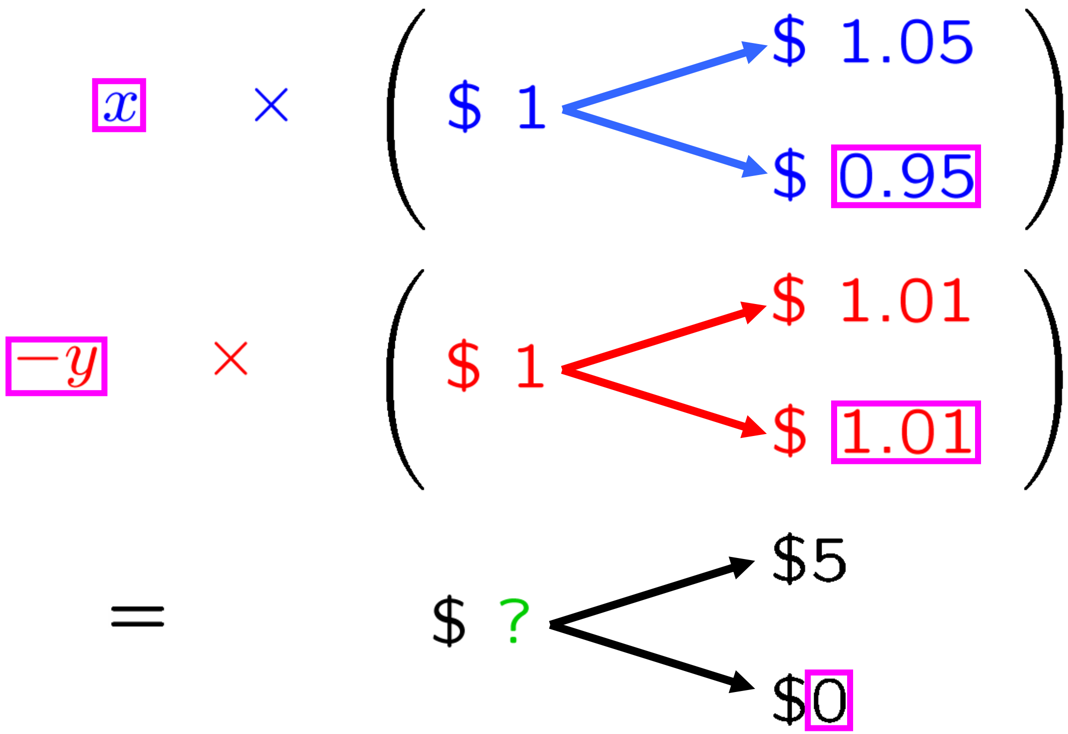

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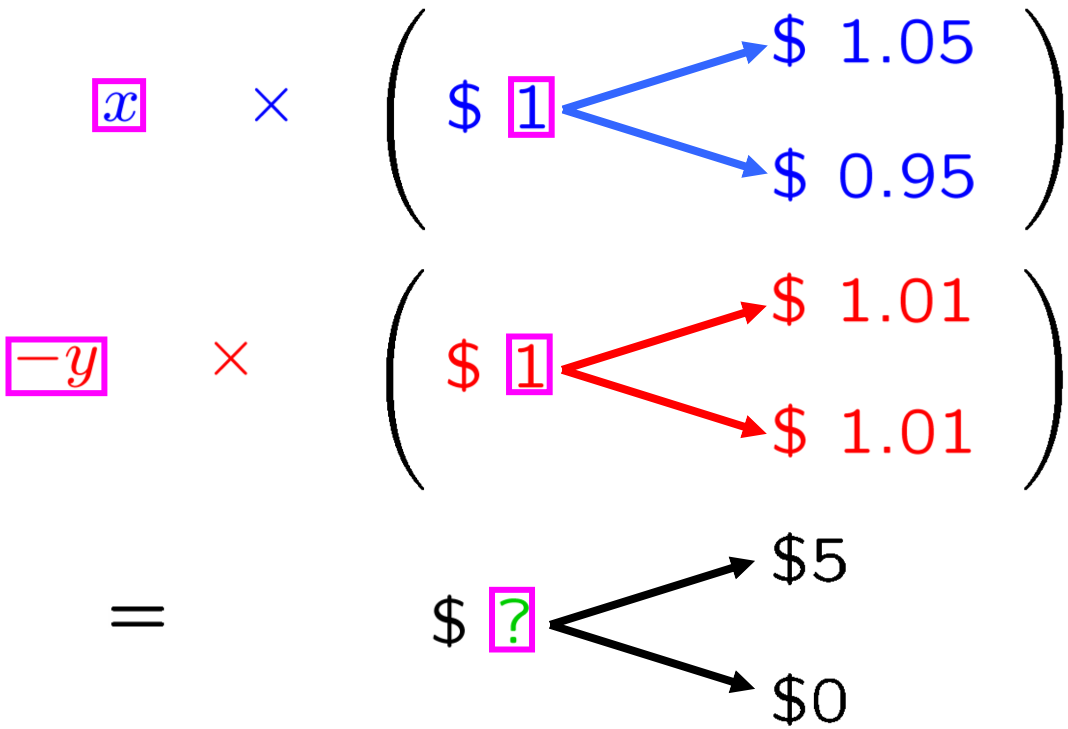

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$$1.05\,x - 1.01\,y = 5$$
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$$x - y = ?$$

$$x = 50$$
$$y = 47.03$$
$$? = 2.97$$

Step 3: Find a perfect hedging strategy.

Ans: Alice charges Dan $2.97
borrows 47.03 from the bank
and buys 50 Euros.

STOP

What price does she charge? More or less than $3?

Ans: $2.97 Ans: less

Alice sets up a hedging portfolio:
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