Financial Mathematics

Testing

the Black-Scholes formula
inputs: $T$, $\sigma_*$, $r_*$, $S_0$, $K$

Let $K' := \frac{K}{e^{r_* T}}$.

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

**first version of Black-Scholes**

Let $d_{\pm} := \frac{[\ln(S_0/K)] + r_* T}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [K e^{-r_* T}][\Phi(d_-)]$

**second version of Black-Scholes**

forward price on stock

Let $F := S e^{r_* T}$.

Let $d_{\pm} := \frac{[\ln(F/K)]}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

output: $e^{-r_* T} \left( F[\Phi(d_+)] - K[\Phi(d_-)] \right)$

**third version of Black-Scholes**

inputs: $\sigma$, $r$, $S_0$, $K$

Let $K' := \frac{K}{e^{r}}$.

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

**version zero of Black-Scholes**

Do these formulas really approximate the CRR price?
Kyle wants right, **but not** obligation, to buy 5000 shares of ABC for $5000, 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

- \( N \)-subperiod 50.001-49.999

CRR model:

\[
S_0 \Phi(d_+) - K^r \Phi(d_-)
\]

PRESENT FORMULA

TIME NORMALIZED

PRESENT FORMULA

TIME NORMALIZED

5000 \( S_0 \Phi(d_+) - K^r \Phi(d_-) \)

Do these formulas really approximate the CRR price?
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, Gail, seller 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

\[ N - \text{subperiod 50.001-49.999} \]

\[ V = e^{-rN}E = 120.7994402 \]

PRESENT FORMULA TIME NORMALIZED

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

close?

Do these formulas really approximate the CRR price?
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, Gail, seller 30 days from now.

\[ N := \text{number of seconds in 30 days} \]

Gail selects:

\[ N \text{-subperiod 50.001-49.999} \]

CRR model

Market analyst: (ann) vol = 0.200002881086

Banker:

\[ V = \text{120.7994402} \]

(annual) continuous compounding nominal rate

\[ = 0.05000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

\[ V = 120.7994402 \]

5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, 30 days from now. $K = 1$, $T = 30/365$

$N := \text{number of seconds in 30 days}$

Gail selects:

- $N$-subperiod 50.001-49.999
- CRR model
- Market analyst: (ann) vol = 0.200002881086

Banker:

(annual) continuous compounding nominal rate = 0.0315359998802

$$V = 120.7994402$$

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

close?
\[ T = \frac{30}{365} \]

Assume: Initial price = $1/share.

\[ r = r_* T \]

\[ \sigma = \sigma_* \sqrt{T} \]

\[ K = 1 \]

\[ K' = \frac{K}{e^r} \]

\[ d_+ = \frac{\ln(S_0/K')}{\frac{\sigma}{2}} + \frac{\text{vol}}{2} \]

\[ d_- = \frac{\ln(S_0/K')}{\frac{\sigma}{2}} - \frac{\text{vol}}{2} \]

Market analyst: (ann) vol = 0.200002881086

Banker:

(annual) continuous compounding nominal rate

\[ = 0.0315359998802 \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} \]
\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} \]
\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} \]

Market analyst: (annual) vol = 0.200002881086

Banker:
(annual) continuous compounding nominal rate = 0.0315359998802

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

Is it close?
\[
T = \frac{30}{365} \\
r = r_T T = 0.00259199999014 \\
\sigma = \sigma_T \sqrt{T} = 0.057339043865 \\
K = 1 \\
K' = \frac{K}{e^r} = 0.997411356345 \\
d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} \\
d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} \\
\text{Market analyst: (ann) vol} = 0.200002881086 \\
\text{Banker: (annual) continuous compounding nominal rate} = 0.0315359998802 \\
V = 120.7994402 \\
5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \\
\text{close?}
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]

\[ K' = \frac{K}{e^{rT}} = 0.997411356345 \]

\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} \]
\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} \]

\[ \ln(S_0/K') = 0.002591999999014 \]

The option is (bogus) "at the money".

Whenever \( S_0 = K \),
\[ \ln(S_0/K') = r \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

close?
\[ T = \frac{30}{365} \]
\[ r = r_0 T = 0.00259199999014 \]
\[ \sigma = \sigma_0 \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]

\[ K' = \frac{K}{e^r} = 0.997411356345 \]

\[ d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} = \begin{pmatrix} 0.0452047996532 \\ +0.0286695219325 \end{pmatrix} \]
\[ d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} = \begin{pmatrix} 0.0452047996532 \\ -0.0286695219325 \end{pmatrix} \]

\[ \ln(S_0/K') = 0.00259199999014 \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]

**close?**
\[ T = 30/365 \]
\[ r = r_*T = 0.00259199999014 \]
\[ \sigma = \sigma_*/\sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ d_+ = \left( -0.02866952193 + \frac{0.04520479965 \times 320}{0.0738743215857} \right) \]
\[ d_- = \left( -0.02866952193 - \frac{0.04520479965 \times 320}{0.0165352777207} \right) \]
\[ V = 120.7994402 \]
\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]

\[ d_+ = \begin{pmatrix} 0.0452047996532 \\ +0.0286695219325 \end{pmatrix} = 0.0738743215857 \]

\[ d_- = \begin{pmatrix} 0.0452047996532 \\ -0.0286695219325 \end{pmatrix} = 0.0165352777207 \]

\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d_-) = 0.50660 \]

\[ V = 120.7994402 \]

\[ 5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)]) \]
\[ T = \frac{30}{365} \]
\[ r = r \times T = 0.00259199999014 \]
\[ \sigma = \sigma \times \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d_-) = 0.50660 \]

\[ S_0 \left[ \Phi(d_+) \right] - K' \left[ \Phi(d_-) \right] = 0.024151406898 \]
\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d_-) = 0.50660 \]

\[ V = 120.7994402 \]

\[ 5000 \left( S_0 \left[ \Phi(d_+) \right] - K' \left[ \Phi(d_-) \right] \right) \]
\[ T = \frac{30}{365} \]
\[ r = r_* T = 0.00259199999014 \]
\[ \sigma = \sigma_* \sqrt{T} = 0.057339043865 \]
\[ K = 1 \]
\[ K' = \frac{K}{e^r} = 0.997411356345 \]
\[ \Phi(d_+) = 0.52944 \]
\[ \Phi(d_-) = 0.50660 \]
\[ S_0[\Phi(d_+)] - K'[\Phi(d_-)] = 0.024151406898 \]
\[
5000 \left( S_0[\Phi(d_+)] - K'[\Phi(d_-)] \right) = 120.7570345
\]
\[ V = 120.7994402 \]
inputs: $T, \sigma_*, r_*, S_0, K$

**PRESENT FORMULA**

Let $K' := \frac{K}{e^{r_* T}}$.

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

**output:** $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

_first version of Black-Scholes_

**NEUTRAL FORMULA**

Let $d_{\pm} := \frac{\ln(S_0/K)}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

**output:** $S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$

_second version of Black-Scholes_

Let $F := Se^{r_* T}$.

Let $d_{\pm} := \frac{\ln(F/K)}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

**forward price on stock**

**output:** $e^{-r_* T} (F[\Phi(d_+)] - K[\Phi(d_-)])$

_third version of Black-Scholes_

inputs: $\sigma, r, S_0, K$

**PRESENT FORMULA**

Let $K' := \frac{K}{e^r}$.

Let $d_{\pm} := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$.

**output:** $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

_version zero of Black-Scholes_

Do these formulas really approximate the CRR price? **YES**
inputs: $T, \sigma_*, r_*$, $S_0, K$

Let $d_+ := \frac{\ln(S_0/K) + r_* T}{\sigma_* \sqrt{T}} \pm \frac{\sigma_* \sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes
inputs: $T, \sigma_*, r_*, S_0, K$

Let $d_\pm := \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \frac{\sigma_*\sqrt{T}}{2}.$

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

$\text{BiSch}(T, \sigma_*, r_*, S_0, K) :=$

$$S_0 \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} + \frac{\sigma_*\sqrt{T}}{2} \right) \right]$$

$- \left[ Ke^{-r_*T} \right] \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} - \frac{\sigma_*\sqrt{T}}{2} \right) \right]$}

Fact: For all $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$, $\sigma_* \mapsto \text{BiSch}(T, \sigma_*, r_*, S_0, K) : (0, \infty) \to (0, \infty)$ is increasing.

Exercise: Prove this.
Definition:
For all $V > 0$, $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$, if $\exists \sigma_* > 0$ such that

$$V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)$$

then this solution $\sigma_*$ is unique and is called the **implied volatility** associated to $V$, $T$, $r_*$, $S_0$ and $K$.

$$\text{BlSch}(T, \sigma_*, r_*, S_0, K) := S_0 \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_* T}{\sigma_* \sqrt{T}} + \frac{\sigma_* \sqrt{T}}{2} \right) \right]$$

$$- \left[ Ke^{-r_* T} \right] \left[ \Phi \left( \frac{[\ln(S_0/K)] + r_* T}{\sigma_* \sqrt{T}} - \frac{\sigma_* \sqrt{T}}{2} \right) \right]$$

Fact: For all $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$, $\sigma_* \mapsto \text{BlSch}(T, \sigma_*, r_*, S_0, K) : (0, \infty) \rightarrow (0, \infty)$ is increasing.

**Exercise**: Prove this.
Definition: For all \( V > 0, T > 0, r_\star > 0, S_0 > 0 \) and \( K > 0 \), if \( \exists \sigma_\star > 0 \) such that
\[
V = \text{BlSch}(T, \sigma_\star, r_\star, S_0, K)
\]
then this solution \( \sigma_\star \) is unique and is called the **implied volatility** associated to \( V, T, r_\star, S_0 \) and \( K \).

Fiction: Black-Scholes works, So why teach BS??
i.e., volatility, drift and risk-free rates are constant.

Fiction: Home mortgage interest rates stay constant over thirty year periods.

Nevertheless: They’re useful, because . . .
they give a way of comparing mortgages.

Similar for Black-Scholes.

Next subtopic: Volatility smiles and skews and volatility surfaces
Definition:
For all $V > 0$, $T > 0$, $r_*>0$, $S_0 > 0$ and $K > 0$, if there exists $\sigma_* > 0$ such that
\[ V = \text{BlSch}(T, \sigma_*, r_*, S_0, K) \]
then this solution $\sigma_*$ is unique and is called the **implied volatility** associated to $V$, $T$, $r_*$, $S_0$ and $K$.

Fiction: Black-Scholes works, i.e., volatility, drift and risk-free rates are constant.

Pick a financial instrument (e.g., a stock). Look up $S_0$. Look up $r_*$. Fix $T$. For various choices of $K$, look up $V$, compute $\sigma_*$ and plot $(K, \sigma_*)$. The result is called the **volatility smile** or the **volatility skew**, depending on whether it's concave up or concave down.
Definition:
For all $V > 0$, $T > 0$, $r_*>0$, $S_0 > 0$ and $K > 0$, if $\exists \sigma_* > 0$ such that

\[ V = \text{BISch}(T, \sigma_*, r_*, S_0, K) \]

then this solution $\sigma_*$ is unique and is called the \textbf{implied volatility} associated to $V$, $T$, $r_*$, $S_0$ and $K$.

Fiction: Black-Scholes works, \textit{i.e.}, volatility, drift and risk-free rates are constant.

Pick a financial instrument (\textit{e.g.}, a stock).
Look up $S_0$. Look up $r_*$.
For various choices of $K$ and $T$,
look up $V$, compute $\sigma_*$ and plot $(K, T, \sigma_*)$.
The result is called the \textbf{volatility surface}.

\[ \text{STOP} \]