Financial Mathematics

Risk-neutrality and Delta-hedging

$$1.05\,x - 1.01\,y = 5$$
$$0.95\,x - 1.01\,y = 0$$
$$x - y = ?$$

Next goal:

Describe a method to compute ? without solving a system of equations.

Key point to remember:

? does not depend on the probability of an uptick or downtick.

Is probability theory then useless? NO:

The trick is to imagine another universe in which the probabilities somehow make the computation of ? easy.

More on this in a moment . . .

According to the selected model,

$$\text{probability} \quad \text{uptick} = 70\%,$$
$$\text{probability downtick} = 30\%$$

Problem:

Find the expected value and return, in our world, after one month, of

(a) \$1 invested in the bank; and

(b) \$1 invested in Euros.

Assume that the bank pays 1% per month on savings accounts. (Same as on loans.)

(a)

According to the selected model,

probability uptick $= 70\%$,

probability downtick $= 30\%$

Bank: $1

According to the selected model,

probability uptick $= 70\%$,

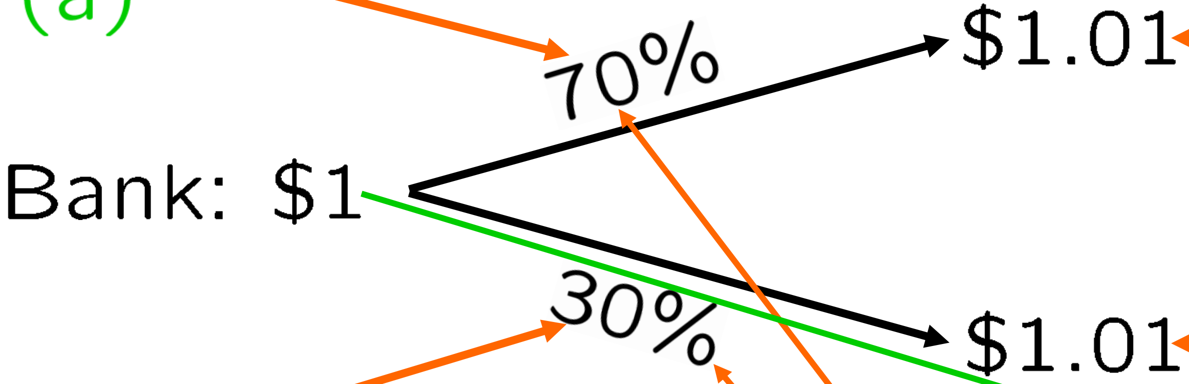
probability downtick $= 30\%$

Assume that the bank pays 1% per month on savings accounts. (Same as on loans.)

s.

(Same as on loans.)

(a)



expected bank value:

$$(70\%)(1.01) + (30\%)(1.01) = 1.01$$

(guaranteed – risk-free) expected bank return: 1%

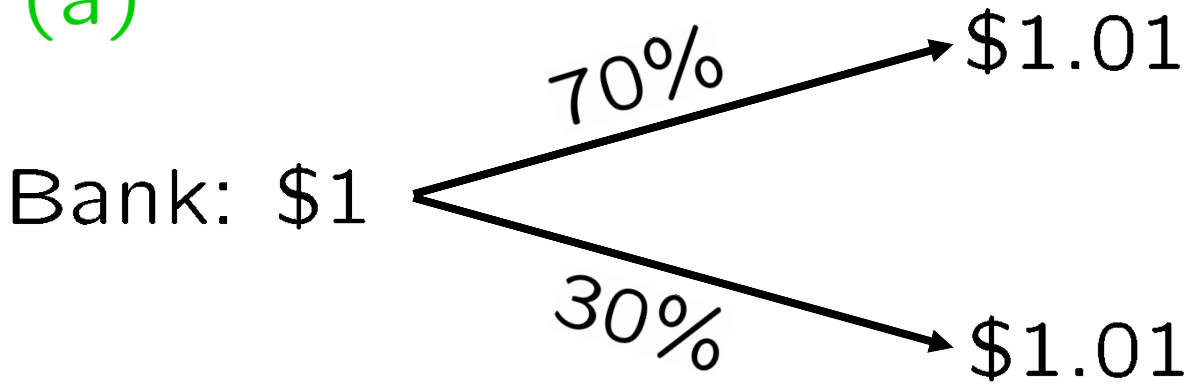
According to the selected model,

probability uptick = 70%,

probability downtick = 30%

Assume that the bank pays 1% per month on savings accounts. (Same as on loans.)

(a)

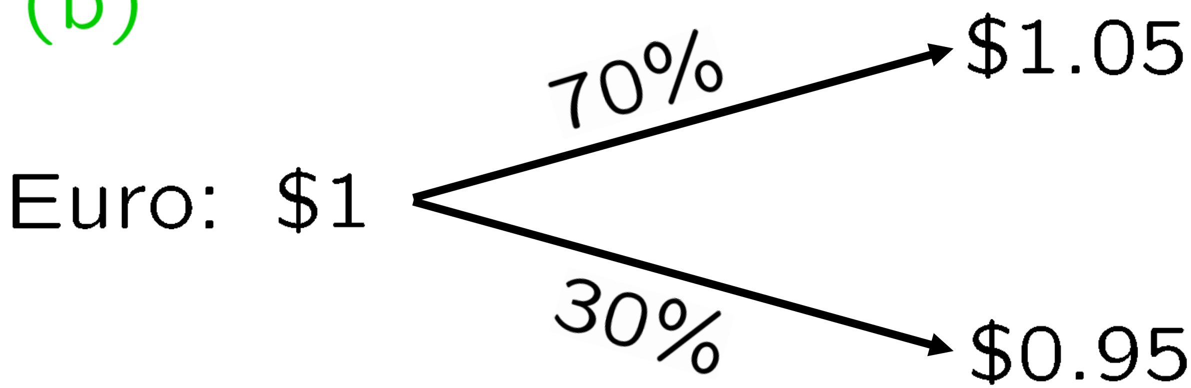


expected bank value:

$$(70\%)(1.01) + (30\%)(1.01) = 1.01$$

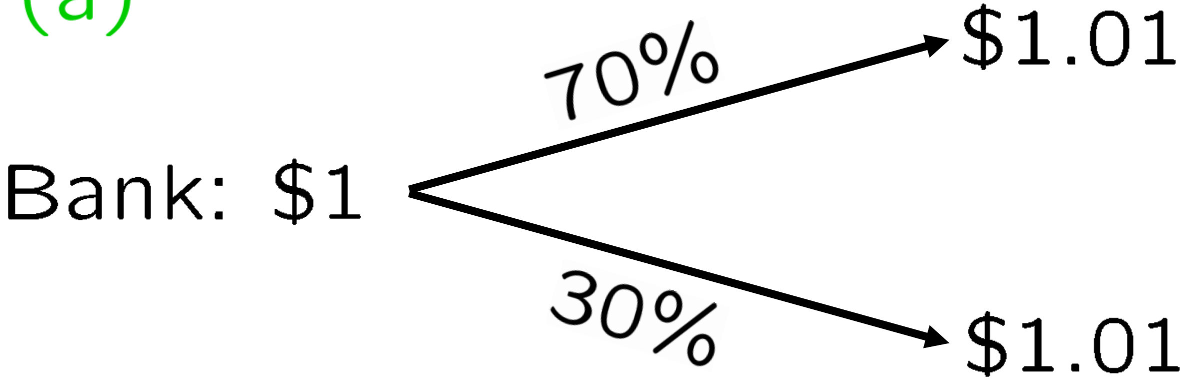
(guaranteed – risk-free) expected bank return: 1%

(b)



According to this model, $S \in \{1.05, 0.95\}$ a.s.

(a)

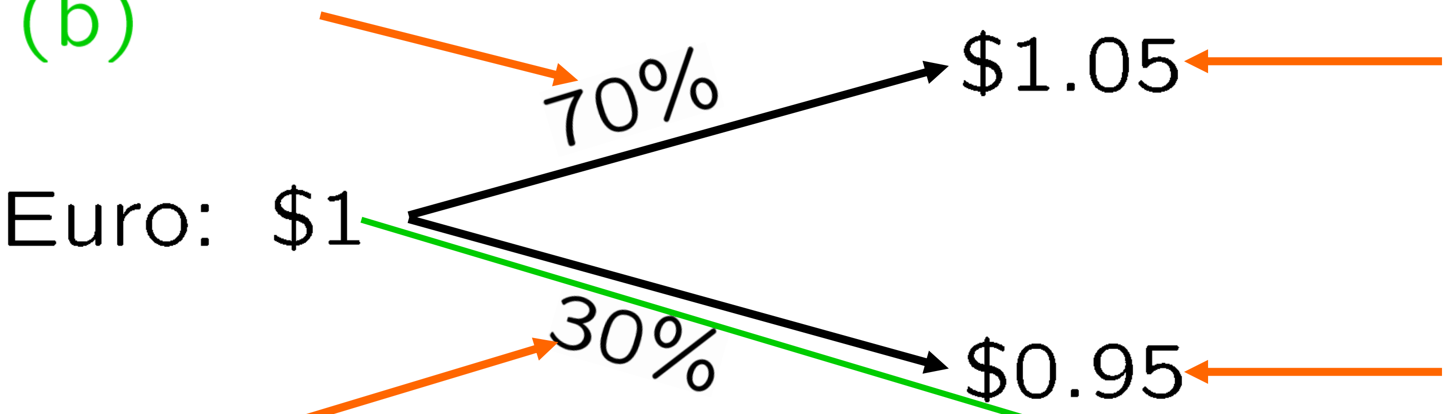


expected bank value:

$$(70\%)(1.01) + (30\%)(1.01) = 1.01$$

(guaranteed – risk-free) expected bank return: 1%

(b)



expected Euro value:

$$(70\%)(1.05) + (30\%)(0.95) = 1.02$$

expected Euro return: 2%

$$1\% < 2\%$$

expected bank return < expected Euro return

Bank is "risk-free". Euros are "risky".

expected bank return 1%

expected Euro return 2%

$$1\% < 2\%$$

expected bank return < expected Euro return

Bank is "risk-free". Euros are "risky".

Economics: Investors are "risk-averse".

So risky investments must have a higher expected rate of return than risk-free investments, or they won't sell.

Imagine a world in which bank and Euros have the same expected return.

This is the "risk-neutral" world.

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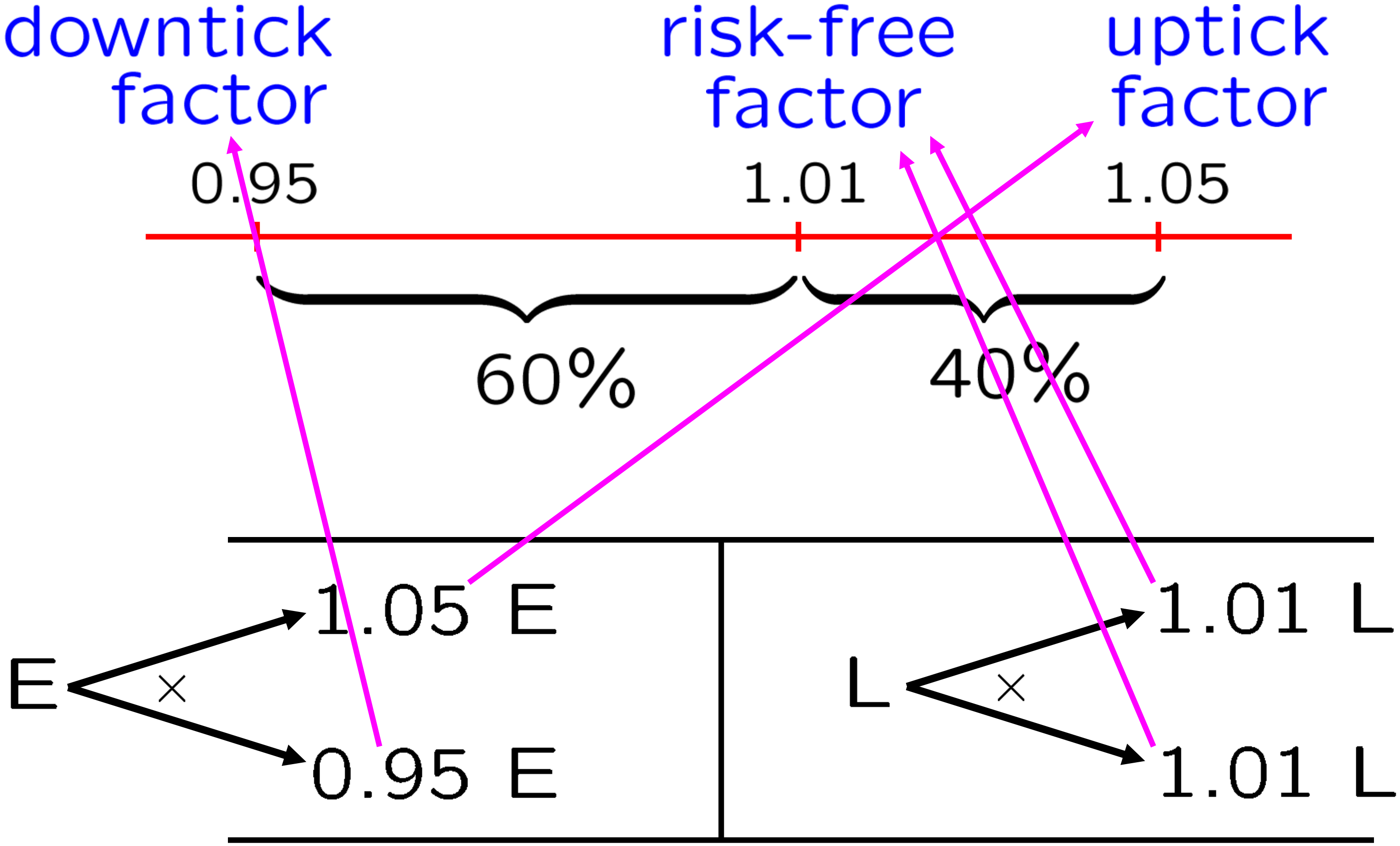
risk-free ≠ risk-neutral


downtick factor
risk-free factor
uptick factor
0.95
1.01
1.05
60%
40%
1.05 E
E
0.95 E
1.01 L
L
1.01 L

downtick factor	risk-free factor	uptick factor
0.95	1.01	1.05

60% | 40%

$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$

(b) 60% 70%

Euro: $1 → $1.05 (70%)

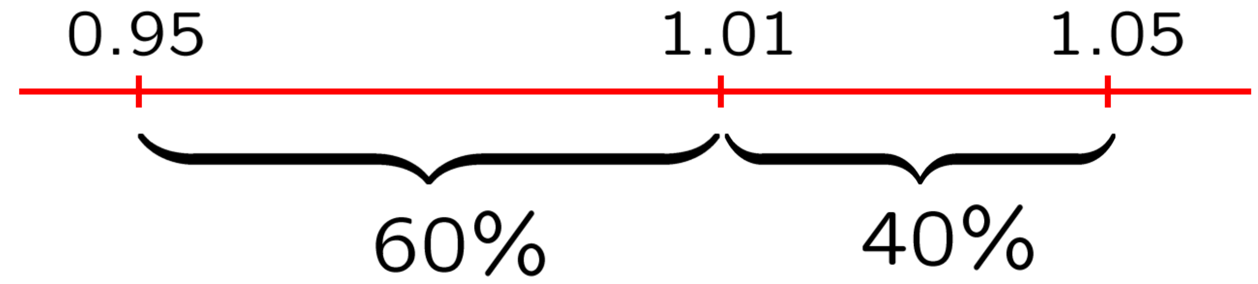
Euro: $1 → $0.95 (30%)

imagine a 60-40 world

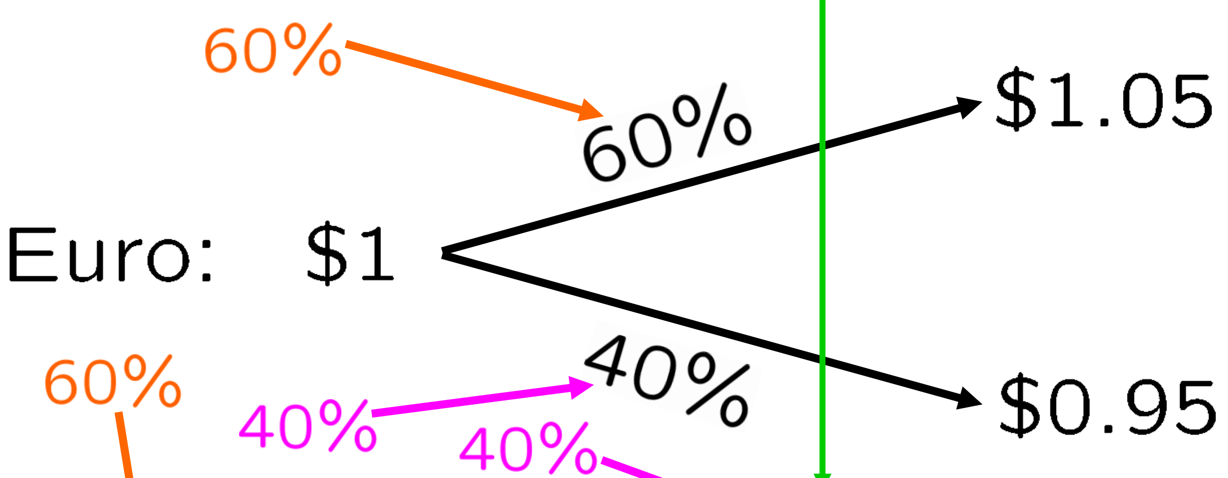
60% 40% 40%

$$(70\%)(1.05) + (30\%)(0.95) = 1.02$$

downtick factor | risk-free factor | uptick factor



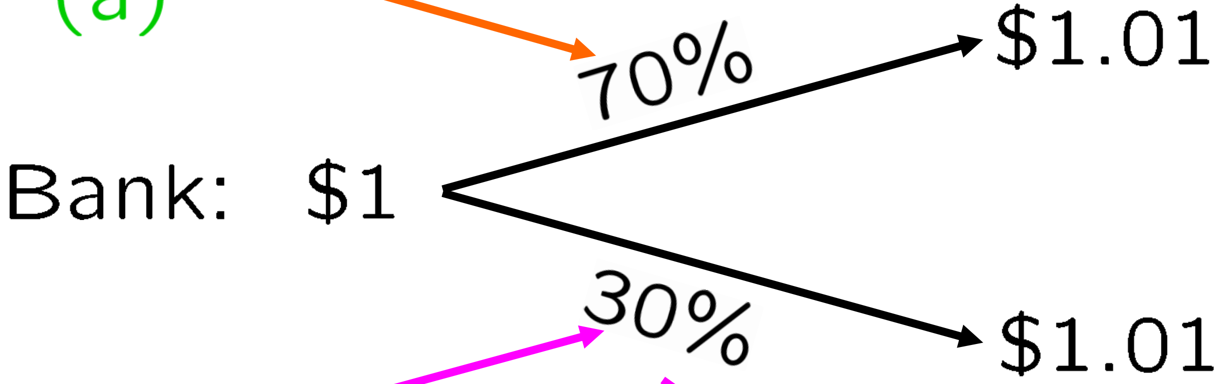
$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$



imagine a 60-40 world

$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$

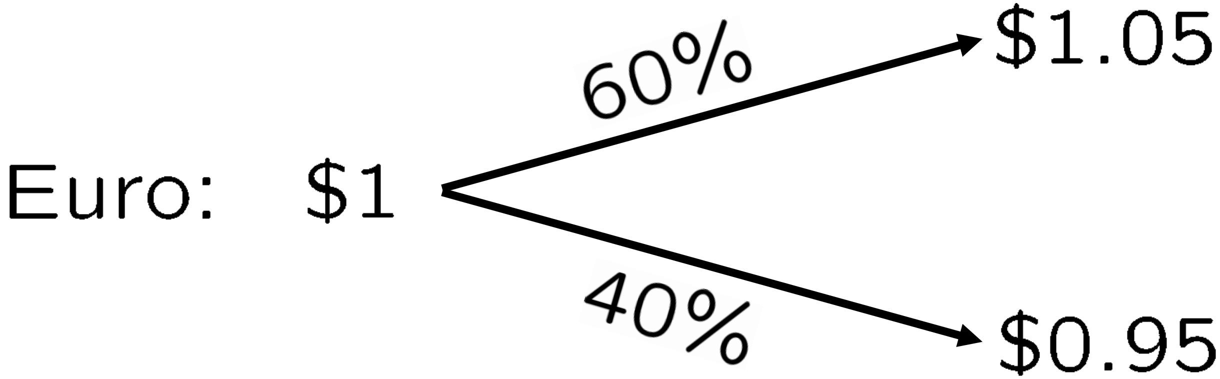
(a)



imagine
a 60-40 world

does not change

$$(70\%)(1.01) + (30\%)(1.01) = 1.01$$

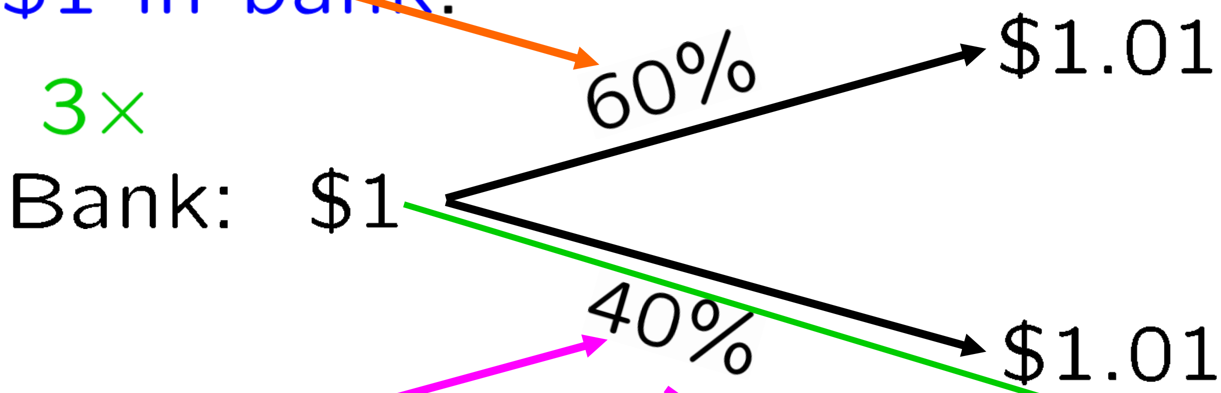


imagine
a 60-40 world

$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$

$1 in bank:

3×



imagine
a 60-40 world

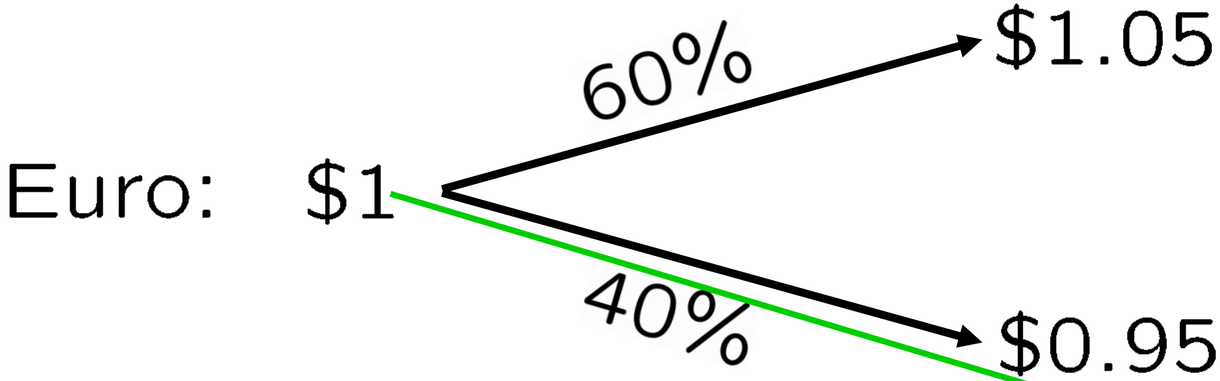
does not change

$$(60\%)(1.01) + (40\%)(1.01) = 1.01$$

expected bank return: 1%

We have achieved risk-neutrality!

$1 in Euros:

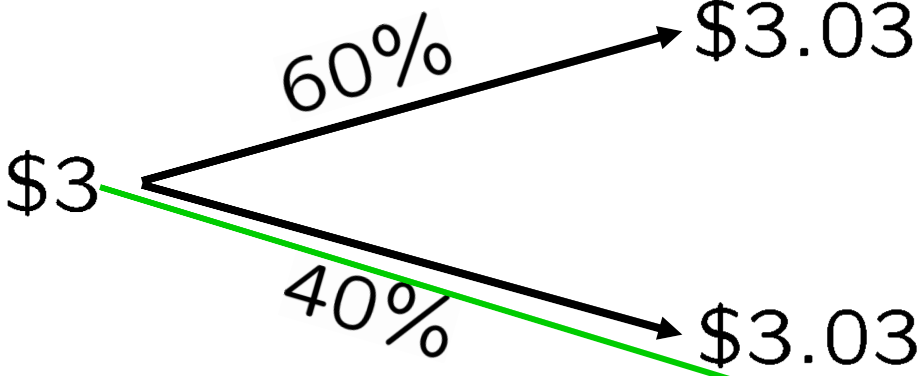


imagine
a 60-40 world

$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$

expected Euro return: 1%

$3 in bank:

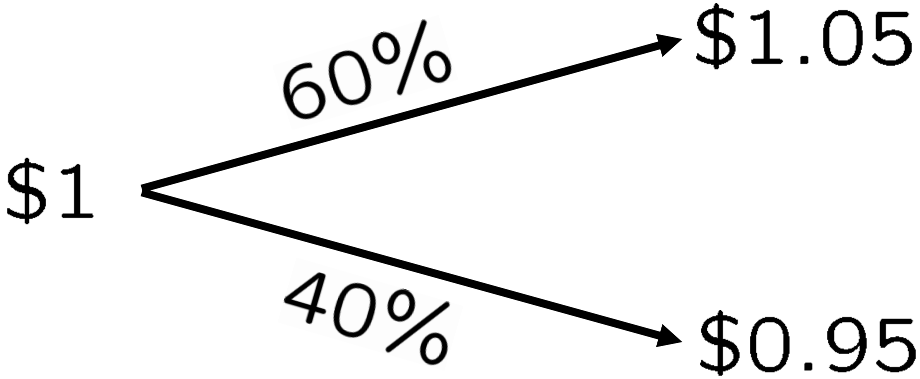


$$(60\%)(3.03) + (40\%)(3.03) = 3.03$$

expected bank return: 1%

$1 in Euros:

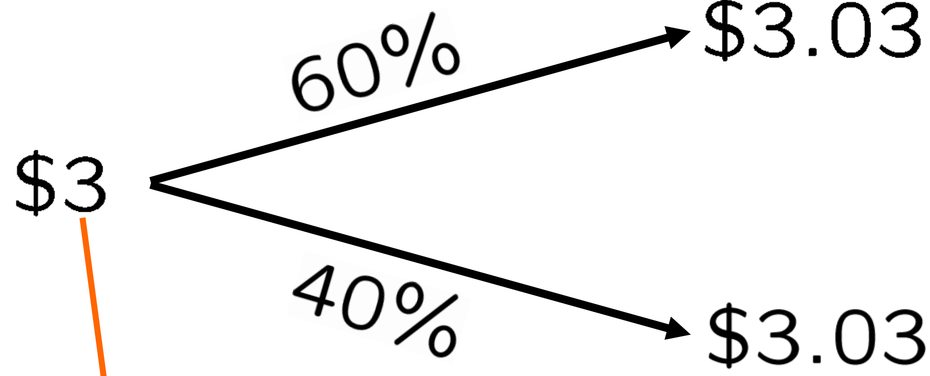
2×



$$(60\%)(1.05) + (40\%)(0.95) = 1.01$$

expected Euro return: 1%

$3 in bank:

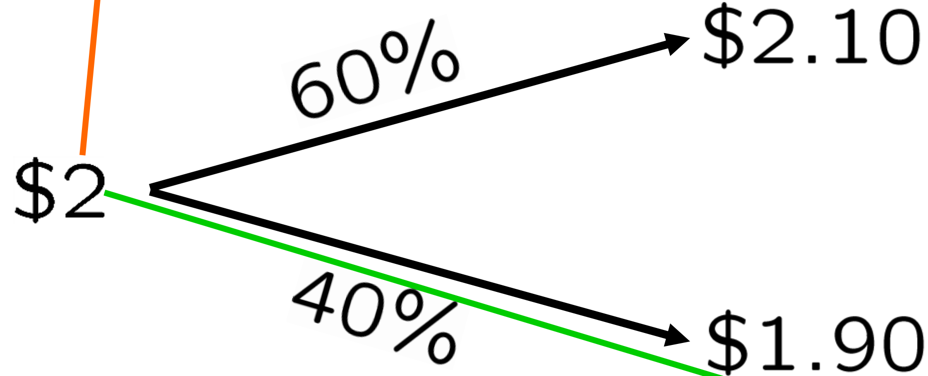


$$(60\%)(3.03) + (40\%)(3.03) = 3.03$$

expected bank return: 1%

$5

$2 in Euros:



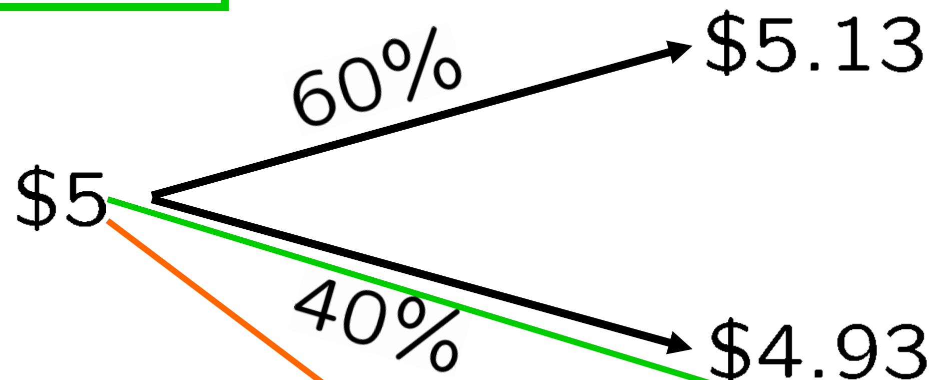
5.05

$$(60\%)(2.10) + (40\%)(1.90) = 2.02$$

expected Euro return: 1%

$3 in bank
and
$2 in Euros:

The same logic will work
on any portfolio.



$$(60\%)(5.13) + (40\%)(4.93) = 5.05$$

Expected Portfolio Return: 1%

$$(1.01)\,5 = 5.05$$

In this risk-neutral world, the expected return on any bank-Euro portfolio is 1% per month.

$-\$y$ in bank

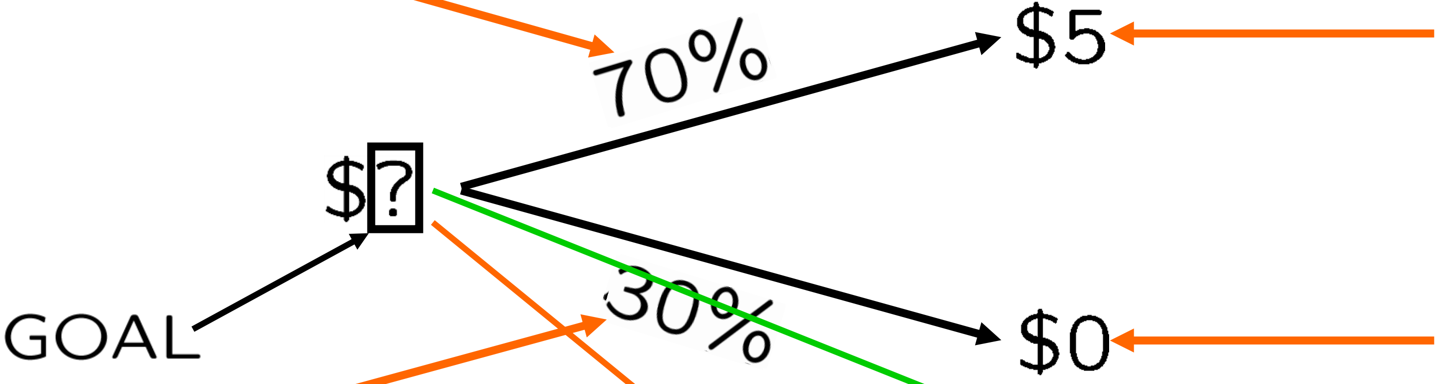
and

$\$x$ in Euros:

“Change of Measure”

Change from the “real” or “physical” world (70–30)

to the “risk-neutral” world. (60–40)



$$(60\%)(5) + (40\%)(0) = 3$$

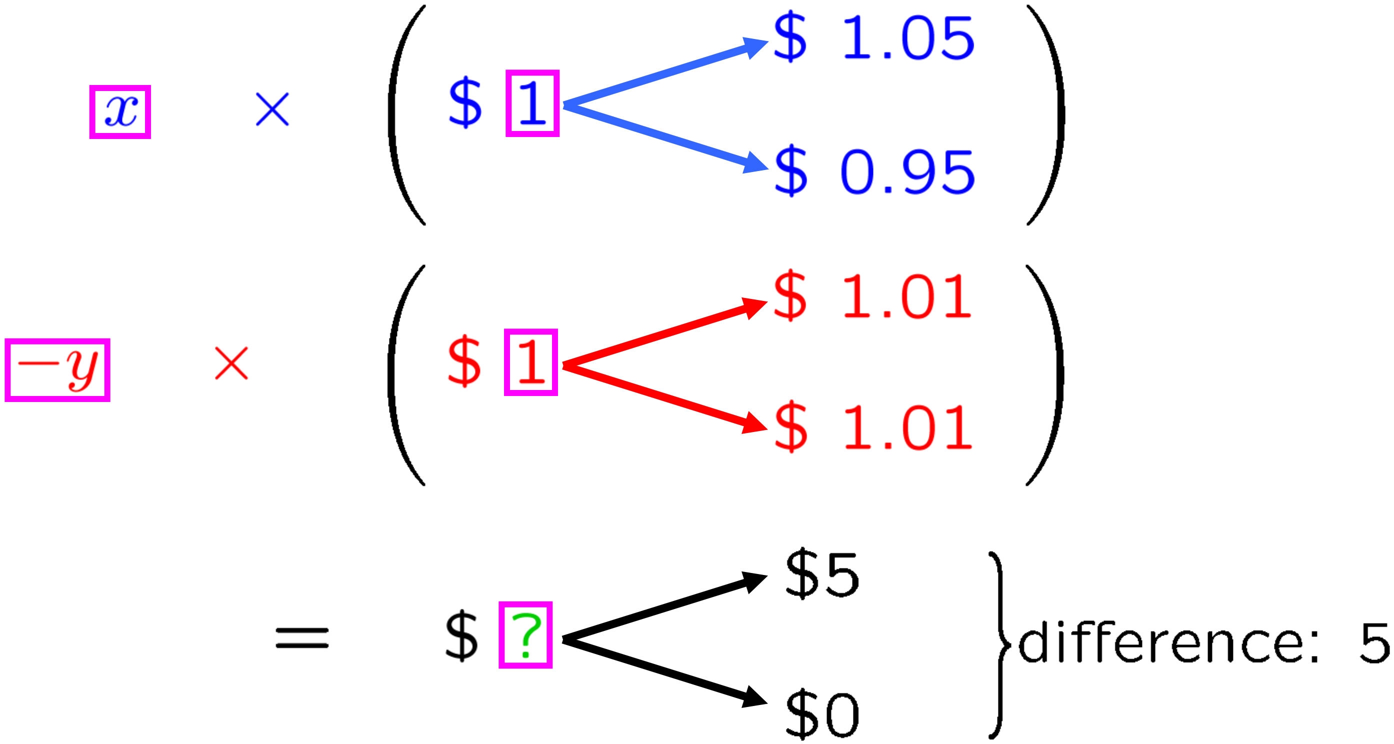
Expected Portfolio Return: 1%

$$(1.01)\,? = 3$$

$$? = 3/1.01 = 2.97 \blacksquare$$

Coin-flippers got price!

How can we figure out
the hedging strategy,
without solving
a system of equations?



$x - y = ? = 2.97$ (We're pricers.)

Want: x (We want to "get hedge".)

$x \times$ (\$ 1 → \$ 1.05 / \$ 0.95) naïve volatility difference: 0.1

$-y \times$ (\$ 1 → \$ 1.01 / \$ 1.01) naïve volatility difference: 0

$=$ \$? → \$5 / \$0 naïve volatility difference: 5

$x - y = ? = 2.97$ (We're pricers.)

Want: x (We want to "get hedge".)

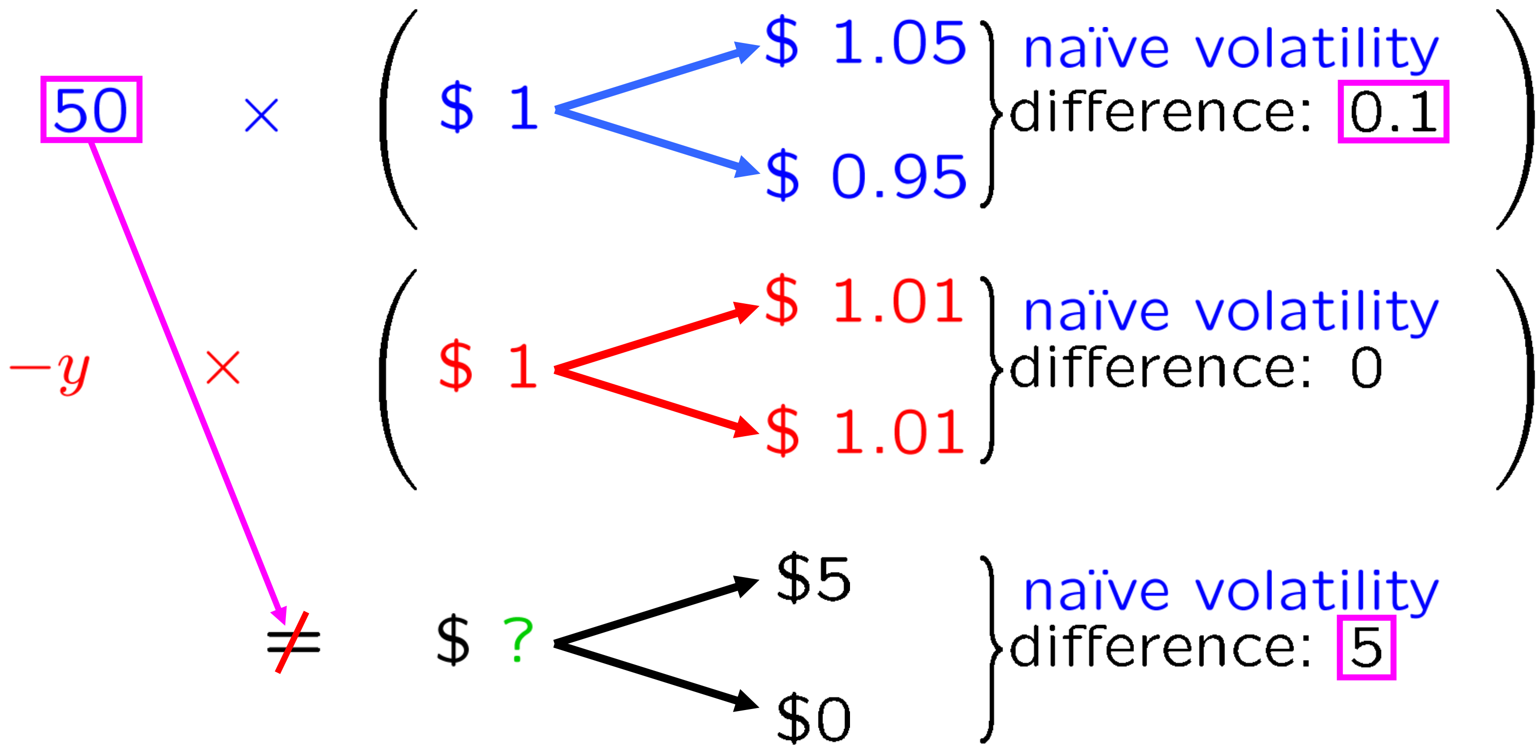
$40 \times \left(\$\,1 \to \begin{cases} \$\,1.05 \\ \$\,0.95 \end{cases} \right)$ naïve volatility difference: 0.1

$-y \times \left(\$\,1 \to \begin{cases} \$\,1.01 \\ \$\,1.01 \end{cases} \right)$ naïve volatility difference: 0

$\neq$ $\$\,? \to \begin{cases} \$5 \\ \$0 \end{cases}$ naïve volatility difference: 5

$x - y = ? = 2.97$ (We're pricers.)

Want: x (We want to "get hedge".)



$x - y = ? = 2.97$ (We're pricers.)

Pricers got hedge

Want: x (We want to "get hedge".)

$$x = 50 = \frac{5}{0.1} = \frac{\text{option naïve volatility}}{\text{Euro naïve volatility}}$$

Pricers got hedge!