# Financial Mathematics

## Preliminaries to
## the Triangular Central Limit Theorem
## and a first proof of Black-Scholes

Fact (Second order Maclaurin approximations):

$f''$ is continuous at 0 $\Rightarrow$ $\exists \varepsilon(x) \to 0$ s.t.

$$f(x) = [f(0)] + [f'(0)]x + \left[\frac{f'(0)}{2}\right] x^2 + \boxed{[\varepsilon(x)]x^2}.$$

error term

$o(x^2)$

Fact (First order Maclaurin approximations):

$f'$ is continuous at 0 $\Rightarrow$ $\exists \varepsilon(x) \to 0$ s.t.

as $x \to 0$

$$f(x) = [f(0)] + [f'(0)]x + \boxed{[\varepsilon(x)]x}.$$

error term

$o(x)$

Fact (Second order Maclaurin approximations):

$f''$ is continuous at 0 $\Rightarrow$ $\exists \varepsilon(x) \to 0$ s.t.

$$f(x) = [f(0)] + [f'(0)]x + \left[\frac{f'(0)}{2}\right] x^2 + [\varepsilon(x)]x^2.$$

Special case: $e^x = 1 + x + \frac{x^2}{2} + [\varepsilon(x)]x^2$

$x :\to x_n, \varepsilon_n := \varepsilon(x_n) \to 0$, as $n \to \infty$

Corollary: $x_n \to 0 \Rightarrow$

$\exists \varepsilon_n \to 0$ s.t. $e^{x_n} = 1 + x_n + \frac{x_n^2}{2} + \varepsilon_n x_n^2$

$x_n :\to 3x_n$

Corollary: $x_n \to 0 \Rightarrow$

$9\varepsilon_n :\to \delta_n$

$\exists \varepsilon_n \to 0$ s.t. $e^{3x_n} = 1 + 3x_n + \frac{9x_n^2}{2} + 9\varepsilon_n x_n^2$

*i.e.*:

$\exists \delta_n \to 0$ s.t. $e^{3x_n} = 1 + 3x_n + \frac{9x_n^2}{2} + \delta_n x_n^2$

"$9[o(x_n^2)] = o(x_n^2)$"

$o(x_n^2)$

Def'n: $\boxed{h(x)} := e^{-x^2/2}/\sqrt{2\pi}$

Def'n: $Z_n \to Z$ in distribution means:

for any contin., bounded $\phi : \mathbb{R} \to \mathbb{R}$,

$$E[\phi(Z_n)] \to E[\phi(\boxed{Z})].$$

$Z$ not yet def'd, so. . .

Replace $E[\phi(Z)]$

by $\displaystyle\int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx$

"Change every $Z$ to $x$
and then integrate against $h(x)\,dx$,
from $-\infty$ to $\infty$.

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$$\int_{-\infty} [\phi(x)][h(x)]\,dx$$

Def'n: $\boxed{h(x)} := e^{-x^2/2}/\sqrt{2\pi}$

Def'n: $Z_n \to Z$ in distribution means:

equivalently: continuous, compactly supported

for any ~~contin., bounded~~ $\phi : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{E}[\phi(Z_n)] \to \int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx.$$

Fact: $Z_n \to Z$ in distribution

$\alpha_n \to 0$ in $\mathbb{R}$

$\Rightarrow Z_n + \alpha_n \to Z$ in distribution.

pf omitted

Fact: $Z_n \to Z$ in distribution

$\alpha_n \to 1$ in $\mathbb{R}$

$\Rightarrow \alpha_n Z_n \to Z$ in distribution.

pf omitted

Def'n: $\boxed{h(x)} := e^{-x^2/2}/\sqrt{2\pi}$

against contin, exp-bdd

Def'n: $Z_n \to Z$ in distribution means:

continuous, exponentially bounded

for any ~~contin., bounded~~ $\phi : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{E}[\phi(Z_n)] \to \int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx.$$

Fact: $Z_n \to Z$ in distribution against contin, exp-bdd

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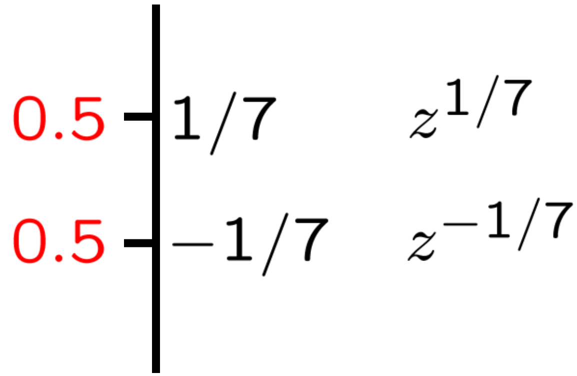
pf omitted

$D_1/7$ :




**What about $D_1/7$?**
**Replace $t$ by $t/7$.**

$$i = \sqrt{-1}$$

Replace $z$ by $e^{-it}$

Generating function: $(0.5)z^{1/7} + (0.5)z^{-1/7}$

Fourier transform: $(0.5)e^{-it/7} + (0.5)e^{it/7}$

$$\| $$

$$\cos(t/7)$$

$$e^{it/7} = \cos(t/7) + i\sin(t/7)$$
$$e^{-it/7} = \cos(t/7) - i\sin(t/7)$$

$D_1/7$ :

| | | |
|---|---|---|
| 0.5 | $1/7$ | $z^{1/7}$ |
| 0.5 | $-1/7$ | $z^{-1/7}$ |

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$$\| $$

$$\cos(t/7)$$

The Fourier transf. of the distr. of $X/7$
is equal to
$\left(\text{the Fourier transf. of the distr. of } X\right)_{t:\to t/7}$

$D_2 = H_2 - T_2$ :

| | | |
|---|---|---|
| 0.25 | 2 | $z^2$ |
| 0.5 | 0 | $z^0 = 1$ |
| 0.25 | $-2$ | $z^{-2}$ |

forget its origin keep the distribution

Generating function:

$$(0.25)z^2 + 0.5 + (0.25)z^{-2}$$
$$= ((0.5)z + (0.5)z^{-1})^2$$

the generating function of the distribution of $D_1$

$$i = \sqrt{-1}$$

Replace $z$ by $e^{-it}$

Fourier transform: $(\cos t)^2 = \cos^2 t$

$D_2 = H_2 - T_2$ :

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Note: $D_2$ is a sum of two independent PCRVs.

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Fourier transform: $(\cos t)^2 = \cos^2 t$

Note: $D_2$ is a sum of two independent PCRVs. The Fourier transf. of the distr. of either one is $\cos t$.

$X$ and $Y$ independent $\Rightarrow$
the Fourier transf. of the distr. of $X + Y$
is the product of
the Fourier transf. of the distr. of $X$
and
the Fourier transf. of the distr. of $Y$.

Computing the distribution of $X+Y$
from the distributions of $X$ and $Y$
is generally impossible;
you need the JOINT distribution of $X$ and $Y$.

If $X$ and $Y$ are independent,
then the joint distribution of $X$ and $Y$
is the "product" of
the distributions of $X$ and $Y$,
and the distribution of $X+Y$
can be obtained from
the distributions of $X$ and $Y$,
by a complicated process called "convolution".

---

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More on all that in a later lecture...
For now, just remember:
Fourier transf. simplifies convolution
to multiplication.

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$Z$:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \quad \Big|\quad x \qquad z^x$$

Do this for *all* $x \in \mathbb{R}$

Generating function:

$$\int_{-\infty}^{\infty} z^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \text{Exercise}$$

Fourier transform:

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$.

Then $Z$ is "close" to $X$. because:(F. transf. of distr. of $X$) $\approx e^{-t^2/2}$

$Z$:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ — $x$     $z^x$

Do this for *all* $x \in \mathbb{R}$

---

If the Fourier transforms
of the distributions of PCRVs $X_1, X_2, \ldots$
approach $e^{-t^2/2}$,
then the distributions of $X_1, X_2, \ldots$
approach the distribution shown above,
*i.e.*, $X_n \to Z$ in distribution.

---

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Then $Z$ is "close" to $X$. because:(F. transf. of distr. of $X$) $\approx e^{-t^2/2}$

Def'n: For any PCRV $Y$,

$$\boxed{\mathcal{F}\delta[Y]} = \boxed{\mathcal{F}\delta_Y} := \mathrm{E}[e^{-itY}] = \left[\mathrm{E}[z^Y]\right]_{z:\to e^{-it}}$$

is **the Fourier transform** $\mathcal{F}$ **of the distribution** $\delta$ **of** $Y$.

---

If the Fourier transforms of the distributions of PCRVs $X_1, X_2, \ldots$ approach $e^{-t^2/2}$,

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*i.e.*, $X_n \to Z$ in distribution.

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Key idea of Central Limit Theorem:

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$.

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is **the Fourier transform** $\mathcal{F}$ **of the distribution** $\delta$ **of** $Y$.

Fact: $\mathcal{F}\delta_{X/7} = [\mathcal{F}\delta_X]_{t:\to t/7}$

Fact: $\mathcal{F}\delta_{cX} = [\mathcal{F}\delta_X]_{t:\to ct}$

Fact: $X_1, \ldots, X_n$ indep. $\Rightarrow$

$$\mathcal{F}\delta_{X_1+\cdots+X_n} = [\mathcal{F}\delta_{X_1}]\cdots[\mathcal{F}\delta_{X_n}]$$

Fact: $\mathcal{F}\delta_{Y_n} \to e^{-t^2/2}$

$\Rightarrow$ $Y_n \to Z$ in distribution.

pf omitted, but here's the idea. . .

$\mathcal{F}\delta_Z = e^{-t^2/2}$

$\mathcal{F}\delta_{Y_n} \to \mathcal{F}\delta_Z$

Apply $\mathcal{F}^{-1}$.

$\delta_{Y_n} \to \delta_Z$

QED

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Def'n: Let $\mathcal{A}$ be a set of PCRVs.

We say $\mathcal{A}$ is **identically distributed** or **i.d.**

if $\forall A, B \in \mathcal{A}$,

$A$ and $B$ have the same distribution.

Def'n: If $\mathcal{A}$ is i.d., then alternate: $\mathcal{F}\delta_{\mathcal{A}}$

$\boxed{\mathrm{E}[\mathcal{A}]}$, $\boxed{\mathrm{Var}[\mathcal{A}]}$, $\boxed{\mathrm{SD}[\mathcal{A}]}$, $\boxed{\mathcal{F}\delta[\mathcal{A}]}$

are defined as alternate: $\mathcal{F}\delta_A$

$\mathrm{E}[A]$, $\mathrm{Var}[A]$, $\mathrm{SD}[A]$, $\mathcal{F}\delta[A]$, resp.

for *any* $A \in \mathcal{A}$.

Def'n: $\forall$set $\mathcal{A}$ of PCRVs, $\forall c \in \mathbb{R}$,

$\boxed{c\mathcal{A}} := \{cA \,|\, A \in \mathcal{A}\}$, $\boxed{c + \mathcal{A}} := \{c + A \,|\, A \in \mathcal{A}\}$

Fact: If $\mathcal{A}$ is i.d., then $c\mathcal{A}$ is i.d., $c{+}\mathcal{A}$?

and

$$\mathrm{E}[c\mathcal{A}] = c(\mathrm{E}[\mathcal{A}])$$

$$\mathrm{Var}[c\mathcal{A}] = c^2(\mathrm{Var}[\mathcal{A}])$$

$$\mathrm{SD}[c\mathcal{A}] = |c|(\mathrm{SD}[\mathcal{A}]).$$

---

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Fact: If $\mathcal{A}$ is i.d., then $c + \mathcal{A}$ is i.d.,

and

E[$c + \mathcal{A}$] $= c +$ (E[$\mathcal{A}$])

Var[$c + \mathcal{A}$] $=$ Var[$\mathcal{A}$]

SD[$c + \mathcal{A}$] $=$ SD[$\mathcal{A}$].

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Def'n: $\mathcal{S}$ := {standard PCRVs}

$$= \{\text{PCRVs } X \mid \text{E}[X] = 0 \ \& \ \text{Var}[X] = 1\}$$

Exercise: Show that $\mathcal{S}$ is not i.d.

---

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Fact: If $\mathcal{A}$ is i.d., then $c + \mathcal{A}$ is i.d.,

and

$$\text{E}[c + \mathcal{A}] = c + (\text{E}[\mathcal{A}])$$
$$\text{Var}[c + \mathcal{A}] = \text{Var}[\mathcal{A}]$$
$$\text{SD}[c + \mathcal{A}] = \text{SD}[\mathcal{A}].$$

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Def'n: Let $\mathcal{A}$ be a set of PCRVs.
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Exercise: Show that $\mathcal{S}$ is not i.d.

Hint: Find two standard binary PCRVs with different distributions.

Fact: If $\mathcal{A}$ is i.d., then

$$\mathcal{A} \subseteq \mathcal{S} \quad \Leftrightarrow \quad \mathrm{E}[\mathcal{A}] = 0 \text{ and } \mathrm{Var}[\mathcal{A}] = 1.$$

Fact: If $\mathcal{A}$ is i.d., and if $\mathcal{A} \cap \mathcal{S} \neq \emptyset$, then $\mathcal{A} \subseteq \mathcal{S}$.

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We say $\mathcal{A}$ is **identically distributed** or **i.d.** if $\forall A, B \in \mathcal{A}$,

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Def'n: For any set $\mathcal{A}$ of PCRVs,
let $\sum^n \mathcal{A}$ denote the set of all $A_1 + \cdots + A_n$
such that $A_1, \ldots, A_n \in \mathcal{A}$
and such that $A_1, \ldots, A_n$ are i.i.d.

Fact: If $\mathcal{A}$ is i.d., and if $n \geq 1$ is an integer
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Fact: If $\mathcal{A}$ is i.d., and if $n \geq 1$ is an integer
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and

$$\mathrm{E}[\textstyle\sum^n \mathcal{A}] = n(\mathrm{E}[\mathcal{A}])$$ ← DIVIDE BY $\sqrt{n}$

$$\mathrm{Var}[\textstyle\sum^n \mathcal{A}] = n(\mathrm{Var}[\mathcal{A}])$$ ← DIVIDE BY $n$

$$\mathrm{SD}[\textstyle\sum^n \mathcal{A}] = \sqrt{n}(\mathrm{SD}[\mathcal{A}])$$

$$\mathcal{F}\delta[\textstyle\sum^n \mathcal{A}] = (\mathcal{F}\delta[\mathcal{A}])^n.$$

Fact: $\sum^n (\frac{c}{n} + \mathcal{A}) = c + (\sum^n \mathcal{A})$

Fact: $\sum^n (c\mathcal{A}) = c(\sum^n \mathcal{A})$

$\mathrm{E}[(\sum^n \mathcal{A})/\sqrt{n}] = \sqrt{n}(\mathrm{E}[\mathcal{A}])$ → 0

$\mathrm{Var}[(\sum^n \mathcal{A})/\sqrt{n}] = \mathrm{Var}[\mathcal{A}]$ → 1

Fact: $\mathcal{A} \subseteq \mathcal{S}$ ?? $\Leftarrow$ $\Rightarrow$ $[\sum^n \mathcal{A}]/\sqrt{n} \subseteq \mathcal{S}$

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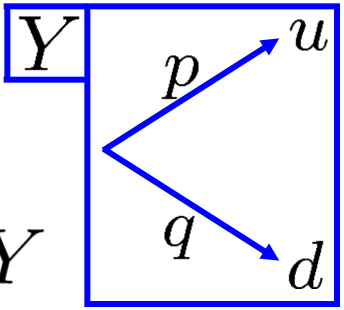
Fact: $\mathcal{A} \subseteq \mathcal{S} \Leftrightarrow [\sum^n \mathcal{A}]/\sqrt{n} \subseteq \mathcal{S}$

renormalized

i.i.d. sum preserves and reflects standardness

Def'n: $\forall p, q \in [0,1]$, s.t. $p+q=1$, $\forall u, d \in \mathbb{R}$, s.t. $d<u$,




let $\mathcal{B}^{p,u}_{q,d}$ be the set of binary PCRVs $Y$ such that $\Pr[Y=u]=p$ and $\Pr[Y=d]=q$.

Fact: $\mathcal{B}^{p,u}_{q,d}$ is i.d.

Fact: $c \neq 0 \quad \Rightarrow \quad c\mathcal{B}^{p,u}_{q,d} = \mathcal{B}^{p,cu}_{q,cd}$

Fact: $c + \mathcal{B}^{p,u}_{q,d} = \mathcal{B}^{p,u+c}_{q,d+c}$

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Fact: $\sum^n(\frac{c}{n} + \mathcal{A}) = c + (\sum^n \mathcal{A})$

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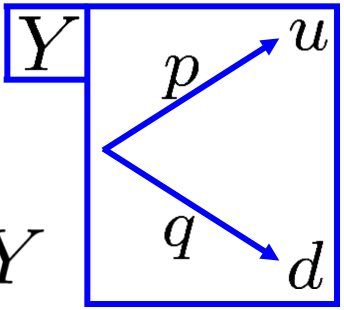


specific

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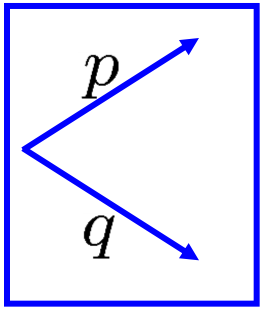
Fact: $c \neq 0 \quad \Rightarrow \quad c\mathcal{B}_{q,d}^{p,u} = \mathcal{B}_{q,cd}^{p,cu}$

Fact: $c + \mathcal{B}_{q,d}^{p,u} = \mathcal{B}_{q,d+c}^{p,u+c}$

Def'n: $\forall p, q \in [0,1]$, s.t. $p + q = 1$ $\quad \mathcal{B}_q^p := \bigcup_{d<u} \mathcal{B}_{q,d}^{p,u}$

general




$X_1, \ldots, X_n \in \bigcup_{d<u} \mathcal{B}_{q,d}^{p,u}$ i.d. $\Rightarrow \exists (d < u)$ s.t. $X_1, \ldots, X_n \in \mathcal{B}_{q,d}^{p,u}$

$$\Sigma^n \mathcal{B}_q^p = \Sigma^n \bigcup_{d<u} \mathcal{B}_{q,d}^{p,u} = \bigcup_{d<u} \Sigma^n \mathcal{B}_{q,d}^{p,u}$$

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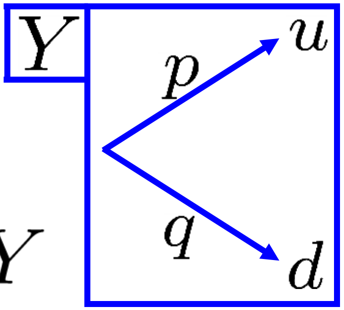


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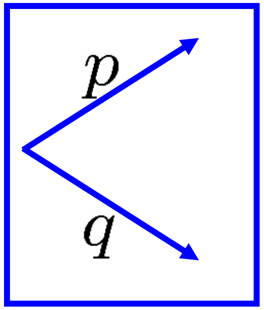


The iid sum of general binaries is the union of the iid sums of specific binaries.

$$\sum^n \mathcal{B}^{p}_{q} = \sum^n \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d} = \bigcup_{d<u} \sum^n \mathcal{B}^{p,u}_{q,d}$$

Lemma: If $X \in \sum^n \mathcal{B}_q^p, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0,$

then $\frac{X-\alpha}{\beta} \in \sum^n \mathcal{B}_q^p.$

Proof: Choose $u, d \in \mathbb{R}$

s.t. $X \in \sum^n \mathcal{B}_{q,d}^{p,u}.$

Then $X - \boxed{\alpha} \in \sum^n \left[\left(\mathcal{B}_{q,d}^{p,u}\right) - \boxed{\frac{\alpha}{n}}\right]$

Def'n: $\forall p, q \in [0, 1]$, s.t. $p + q = 1$

$$\boxed{\mathcal{B}_q^p} := \bigcup_{d<u} \mathcal{B}_{q,d}^{p,u}$$

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$$\sum^n \mathcal{B}_q^p = \sum^n \bigcup_{d<u} \mathcal{B}_{q,d}^{p,u} = \bigcup_{d<u} \sum^n \mathcal{B}_{q,d}^{p,u}$$

Lemma: If $X \in \sum^n \mathcal{B}_q^p$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$,

then $\frac{X-\alpha}{\beta} \in \sum^n \mathcal{B}_q^p$.

Special case:

$\alpha = \mathrm{E}[X]$,

$\beta = \mathrm{SD}[X]$.

Proof: Choose $u, d \in \mathbb{R}$

s.t. $X \in \sum^n \mathcal{B}_{q,d}^{p,u}$.

Then $X - \alpha \in \sum^n \left[ \left( \mathcal{B}_{q,d}^{p,u} \right) - \frac{\alpha}{n} \right]$

$$= \sum^n \mathcal{B}_{q,d-(\alpha/n)}^{p,u-(\alpha/n)},$$

so $\frac{X-\alpha}{\beta} \in \sum^n \left[ \frac{1}{\beta} \left( \mathcal{B}_{q,d-(\alpha/n)}^{p,u-(\alpha/n)} \right) \right]$

$$= \sum^n \mathcal{B}_{q,[d-(\alpha/n)]/\beta}^{p,[u-(\alpha/n)]/\beta} \subseteq \sum^n \mathcal{B}_q^p.$$ QED

Uptick/downtick *VALUES* change,
but uptick/downtick *PROBABILITIES* do not.

Lemma: If $X \in \sum^n \mathcal{B}_q^p$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$,

then $\frac{X-\alpha}{\beta} \in \sum^n \mathcal{B}_q^p$.

Special case:

$\alpha = \mathrm{E}[X]$,

$\beta = \mathrm{SD}[X]$.

Def'n: $X_\circ = \dfrac{X - (\mathrm{E}[X])}{\mathrm{SD}[X]}$

= the **standardization of** $X$

Lemma: If $X \in \sum^n \mathcal{B}_q^p$,

then $X_\circ \in \sum^n \mathcal{B}_q^p$.

Def'n: For any set $\mathcal{A}$ of PCRVs,
let $\sum^n \mathcal{A}$ denote the set of all $A_1 + \cdots + A_n$
such that $A_1, \ldots, A_n \in \mathcal{A}$
and such that $A_1, \ldots, A_n$ are i.i.d.

Def'n: For any set $\mathcal{A}$ of PCRVs,
let $\prod^n \mathcal{A}$ denote the set of all $A_1 \cdots A_n$
such that $A_1, \ldots, A_n \in \mathcal{A}$
and such that $A_1, \ldots, A_n$ are i.i.d.

Def'n: $\exp(\mathcal{A}) := e^{\mathcal{A}} := \{e^A \,|\, A \in \mathcal{A}\}$.

Fact: $e^{\sum^n \mathcal{A}} = \prod^n e^{\mathcal{A}}$,
*i.e.*, $\exp(\sum^n \mathcal{A}) = \prod^n [\exp(\mathcal{A})]$

Fact: $\mathcal{A}$ is i.d. $\Rightarrow$ $\prod^n \mathcal{A}$ is i.d.
and $\mathrm{E}[\prod^n \mathcal{A}] = (\mathrm{E}[\mathcal{A}])^n$.

STOP