Financial Mathematics
One period pricing and hedging
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to $1. Take out a loan? Loan rate: 1% per month!

Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown.

Call it $S$. Dan wants a contract that will pay him

$$100(S - 1), \text{ if } S > 1.$$ 

Alice agrees to sell Dan a contract of this form. What if $S \leq 1$?

(Money burns a hole in Dan’s pocket, and he knows he’ll spend the $3 by the end of the month. if he doesn’t spend it now. (So he can’t count on having more than $100 at the end of the month.)
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Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan wants a contract that will pay him
\[ 100(S - 1), \text{ if } S > 1. \]

Alice agrees to sell Dan a contract of this form. What if $S \leq 1$?

Futures or forward: \[ 100(S - 1), \text{ if } S \leq 1. \]

i.e., Dan pays Alice \[ 100(1 - S), \text{ if } S \leq 1. \]

Knowing Dan is irresponsible,

Alice refuses to agree to this.

Option: \[ 0, \text{ if } S \leq 1. \]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to \( > $1 \).

Take out a loan? Loan rate: 1% per month!

Dan has poor credit ... No loans for Dan!

Dollar price of a Euro a month from now is unknown. Call it \( S \). Dan and Alice agree on an option that will pay him

\[
\begin{cases} 
100(S - 1), & \text{if } S > 1 \\ 
0, & \text{if } S \leq 1 
\end{cases}
\]

one month from now.
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to > $1. Take out a loan? **Loan rate:** 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. **Call it** $S$. Dan and Alice agree on an option that will pay him \[
\begin{cases} 
100(S-1), & \text{if } S > 1 \\
0, & \text{if } S \leq 1
\end{cases}
\] one month from now.

This is the **payoff or claim**.

The claim is **contingent**!
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to $>$ $1. Take out a loan? **Loan rate:** 1% per month!

Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him

$$\begin{cases} 100(S - 1), & \text{if } S > 1 \\ 0, & \text{if } S \leq 1 \end{cases}$$

one month from now.

$$\begin{cases} 100(S - 1), & \text{if } S - 1 > 0 \\ 0, & \text{if } S - 1 \leq 0 \end{cases}$$

$$100 \begin{cases} S - 1, & \text{if } S - 1 > 0 \\ 0, & \text{if } S - 1 \leq 0 \end{cases}$$
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to > $1.

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\[
\begin{cases} 
100(S - 1), & \text{if } S > 1 \\
0, & \text{if } S \leq 1 
\end{cases}
\]

\[
100 \begin{cases} 
S - 1, & \text{if } S - 1 > 0 \\
0, & \text{if } S - 1 \leq 0 
\end{cases}
\]

\[
100(S - 1) + x^+ := \begin{cases} 
x, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases}
\]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to $1. Take out a loan? **Loan rate:** 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him $100(S - 1) + x$ one month from now. What price does she charge?

\[ 100(S - 1) + x^+ := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \]
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. **Worry:** Rises to $1. Take out a loan? **Loan rate:** 1% per month! Dan has poor credit . . . No loans for Dan! Dollar price of a Euro a month from now is unknown. **Call it** $S$. Dan and Alice agree on an option that will pay him $100(S - 1)_+$ one month from now. **What price does she charge?** More or less than $3? **Step 1:** Model “the underlying”, *i.e.*, the Euro, *i.e.*, $S$. Alice selects: A 1-subperiod 70 – 30 CRR model,

$$x_+ := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$
Dan wants 100 Euros one month from now. He’ll receive $100 one month from now from some source, but only has $3 right now. Current price is $1/Euro. Worry: Rises to $1.

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Dollar price of a Euro a month from now is unknown. Call it $S$. Dan and Alice agree on an option that will pay him $100(S - 1)_+$ one month from now.

What price does she charge? More or less than $3?

Step 1: Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod 70 – 30 CRR model, in which one ASSUMES that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$ and a 30% chance of changing from 1 to $1 \times e^d$. 
Dollar price of a Euro a month from now is $S$.

**Step 1:** Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod 70 – 30 CRR model, in which one *ASSUMES* that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$

and a 30% chance of changing from 1 to $1 \times e^d$.

**NOTE:** $S$ is a binary random variable, whose distribution is described by:

$$\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.$$
Dollar price of a Euro a month from now is $S$.

Step 1: Model “the underlying”, i.e., the Euro, i.e., $S$. Alice selects: A 1-subperiod $70 - 30$ CRR model, in which one **ASSUMES** that $\exists d, u \in \mathbb{R}$ s.t. $d < u$ and s.t. the dollar price of one Euro has a 70% chance of changing from 1 to $1 \times e^u$ and a 30% chance of changing from 1 to $1 \times e^d$.

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Step 2: Calibrate the model.

Alice asks her market analyst for the (one-month) drift $:=$ $\mathbb{E}[\ln S]$ and volatility $:=$ $\text{SD}[\ln S]$. She gets this answer:

- drift $= 0.018765126$ and volatility $= 0.045864002$

unrealistically high // CRR assumes independence //

- $0.225181512/12$ low $0.158877565/\sqrt{12}$
Step 2: Calibrate the model.
Alice asks her market analyst for the one-year drift := \( E[\ln S] \) and volatility := \( SD[\ln S] \).

She gets this answer:
\[
\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

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\]

**NOTE:** The random variable \( S \), whose distribution is described by:

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\]

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Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift := $E[\ln S]$ and volatility := $SD[\ln S]$.

She gets this answer:

$\text{drift} = 0.018765126$ and $\text{vol} = 0.045864002$

**NOTE:** $S$ is a binary random variable, whose distribution is described by:

$\Pr[S = e^u] = 0.7$ and $\Pr[S = e^d] = 0.3$.

**NOTE:** $\ln S$ is a binary random variable, whose distribution is described by:

$\Pr[\ln S = u] = 0.7$ and $\Pr[\ln S = d] = 0.3$.

$E[\ln S] = (0.7)u + (0.3)d$

$SD[\ln S] = \sqrt{(0.7)(0.3)(u - d)}$
Step 2: Calibrate the model.  
Alice asks her market analyst for the one-year drift \( \text{drift} := E[\ln S] \) and volatility \( \text{volatility} := SD[\ln S] \).

She gets this answer:

\[
\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

\[
\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

**NOTE:** \( \ln S \) is a binary random variable, whose distribution is described by:

\[
\Pr[\ln S = u] = 0.7 \quad \text{and} \quad \Pr[\ln S = d] = 0.3.
\]

\[
0.018765126 = (0.7)u + (0.3)d \quad \text{and} \quad \sqrt{(0.7)(0.3)(u - d)} = 0.045864002
\]

\[
\begin{align*}
&u = 0.0487902 \\
d &\approx -0.0512933
\end{align*}
\]
Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift \( \text{drift} := \mathbb{E}[\ln S] \) and volatility \( \text{volatility} := \text{SD}[\ln S] \).

She gets this answer:

\[
\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

\[
\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

\[
\begin{align*}
u &= 0.0487902 & \implies & e^u &= 1.05000000 \\
d &= -0.0512933 & \implies & e^d &= 0.95000000
\end{align*}
\]

\[
u = 0.0487902 \\
d = -0.0512933
\]
Step 2: Calibrate the model.

Alice asks her market analyst for the one-year drift \( \text{drift} := \mathbb{E}\left[\ln S\right] \) and volatility \( \text{volatility} := \text{SD}\left[\ln S\right] \).

She gets this answer:

\[
\text{drift} = 0.018765126 \quad \text{and} \quad \text{vol} = 0.045864002
\]

**NOTE:** \( S \) is a binary random variable, whose distribution is described by:

\[
\Pr[S = e^u] = 0.7 \quad \text{and} \quad \Pr[S = e^d] = 0.3.
\]

\[
\begin{align*}
\quad u &= 0.0487902 \\
\Rightarrow \quad e^u &= 1.0500000 \\
\quad d &= -0.0512933 \\
\Rightarrow \quad e^d &= 0.9500000
\end{align*}
\]

According to this model, \( S \in \{1.05, 0.95\} \) a.s.

Recall: Dollar price of a Euro a month from now is \( S \).

Step 3: Find a perfect hedging strategy.
According to this model, \( S \in \{1.05, 0.95\} \text{ a.s.} \)

Recall: Dollar price of a Euro a month from now is \( S \).

**Step 3:** Find a perfect hedging strategy.

Alice sets up a hedging portfolio:

\( x \) Euros and a \( y \) dollar bank loan.

\[
\begin{align*}
\times & \quad \left( \begin{array}{c}
$1 \\
$1.05 \\
$0.95 \\
\end{array} \right) \\
\end{align*}
\]

**NOTE:** Alice does not have access to a bank that holds Euros.

Her Euros all go “under the mattress”.

According to this model, \( S \in \{1.05, 0.95\} \text{ a.s.} \)

Recall: Dollar price of a Euro a month from now is \( S \).

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According to this model, \( S \in \{1.05, 0.95\} \text{ a.s.} \)

Recall: Dollar price of a Euro a month from now is \( S \).

**Step 3: Find a perfect hedging strategy.**

Alice sets up a **hedging portfolio:**

- \( x \) Euros and a \( y \) dollar bank loan.

\[
\begin{align*}
    x \ \times & \quad \begin{pmatrix}
    \$ 1 \\
    \$ 0.95
\end{pmatrix} \\
- y \ \times & \quad \begin{pmatrix}
    \$ 1 \\
    \$ 1.01
\end{pmatrix}
\end{align*}
\]

- Loan rate: 1% per month!
According to this model, $S \in \{1.05, 0.95\}$ a.s.

Recall: Dollar price of a Euro a month from now is $S$.

**Step 3: Find a perfect hedging strategy.**

Alice sets up a hedging portfolio:

- $x$ Euros and a $y$ dollar bank loan.

\[
\begin{align*}
    x \times \begin{pmatrix} 1 \\ 0.95 \end{pmatrix} + \quad & \quad y \times \begin{pmatrix} 1 \\ 1.01 \end{pmatrix} \\
    \quad = & \quad ? \\
\end{align*}
\]

\[
\begin{align*}
    \text{Dan and Alice agree on an option that will pay him } & 100(S - 1)^+ \text{ one month from now.}
\end{align*}
\]
1.05 \( x \) – 1.01 \( y \) = 5

Alice sets up a hedging portfolio: 

\( x \) Euros and a \( y \) dollar bank loan.

\[ \begin{align*}
    x & \times \left( \begin{array}{c}
    \$1 \\
    \$0.95 \\
    \end{array} \right) \\
    -y & \times \left( \begin{array}{c}
    \$1 \\
    \$1.01 \\
    \end{array} \right) \\
    &= \left( \begin{array}{c}
    \$1.05 \\
    \$0.95 \\
    \$1.01 \\
    \$1.01 \\
    \end{array} \right)
\end{align*} \]

Alice sets up a hedging portfolio: 

\( x \) Euros and a \( y \) dollar bank loan.
1.05 \(x - 1.01 \ y = 5\)
0.95 \(x - 1.01 \ y = 0\)

Alice sets up a hedging portfolio:
\(x\) Euros and a \(y\) dollar bank loan.
1.05 \, x - 1.01 \, y = 5 \\
0.95 \, x - 1.01 \, y = 0 \\
\text{or} \\
x - y = ?

\[ \begin{align*}
\begin{pmatrix} x \\ -y \end{pmatrix} \times & \begin{pmatrix} 1 \\ 0.95 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 1.01 \end{pmatrix} \times & \begin{pmatrix} 1.05 \\ 0.95 \end{pmatrix}
\end{align*} \]

\[ \begin{array}{c}
\begin{pmatrix} x \\ -y \end{pmatrix} \times \\
\begin{pmatrix} 1.05 \\ 0.95 \end{pmatrix}
\end{array} \]

Alice sets up a hedging portfolio: 
\[ x \text{ Euros} \quad \text{and} \quad y \text{ dollar bank loan.} \]
1.05 \( x - 1.01 \) \( y = 5 \)
0.95 \( x - 1.01 \) \( y = 0 \)
\[ x - y = ? \]

\[ x = 50 \]
\[ y = 47.03 \]
\[ ? = 2.97 \]

Step 3: Find a perfect hedging strategy.

Ans: Alice charges Dan \$2.97 \text{ \( \rightarrow \) } borrows 47.03 \text{ from the bank and buys 50 Euros.}

\textbf{What price does she charge? More or less than \$3?}

Ans: \$2.97 \text{ Ans: less}

Alice sets up a \textbf{hedging portfolio}: \( x \text{ Euros and } y \text{ dollar bank loan.} \)