Financial Mathematics

Central Limit Theorem

Coin-flipping game: Flip a fair coin N times.

$N = 2,592,000$

If H heads and T tails,

$f(S) = (5000S - 5000)_+$

pay $f(e^{Hu+Td})$,

30 days from now.

expected payout $=: E = ???$

Easier problem:

probability problems,

then expected value problems

Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$

probability problems,
then expected value problems

Easier problem:

Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$

DIVIDE BY $\sqrt{N}$

$$X := (H - T)/\sqrt{N}$$

standardization of $H - T$

mean $= 0$
variance $= 1$
(standard)

probability problems,
then expected value problems

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DIVIDE BY $\sqrt{N}$

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Easier problem after standardization:

Compute the probability that

$$-1 < X < 1.$$

$H_1 :=$ number of heads after first flip

$H_2 :=$ number of heads after second flip

$\vdots$

$H_N :=$ number of heads after Nth flip $= H$

Easier problem:

Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$

$$X := (H - T)/\sqrt{N}$$

Easier problem after standardization:

Compute the probability that

$$-1 < X < 1.$$

X is hard . . .

For all integers $j \in [1, N]$,

$H_j :=$ number of heads after jth flip

$T_j :=$ number of tails after jth flip

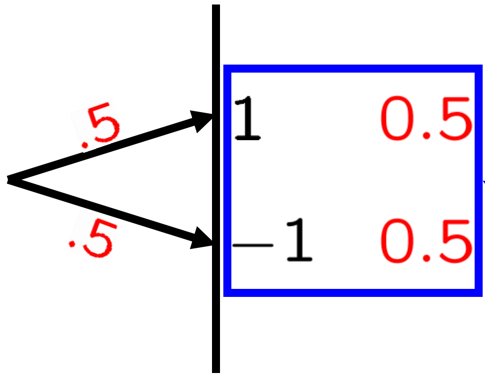
$D_j := H_j - T_j$

Easier: D_1, $D_1/7$, D_2, D_N

$H = H_N$, $T = T_N$,

$$X = (H_N - T_N)/\sqrt{N}$$
$$= D_N/\sqrt{N}$$

$$D_1 = \overset{0}{H_1} - \overset{1}{T_1} :$$

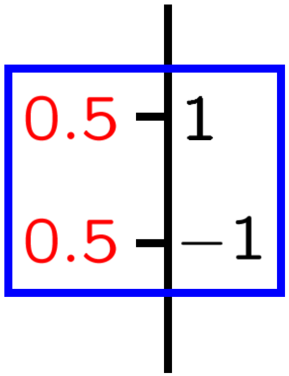


distribution of D_1

keep the distribution
forget its origin

distribution of $T_1 - H_1$
is exactly the same

$$D_1 = H_1 - T_1 :$$



keep the distribution
forget its origin

$$D_1 = H_1 - T_1 :$$

0.5	1	$z^1 = z$
0.5	-1	z^{-1}

$$i = \sqrt{-1}$$

Replace z by e^{-it}

(expression of z)

Generating function: $(0.5)z + (0.5)z^{-1}$

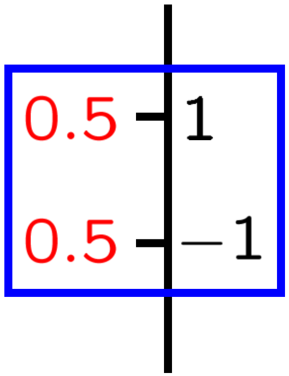
Fourier transform: $(0.5)e^{-it} + (0.5)e^{it}$

ξ t not time

$$\| \quad \cos t$$

$$0.5 \times [\; e^{it} = \cos t + i \sin t \;] \; +$$
$$0.5 \times [\; e^{-it} = \cos t - i \sin t \;]$$

$D_1 = H_1 - T_1$:



Cannot recover the random variable!
Only its distribution.

form of the distribution of D_1 is $\cos t$

$T_1 - H_1$ has the same distribution.

Generating function: $(0.5)z + (0.5)z^{-1}$

Fourier transform: $(0.5)e^{-it} + (0.5)e^{it}$

$\|$

$\cos t$

$0.5 \times [\ e^{it} = \cos t + i \sin t\ ] +$

$0.5 \times [\ e^{-it} = \cos t - i \sin t\ ]$

Inverse Fourier transform

divide by 7

$D_1 = H_1 - T_1$:

0.5	1	z^1
0.5	-1	z^{-1}

What about $D_1/7$?

Generating function: $(0.5)z + (0.5)z^{-1}$

Fourier transform: $(0.5)e^{-it} + (0.5)e^{it}$

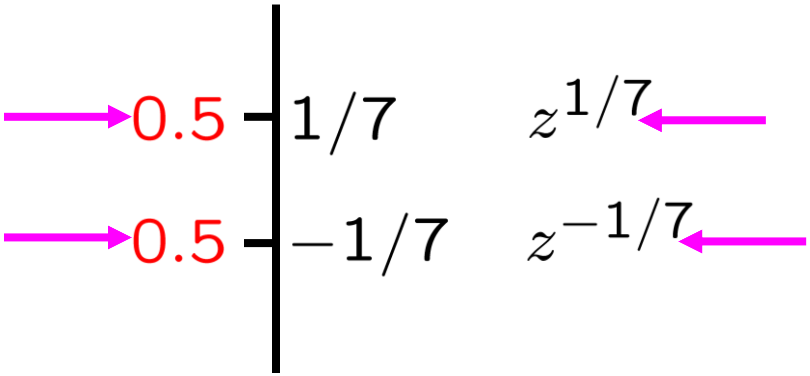
$$\| $$

$\cos t$

Repl. t by $t/7$

$$e^{it} = \cos t + i \sin t$$
$$e^{-it} = \cos t - i \sin t$$

$D_1/7$:



What about $D_1/7$?

Replace t by $t/7$.

$$i = \sqrt{-1}$$

Replace z by e^{-it}

Generating function: $(0.5)z^{1/7} + (0.5)z^{-1/7}$

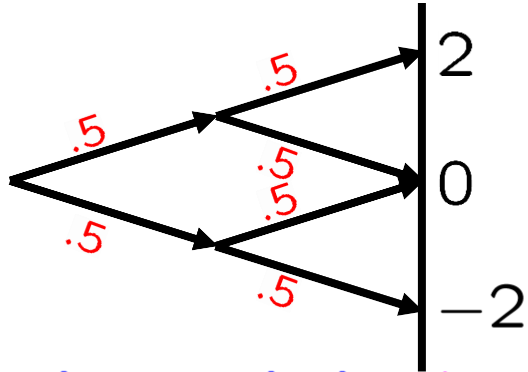
Fourier transform: $(0.5)e^{-it/7} + (0.5)e^{it/7}$

$$\| $$

$$\cos(t/7)$$

$$e^{it/7} = \cos(t/7) + i \sin(t/7)$$
$$e^{-it/7} = \cos(t/7) - i \sin(t/7)$$

$D_2 = \overset{0}{H_2} - \overset{2}{T_2}$:



0.25

$0.25 + 0.25 = 0.5$

0.25

forget its origin keep the distribution

$D_2 = H_2 - T_2$:

0.25	2	z^2
0.5	0	$z^0 = 1$
0.25	-2	z^{-2}

forget its origin keep the distribution

Generating function:

$$(0.25)z^2 + 0.5 + (0.25)z^{-2}$$
$$= ((0.5)z + (0.5)z^{-1})^2$$

the generating function of the distribution of D_1

$i = \sqrt{-1}$

Replace z by e^{-it}

Fourier transform: $(\cos t)^2 = \cos^2 t$

$D_N = H_N - T_N$:

NO WAY!!

Goal: $X = D_N/\sqrt{N}$?

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

Generating function:

NO WAY!!

$$= ((0.5)z + (0.5)z^{-1})^N$$

the generating function of the distribution of D_1

$i = \sqrt{-1}$

Replace z by e^{-it}

Fourier transform: $(\cos t)^N = \cos^N t$

$X = D_N/\sqrt{N}$:

Goal: $X = D_N/\sqrt{N}$

NO WAY!!

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

Fourier transform: $\cos^N(t/\sqrt{N})$

Generating functions
Fourier transforms

$X = D_N/\sqrt{N}$:

NO WAY!!

Fourier transform: $\cos^N(t/\sqrt{N})$

Fourier transform: $\cos^N(t/\sqrt{N})$

$X = D_N/\sqrt{N}$: NO WAY!!! NO WAY!!!

Generating functions
Fourier transforms
Fourier analysis
Spectral theory

Useful?

Easier problem after standardization:

Compute the probability that

$$-1 < X < 1.$$

Exercise: $\lim_{n\to\infty} \cos^n(3/\sqrt{n}) = e^{-3^2/2}$

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Verify for $t = 3$.

$X = D_N/\sqrt{N}$:

Fourier transform of distr. of X

Fourier transform: $\cos^N(t/\sqrt{N})$

Fourier transform of distr. of Z

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X. (in distribution)

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

$X = D_N/\sqrt{N}$:

Fourier transform: $\cos^N(t/\sqrt{N})$

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Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

How to find Z?
Inverse Fourier Transform
Its distribution . . .

Easier problem after standardization:

Compute the probability that

$$-1 < X < 1.$$

Approximately equal to the probability that

$$-1 < Z < 1.$$

Z:

$e^{-x^2/2}\,dx$ — x

infinitesimal

Do this for *all* $x \in \mathbb{R}$

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Z:

$$e^{-x^2/2}\,dx \;\Big|\; x$$

Do this for *all* $x \in \mathbb{R}$

NOTES

$D_2 \in \{2, 0, -2\}$
distribution supported on three points

$D_N \in \{-N, -N+2, \ldots, N-2, N\}$
distribution supported on $N+1$ points

By contrast, the distribution of Z
does not have finite support.

Z:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ — x

Do this for *all* $x \in \mathbb{R}$

$\exists$ RV Z with this dist.

NOTES There's a mistake:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

probability theory: should get 1, not $\sqrt{2\pi}$

Z:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \quad \Big|\, x$$

Do this for *all* $x \in \mathbb{R}$

Problem: Compute the probability that

$$Z = 7$$

Solution: $$\int_7^7 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = 0$$

Z:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \;\Big|\; x$$

Do this for *all* $x \in \mathbb{R}$

Problem: Compute the probability that

$$2 < Z < 3$$

Solution:

$$\int_2^3 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = [\Phi(x)]_{x=2}^{x=3}$$

$$= [\Phi(3)] - [\Phi(2)] = 0.0214$$

$$= 2.14\%$$

Z:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \quad \Big| \quad x \qquad z^x$$

Do this for *all* $x \in \mathbb{R}$

Generating function:

$$\int_{-\infty}^{\infty} z^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \text{Exercise}$$

Fourier transform:

Verify for $t = 3$.

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

Z:
$X \approx$ $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ | x z^x

Do this for all $x \in \mathbb{R}$

Exercise: $\int_{-\infty}^{\infty} e^{-3ix} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = e^{-3^2/2}$

Fourier transform: Verify for $t = 3$.

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

probability problems,
then expected value problems

Z:
$\underset{X}{\wr\wr}$

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \qquad x \qquad z^x$$

Do this for all $x \in \mathbb{R}$

Easier problem after standardization:

Compute the probability that
$-1 < X < 1$.

Approximately equal to the probability that
$-1 < Z < 1$.

Approximate solution:

Berry-Esseen Theorem

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = [\Phi(x)]_{x=-1}^{x=1}$$
$$= 68.27\% \quad ■$$

probability problems,
then expected value problems

Z:

$X \approx$

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \Big|\ x \qquad z^x$

Do this for *all* $x \in \mathbb{R}$

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If H heads and T tails,

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STOP expected payout $=: E = ???$