Financial Mathematics
Central Limit Theorem
Coin-flipping game: Flip a fair coin $N$ times. If $H$ heads and $T$ tails, pay $f(e^{Hu+Td})$, 30 days from now.

expected payout $=: E = ???$

Easier problem: Compute the probability that $-\sqrt{N} < H - T < \sqrt{N}$. 

probability problems, then expected value problems
Easier problem: Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$ 

\[ X := \frac{H - T}{\sqrt{N}} \]  

standardization of \( H - T \)

Easier problem: Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$
Easier problem:
Compute the probability that $-\sqrt{N} < H - T < \sqrt{N}$.

$X := (H - T)/\sqrt{N}$

Easier problem after standardization:
Compute the probability that $-1 < X < 1$.

$H_1 :=$ number of heads after first flip
$H_2 :=$ number of heads after second flip
$\vdots$
$H_N :=$ number of heads after $N$th flip $= H$
Easier problem: Compute the probability that 

\[-\sqrt{N} < H - T < \sqrt{N}\].

\[X := (H - T)/\sqrt{N}\]

Easier problem after standardization: Compute the probability that 

\[-1 < X < 1\]. \(X\) is hard …

For all integers \(j \in [1, N]\),

\[H_j := \text{number of heads after } j\text{th flip}\]

\[T_j := \text{number of tails after } j\text{th flip}\]

\[D_j := H_j - T_j\]

\[H = H_N, \quad T = T_N, \quad X = (H_N - T_N)/\sqrt{N} = D_N/\sqrt{N}\]
$D_1 = H_1 - T_1$:

Keep the distribution of $D_1$

Forget its origin

Distribution of $T_1 - H_1$ is exactly the same
\[ D_1 = H_1 - T_1 \]
\[ D_1 = H_1 - T_1 : \]

- \(0.5 \rightarrow 1\)
- \(0.5 \rightarrow -1\)

\[ z^1 = z \]
\[ z^{-1} \]

Generating function:

\[ (\text{expression of } z) \]

Replace \( z \) by \( e^{-it} \)

\[ (0.5)z + (0.5)z^{-1} \]

Fourier transform:

\[ \xi \quad \text{not time} \]

\[ (0.5)e^{-it} + (0.5)e^{it} \]

\[ \| \cos t \]

\[ 0.5 \times \begin{bmatrix} e^{it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} \cos t + i \sin t \\ \cos t - i \sin t \end{bmatrix} + \]

\[ 0.5 \times \begin{bmatrix} e^{it} \\ e^{-it} \end{bmatrix} = \begin{bmatrix} \cos t + i \sin t \\ \cos t - i \sin t \end{bmatrix} \]
\[ D_1 = H_1 - T_1 : \]

Cannot recover the random variable!
Only its distribution.

The distribution of \( D_1 \) is \( \cos t \)

\( T_1 - H_1 \) has the same distribution.

Generating function:
\[
(0.5)z + (0.5)z^{-1}
\]

Fourier transform:
\[
(0.5)e^{-it} + (0.5)e^{it}
\]

Inverse Fourier transform
\[
0.5 \times \left[ e^{it} = \cos t + i \sin t \right] + 0.5 \times \left[ e^{-it} = \cos t - i \sin t \right]
\]
$D_1 = H_1 - T_1$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$z^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-1</td>
<td>$z^{-1}$</td>
</tr>
</tbody>
</table>

What about $D_1/7$?

Generating function:

Fourier transform:

$e^{it} = \cos(t) + i \sin(t)$

$e^{-it} = \cos(t) - i \sin(t)$

$(0.5)z + (0.5)z^{-1}$

$(0.5)e^{-it} + (0.5)e^{it}$

$\cos(t)$

Repl. $t$ by $t/7$
What about $D_{1/7}$?
Replace $t$ by $t/7$.

Generating function:

Fourier transform:

\[
egin{align*}
i &= \sqrt{-1} \\
\text{Replace } z &\text{ by } e^{-it} \\
(0.5)z^{1/7} + (0.5)z^{-1/7} &\equiv (0.5)e^{-it/7} + (0.5)e^{it/7} \\
\cos(t/7) &
\end{align*}
\]

\[
egin{align*}
e^{it/7} &= \cos(t/7) + i \sin(t/7) \\
e^{-it/7} &= \cos(t/7) - i \sin(t/7)
\end{align*}
\]
\[ D_2 = \bar{H}_2 - \bar{T}_2 : \]

\[
\begin{array}{c|c}
  & 2 & 0.25 \\
  / & \\
 0 & 0.25 + 0.25 = 0.5 \quad & \quad 0.25 \\
  & 0 & \quad & \quad -2 \\
  \rightarrow & / & \rightarrow & / \\
  & 0.5 & \rightarrow & 0.5 \\
  & 0.5 & \rightarrow & 0.5 \\
  & 0.5 & \rightarrow & 0.5 \\
  & 0.5 & \rightarrow & 0.5 \\
\end{array}
\]

*forget its origin keep the distribution*
\( D_2 = H_2 - T_2 : \)

| 0.25 | 2 | \( z^2 \) |
| 0.5  | 0 | \( z^0 = 1 \) |
| 0.25 | -2 | \( z^{-2} \) |

forget its origin keep the distribution

Generating function:

\[
(0.25)z^2 + 0.5 + (0.25)z^{-2} = ((0.5)z + (0.5)z^{-1})^2
\]

the generating function of the distribution of \( D_1 \)

\[ i = \sqrt{-1} \]

Replace \( z \) by \( e^{-it} \)

Fourier transform:

\[
(\cos t)^2 = \cos^2 t
\]
\[ D_N = H_N - T_N : \]

**Goal:** \( X \equiv D_N / \sqrt{N} \) ?

Replace \( t \) by \( t / \sqrt{N} \).

**Generating function:**

\[
= \left( (0.5)z + (0.5)z^{-1} \right)^N
\]

the generating function of the distribution of \( D_1 \)

\[
i = \sqrt{-1}
\]

Replace \( z \) by \( e^{-it} \)

**Fourier transform:**

\[
(\cos t)^N = \cos^N t
\]
Goal: $X \equiv \frac{D_N}{\sqrt{N}}$?

What about $D_N/\sqrt{N}$?

Replace $t$ by $t/\sqrt{N}$?

No way!!

Fourier transform: $\cos^N(t/\sqrt{N})$
\[ X = D_N / \sqrt{N} : \]

Generating functions
Fourier transforms

NO WAY!!

Fourier transform: \[ \cos^N(t / \sqrt{N}) \]

Fourier transform: \[ \cos^N(t / \sqrt{N}) \]
\[ X = \frac{D_N}{\sqrt{N}} : \text{NO WAY!} \]

Generating functions
Fourier transforms
Fourier analysis
Spectral theory

Useful?

Easier problem after standardization:
Compute the probability that
\(-1 < X < 1.\)

Exercise: \( \lim_{n \to \infty} \cos^n \left( \frac{3}{\sqrt{n}} \right) = e^{-3^2/2} \)

Fourier transform:
\[ \cos^N \left( \frac{t}{\sqrt{N}} \right) \]
\[ \approx \lim_{n \to \infty} \cos^n \left( \frac{t}{\sqrt{n}} \right) = e^{-t^2/2} \]

Verify for \( t = 3. \)
Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$. Then $Z$ is “close” to $X$ in distribution.

Fourier transform: $\cos^N(t/\sqrt{N})$

$\approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$
$X = D_N / \sqrt{N} :$

**Fourier transform:** \( \cos^N(t / \sqrt{N}) \)

\[ \approx \lim_{n \to \infty} \cos^n(t / \sqrt{n}) = e^{-t^2/2} \]

**Key idea of Central Limit Theorem:**

Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \).

Then \( Z \) is “close” to \( X \).

**Easier problem after standardization:**

Compute the probability that \(-1 < X < 1\).

Approximately equal to the probability that \(-1 < Z < 1\).
Key idea of Central Limit Theorem:

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$. Then $Z$ is “close” to $X$. How to find $Z$? Inverse Fourier Transform Its distribution ...

Easier problem after standardization:

Compute the probability that $-1 < X < 1$.

Approximately equal to the probability that $-1 < Z < 1$. 

$Z$:

$$e^{-x^2/2} \text{ dx}$$

infinitesimal

Do this for all $x \in \mathbb{R}$.
\[ Z: \quad \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \]

Do this for all \( x \in \mathbb{R} \)

NOTES

\[ D_2 \in \{2, 0, -2\} \]

distribution supported on three points

\[ D_N \in \{-N, -N + 2, \ldots, N - 2, N\} \]

distribution supported on \( N + 1 \) points

By contrast, the distribution of \( Z \) does not have finite support.
$Z$: \[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \]

Do this for all $x \in \mathbb{R}$.

NOTES There's a mistake:

\[ \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

probability theory: should get 1, not \( \sqrt{2\pi} \)
Z: \[
\frac{1}{\sqrt{2\pi}} \int_{x} e^{-\frac{x^2}{2}} \, dx
\]

Do this for all \( x \in \mathbb{R} \)

Problem: Compute the probability that \( Z = 7 \)

Solution: \[
\int_{7}^{7} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = 0
\]
Problem: Compute the probability that 
\[ 2 < Z < 3 \]

Solution: 
\[
\int_{2}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\Phi(x)]_{x=2}^{x=3} \\
= [\Phi(3)] - [\Phi(2)] = 0.0214 \\
= 2.14\% 
\]
$Z: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \quad \text{Do this for all } x \in \mathbb{R}$

Generating function:

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \text{Exercise}$

Fourier transform:

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-itx} e^{-x^2/2} \, dx = e^{-t^2/2}$

Verify for $t = 3$.

Key idea of Central Limit Theorem:

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$. Then $Z$ is “close” to $X$. 

25
$Z:\quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx$  

Do this for all $x \in \mathbb{R}$

**Exercise:**

\[
\int_{-\infty}^{\infty} e^{-3ix} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-3^2/2}
\]

**Fourier transform:**

\[
\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-t^2/2}
\]

**Key idea of Central Limit Theorem:**

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$. Then $Z$ is "close" to $X$. 

[Verify for $t = 3$.]
Easier problem after standardization:
Compute the probability that
\(-1 < X < 1.\)

Approximately equal to the probability that
\(-1 < Z < 1.\)

Approximate solution:
\[
\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\Phi(x)]_{x=-1}^{x=1} = 68.27\%\]
Easier problem after standardization:
Compute the probability that 
\(-1 < X < 1\).

Coin-flipping game: 
Flip a fair coin \(N\) times. 
If \(H\) heads and \(T\) tails, 
pay \(f(u^H d^T)\), 
30 days from now.

\[ N = 2,592,000 \]
\[ f(S) = (5000S - 5000)_+ \]