Financial Mathematics

Introduction to
the Black-Scholes formula

Description of Black-Scholes: We price and sell a T-year (European) call option, struck at K, on one share of a stock with current price S_0.

(Typically, $T < 1$, e.g., $T = 1/4$.)

Payoff: $f(S) := (S - K)_+$

Market analyst: annual drift $= \mu_*$; annual volatility $= \sigma_*$

$\mu :=$ drift over T years $= \mu_* T$ → $\mu_*/4$

$\sigma :=$ volatility over T years $= \sigma_* \sqrt{T}$ → $\sigma_*/\sqrt{4}$

Banker: ann. logarithmic risk-free factor $= r_*$

$r :=$ log. risk-free factor over T years $= r_* T$ → $r_*/4$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

ANNUAL

We select: $p \in (0, 1)$ $q := 1 - p$

$\forall$integers $n \geq 1$,

Centrality of BS!
For any choice of $p \in (0, 1)$,

n-subperiod (p, q) CRR model —$n \to \infty$→ Black-Scholes model

Goal: lim nth option price, as $n \to \infty$

Describe the model, its pricing and its hedging.

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Market analyst: $\left\{ \begin{array}{r} \text{annual drift} = \mu_* \\ \text{annual volatility} = \sigma_* \end{array} \right.$

$$\mu := \text{drift over } T \text{ years} = \mu_* T$$

$$\sigma := \text{volatility over } T \text{ years} = \sigma_* \sqrt{T}$$

Banker: ann. logarithmic risk-free factor $= r_*$

$$r := \text{log. risk-free factor over } T \text{ years} = r_* T$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$,

stock price: S — with probability p → Se^{u_n}; with probability q → Se^{d_n} ($\times$)

n-subperiod (p, q) CRR model

change to ln(stock pr): ~~S~~ — with probability p → ~~S~~ $+ u_n$; with probability q → ~~S~~ $+ d_n$ ($+$)

Goal: lim nth option price, as $n \to \infty$

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Banker: ann. logarithmic risk-free factor $= r_*$

$r :=$ log. risk-free factor over T years $= r_* T$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0, 1)$ $\quad$ $q := 1 - p$

$\forall$integers $n \geq 1$, $\quad$ drift over T/n years $= \mu/n$

volatility over T/n years $= \sigma/\sqrt{n}$

n-subperiod (p, q) CRR model

change to ln(stock pr): ~~s~~ $\to$ (p) ~~s~~ $+ u_n$; (q) ~~s~~ $+ d_n$

Goal: lim nth option price, as $n \to \infty$

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Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, drift over T/n years $= \mu/n$

volatility over T/n years $= \sigma/\sqrt{n}$

$pu_n + qd_n = \mu/n$

change to ln(stock pr): s → (p) $s + u_n$; (q) $s + d_n$; $+$

Goal: lim nth option price, as $n \to \infty$

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Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$,

drift over T/n years $= \mu/n$

volatility over T/n years $= \sigma/\sqrt{n}$

$$p u_n + q d_n = \mu/n$$

$$\sqrt{pq}\,(u_n - d_n) = \sigma/\sqrt{n}$$

change to ln(stock pr): $s \xrightarrow{p} s + u_n$, $\quad s \xrightarrow{q} s + d_n$

Goal: lim nth option price, as $n \to \infty$

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Banker: ann. logarithmic risk-free factor $= r_*$

$r :=$ log. risk-free factor over T years $= r_* T$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad pu_n + qd_n = \mu/n$

$\sqrt{npq}(u_n - d_n) = \sigma/\sqrt{n}$

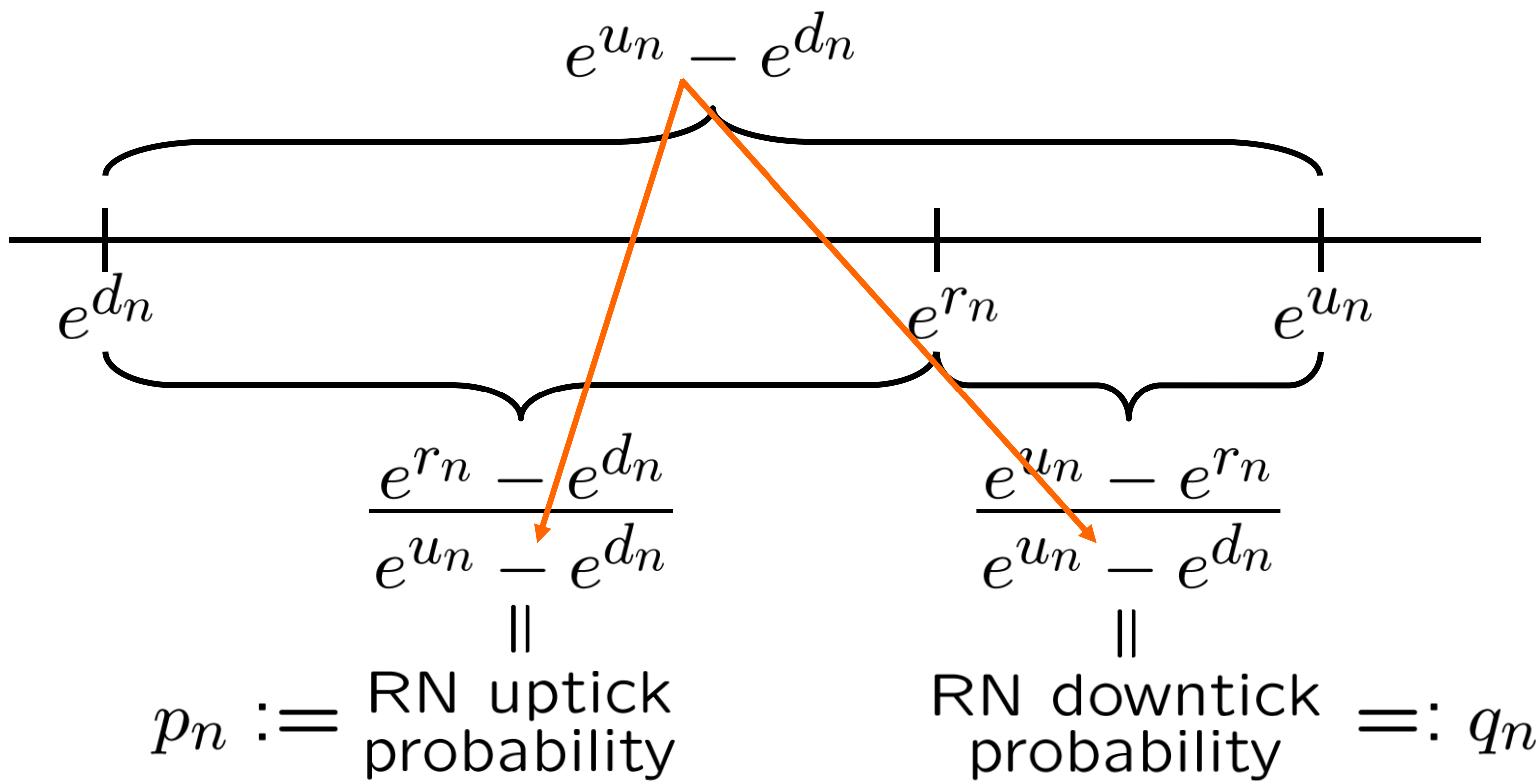
Calibration: Solve for u_n, d_n later. . .

$pu_n + qd_n = \mu/n$

$\sqrt{npq}(u_n - d_n) = \sigma/\sqrt{n}$

$r_n :=$ logarithmic interest rate over T/n years $= r/n$

Goal: lim nth option price, as $n \to \infty$ option price, as $n \to \infty$



Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$,

$$\left.\begin{aligned} pu_n + qd_n &= \mu/n \\ \sqrt{pq}(u_n - d_n) &= \sigma/\sqrt{n} \end{aligned}\right\}$$

Calibration: Solve for u_n, d_n later. . .

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Goal: lim nth option price, as $n \to \infty$

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e^{d_n} $\quad$ e^{r_n} $\quad$ e^{u_n}

$$\frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}}$$

$p_n :=$ RN uptick probability $\qquad$ RN downtick probability $=: q_n$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0, 1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad pu_n + qd_n = \mu/n$

$$r_n = r/n$$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n}$$

r_n $\qquad p_n = \dfrac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}}$ $\qquad q_n = \dfrac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$ r/n

Goal: lim nth option price, as $n \to \infty$

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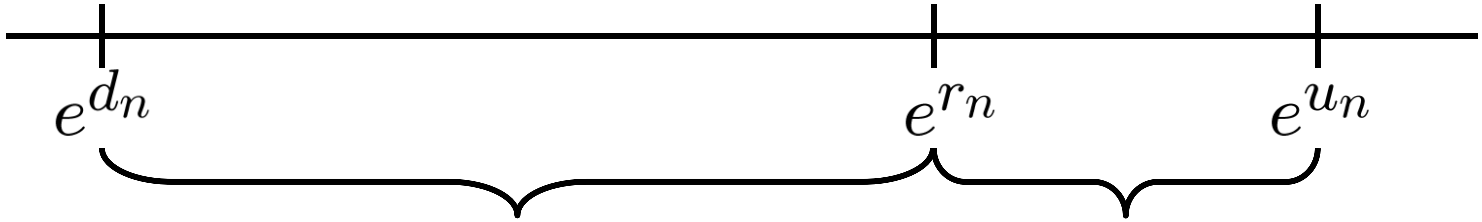

$$p_n := \text{RN uptick probability} = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = \text{RN downtick probability} =: q_n$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0, 1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad p u_n + q d_n = \mu / n$

$$\sqrt{pq}(u_n - d_n) = \sigma / \sqrt{n} \qquad r_n = r/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

risk neutral probabilities

Goal: lim nth option price, as $n \to \infty$

$V_n :=$ initial value of hedge $=$ nth option price

Goal: $\lim_{n\to\infty} V_n$

final price of underlying $= S_0 (e^{u_n})^j (e^{d_n})^{n-j}$

$r :=$ logarithmic interest rate over T years

$V_n e^r =$ RN expected final value of hedge

$$= \sum_{j=0}^{n} \begin{bmatrix} \text{RN prob.} \\ j, n-j \end{bmatrix} \begin{bmatrix} \text{payout} \\ j, n-j \end{bmatrix}$$

$$\binom{n}{j} [p_n^j q_n^{n-j}]$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu/n$

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risk neutral probabilities

Payoff: $f(S) := (S - K)_+$

$V_n :=$ initial value of hedge

Goal: $\lim_{n\to\infty} V_n$

$S_0 e^{ju_n+(n-j)d_n}$
$\|$
final price of underlying $= S_0 (e^{u_n})^j (e^{d_n})^{n-j}$

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$$= \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}] \quad [f(S_0 e^{ju_n+(n-j)d_n})]$$

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$\forall$integers $n \geq 1$, $\qquad pu_n + qd_n = \mu/n$

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$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

risk neutral probabilities

$$f(S) := (S-K)_+$$

Goal: $\lim_{n\to\infty} V_n$

$r :=$ logarithmic interest rate over T years

$V_n e^r$

$$f(S) := (S-K)_+$$

$$V_n e^r = \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n+(n-j)d_n})]$$

$$= \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}] \quad [f(S_0 e^{ju_n+(n-j)d_n})]$$

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CRR Option Pricing Formula: $f(S) := (S - K)_+$

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This gives us V_n. Next . . .

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

Goal: $\lim_{n \to \infty} V_n$

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risk neutral probabilities

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$r :=$ logarithmic interest rate over T years

Let $K' :=$ present value of strike $= \dfrac{K}{e^r}$.

bogus at the money quotient $:= S_0/K$

at the money quotient $:= S_0/K'$

logarithmic at the money quotient $:= \ln(S_0/K')$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

Goal: $\lim_{n\to\infty} V_n$

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risk neutral probabilities

Description of Black-Scholes: We price and sell a T-year (European) call option, struck at K, on one share of a stock with current price S_0.

Let $K' := \dfrac{K}{e^r}$.

Let $K' :=$ K

$$\textbf{Black-Scholes center} := \frac{\ln(S_0/K')}{\sigma}$$

$$\textbf{logarithmic at the money quotient} := \ln(S_0/K')$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$ Goal: $\lim\limits_{n\to\infty} V_n$

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu/n$ $\quad r_n = r/n$

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risk neutral probabilities

Description of Black-Scholes: We price and sell a T-year (European) call option, struck at K, on one share of a stock with current price S_0.

Let $K' := \dfrac{K}{e^r}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

$$\textbf{Black-Scholes center} := \frac{\ln(S_0/K')}{\sigma}$$

$$\textbf{Black-Scholes interval} := (d_-, d_+)$$

$$\textbf{logarithmic at the money quotient} := \ln(S_0/K')$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$ Goal: $\lim_{n\to\infty} V_n$

We select: $p \in (0, 1)$ $q := 1 - p$

$\forall$integers $n \geq 1$, $pu_n + qd_n = \mu/n$

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Let $K' := \frac{K}{e^r}$. Let $d_\pm := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$. Goal: $\lim_{n\to\infty} V_n$

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad p u_n + q d_n = \mu/n \qquad r_n = r/n$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n}$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $\quad f(S) := (S - K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{j u_n + (n-j) d_n})]$$

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$. Goal: $\lim_{n\to\infty} V_n$

We select: p

$\forall$int

$r_n = r/n$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} = \frac{K}{e^r}. \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

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CRR Option Pricing Formula: $f(S) := (S - K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n + (n-j)d_n})]$$

Theorem: Let $K' := \frac{K}{e^r}$. Let $d_\pm := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: verison zero

This is the Black-Scholes Option Pricing Formula.

Several proofs offered, in later lectures.

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

Goal: $\lim_{n\to\infty} V_n$

We select: $p \in (0,1)$ $\quad q := 1 - p$

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Remarks: verison zero

This is the Black-Scholes Option Pricing Formula.

Several proofs offered, in later lectures.

simplicity

NOTE: This does not depend on μ or p. centrality

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad pu_n + qd_n = \mu/n$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n} \qquad r_n = r/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $\qquad f(S) := (S - K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n + (n-j)d_n})]$$

Theorem: Let $K' := \dfrac{K}{e^r}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: μ, σ, r, p, S_0, K

outputs: V_1, V_2, V_3, . . .

asymptotic output: $V := \lim_{n\to\infty} V_n$

NOTE: This does not depend on μ or p.

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu/n$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n} \qquad r_n = r/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $\qquad f(S) := (S-K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n + (n-j)d_n})]$$

Theorem: Let $K' := \dfrac{K}{e^r}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: μ, σ, r, p, S_0, K

outputs: V_1, V_2, V_3, ...

asymptotic output: $V := \lim_{n\to\infty} V_n$

NOTE: This does not depend on μ or p.

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\quad q := 1 - p$

std to use ANNUAL drift, vol and interest

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu/n$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n}$$

$$r_n = r/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $\quad f(S) := (S - K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n + (n-j)d_n})]$$

Theorem: Let $K' := \dfrac{K}{e^r}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: μ, σ, r, p, S_0, K

$$\mu := \text{lesser drift over } T \text{ years} = \mu_* T$$
$$\sigma := \text{volatility over } T \text{ years} = \sigma_* \sqrt{T}$$

std to use ANNUAL drift, vol and interest

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad pu_n + qd_n = \mu/n \qquad r_n = r/n$

$$\sqrt{pq}(u_n - d_n) = \sigma/\sqrt{n}$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

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Then $\lim_{n \to \infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: μ, σ, r, p, S_0, K

$$\mu = \mu_* T$$
$$\sigma = \sigma_* \sqrt{T}$$

$r :=$ logarithmic interest rate over T years $= r_* T$

$\sigma = \sigma_* \sqrt{T}$

std to use ANNUAL drift, vol and interest

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\qquad p u_n + q d_n = \mu / n$ $\qquad r_n = r/n$

$$\sqrt{pq}(u_n - d_n) = \sigma / \sqrt{n}$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

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Then $\lim\limits_{n \to \infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: μ, σ, r, p, S_0, K

$\mu = \mu_* T$

$\sigma = \sigma_* \sqrt{T}$

$r = r_* T$

r inputs: T, μ_*, σ_*, r_*, p, S_0, K

Given to us: $\mu \in \mathbb{R}$, $\sigma > 0$, $r > 0$

std to use ANNUAL drift, vol and interest

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu/n$ $\qquad r_n = r/n$

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Then $\lim_{n \to \infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks: inputs: $\mu,\ \sigma,\ r,\ p,\ S_0,\ K$

inputs: $T,\ \mu_*,\ \sigma_*,\ r_*,\ p,\ S_0,\ K$

$$\mu = \mu_* T$$
$$\sigma = \sigma_* \sqrt{T}$$
$$r = r_* T$$

Given to us: $\mu_* \in \mathbb{R},\ \sigma_* > 0,\ r_* > 0,\ T > 0$

We select: $p \in (0,1) \qquad q := 1 - p$

std to use ANNUAL drift, vol and interest

$\forall$integers $n \geq 1$,
$$p u_n + q d_n = \mu_* T/n$$
$$\sqrt{pq}(u_n - d_n) = \sigma_* \sqrt{T}/\sqrt{n}$$
$$r_n = r_* T/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $f(S) := (S-K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{j u_n + (n-j) d_n})]$$

Theorem: Let $K' := \dfrac{K}{e^{r_* T}}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks:

inputs: T, μ_*, σ_*, r_*, p, S_0, K

asymptotic output: $V := \lim_{n\to\infty} V_n$

NOTE: V does not depend on μ_* or p.

Given to us: $\mu_* \in \mathbb{R}$, $\sigma_* > 0$, $r_* > 0$, $T > 0$

We select: $p \in (0,1)$ $\qquad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu_* T/n$

$$\sqrt{pq}(u_n - d_n) = \sigma_* \sqrt{T}/\sqrt{n} \qquad r_n = r_* T/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

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Theorem: Let $K' := \dfrac{K}{e^{r_* T}}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks:

Centrality and simplicity of BS!

inputs: T, μ_*, σ_*, r_*, p, S_0, K

asymptotic output: $V := \lim_{n\to\infty} V_n$

NOTE: V does not depend on μ_* or p.

Given to us: $\mu_* \in \mathbb{R}$, $\sigma_* > 0$, $r_* > 0$, $T > 0$

We select: $p \in (0,1)$ $\quad q := 1 - p$

$\forall$integers $n \geq 1$, $\quad pu_n + qd_n = \mu_* T/n$

$$\sqrt{pq}(u_n - d_n) = \sigma_* \sqrt{T}/\sqrt{n} \qquad r_n = r_* T/n$$

$$p_n = \frac{e^{r_n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r_n}}{e^{u_n} - e^{d_n}} = 1 - p_n$$

CRR Option Pricing Formula: $\quad f(S) := (S - K)_+$

$$V_n = e^{-r} \sum_{j=0}^{n} \binom{n}{j} [p_n^j q_n^{n-j}][f(S_0 e^{ju_n + (n-j)d_n})]$$

Theorem: Let $K' := \dfrac{K}{e^{r_* T}}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

Then $\lim_{n\to\infty} V_n = S_0[\Phi(d_+)] - K'[\Phi(d_-)]$.

Remarks:

Centrality and simplicity of BS!

inputs: T, μ_*, σ_*, r_*, p, S_0, K

asymptotic output: $V := \lim_{n \to \infty} V_n$

NOTE: V does not depend on μ_* or p.

inputs: T, ~~μ_*~~, σ_*, r_*, ~~p~~, S_0, K

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

Let $K' := \dfrac{K}{e^{r_* T}}$. Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

$S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

Remarks:

Centrality and simplicity of BS!

asymptotic output: $V := \lim_{n \to \infty} V_n$

NOTE: V does not depend on μ_* or p.

inputs: T, σ_*, r_*, S_0, K

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

inputs: T, σ_*, r_*, S_0, K

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

TAKE ln OF BOTH SIDES → $\frac{S_0}{K'} = \frac{S_0 e^{r_* T}}{K}$

$$\ln\left(\frac{S_0}{K'}\right) = \left[\ln\left(\frac{S_0}{K}\right)\right] + r_* T$$

inputs: $T,\ \sigma_*,\ r_*,\ S_0,\ K$

Let $K' := \frac{K}{e^{r_* T}} = Ke^{-r_* T}$.

NONSTD

Let $d_\pm := \frac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$. $= \frac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

$$= S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$$

inputs: T, σ_*, r_*, S_0, K

$$\frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$$

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

$$S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$$

inputs: $T,\ \sigma_*,\ r_*,\ S_0,\ K$

$$\frac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$$

forward price on stock

Let $F := S_0 e^{r_* T}$.

$e^{r_* T}$

$\parallel$

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$

second version of Black-Scholes

$\parallel$

GONE

$$e^{-r_* T}\left([S_0 e^{r_* T}][\Phi(d_+)] - K[\Phi(d_-)]\right)$$

$\parallel$

$$e^{-r_* T}\left(F[\Phi(d_+)] - K[\Phi(d_-)]\right)$$

inputs: T, σ_*, r_*, S_0, K

$$\frac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}$$

forward price on stock

Let $\boxed{F} := S_0 e^{r_* T}$.

Let $d_\pm := \quad \dfrac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

$$e^{-r_* T}\left(F[\Phi(d_+)] - K[\Phi(d_-)]\right)$$

inputs:
T, σ_*, r_*, S_0, K

inputs: T, σ_*, r_*, S_0, K

forward price on stock

Let $\boxed{F} := S_0 e^{r_* T}$.

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_* \sqrt{T}} \pm \dfrac{\sigma_* \sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

FORWARD FORMULA

inputs:
$T,\ \sigma_*,\ r_*,\ S_0,\ K$

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

PRESENT FORMULA

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [K e^{-r_* T}][\Phi(d_-)]$

second version of Black-Scholes

NEUTRAL FORMULA

forward price on stock

Let $\boxed{F} := S e^{r_* T}$.

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

FORWARD FORMULA

inputs:
$T,\ \sigma_*,\ r_*,\ S_0,\ K$

PRESENT FORMULA

inputs:
$\sigma,\ r,\ S_0,\ K$

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

Let $K' := \dfrac{K}{e^{r}}$.

PRESENT FORMULA

$:= \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

version zero of Black-Scholes

PRESENT FORMULA
TIME NORMALIZED

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$

second version of Black-Scholes

NEUTRAL FORMULA

forward price on stock

Let $\boxed{F} := Se^{r_* T}$.

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

FORWARD FORMULA